ON A THEOREM OF FEJÉR

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1. Let

$$T: (\tau_{n\nu}) \qquad (n = 0, 1, 2, \cdots; \nu = 0, 1, 2, \cdots)$$

be an infinite Toeplitz matrix satisfying the conditions

(i) $$\lim \tau_{n\nu} = 0$$

for every fixed ν,

(ii) $$\lim \sum_{\nu=0}^{\infty} \tau_{n\nu} = 1$$

and

(iii) $$\sum_{\nu=0}^{\infty} |\tau_{n\nu}| \leqq K ,$$

K being an absolute constant independent of n.

Given a sequence (S_n) if

$$\lim \sum_{\nu=0}^{\infty} \tau_{n\nu} S_\nu = S ,$$

then we say that the sequence (S_n) or the series with partial sums S_n is summable (T) to the sum S.

2. Suppose that $f(x)$ is integrable in the Lebesgue sense and periodic with period 2π. Let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

Let

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum B_n(x)$$

be the derived series of the Fourier series of $f(x)$. Fixing x, we write

$$\psi_x(t) = f(x + t) - f(x - t) .$$

Fejér [1] has proved the following

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THEOREM A. *If $f(x)$ is of bounded variation in $(0, 2\pi)$, then $\{B_n(x)\}$ is summable (C, r) to the jump $l(x) = \{f(x+0) - f(x-0)\}/\pi$ for every $r > 0$ at each point x.*

Recently, Siddiqi [3] extended Fejér's result and established the following

THEOREM B. *Let $\Lambda\colon (\lambda_{n\nu})$ be a triangular Toeplitz matrix,* i.e., *$\lambda_{n\nu} = 0$ for $\nu > n$. If it satisfies, in addition, the condition*

$$\text{(iv)} \qquad \sum_{\nu=0}^{n} | \Delta\lambda_{n\nu} | = 0(1)$$

as $n \to \infty$, then $\{B_n(x)\}$ is summable (Λ) to $l(x)$.

It is known [2] that a series which is summable by the harmonic means is also summable (C, r) for every $r > 0$ but not conversely. We take, for the (C, r) means, $\lambda_{n\nu} = A_{n-\nu}^{r-1}/A_n^r$,

$$A_n^r = \Gamma(n + r + 1)/\Gamma(n + 1)\Gamma(r + 1) ,$$

and for the harmonic means, $\lambda_{n\nu} = 1/(n - \nu + 1)$. Both satisfy (iv). Thus, we infer that Siddiqi's theorem contains Fejér's as a special case.

In this note, we develop Siddiqi's theorem into the following general form for the summability (T) of $\{B_n(x)\}$ at a given point.

THEOREM. *If $\psi_x(t)$ is of bounded variation in the neighborhood of $t=0$ and absolutely continuous in (η, π) for any $0 < \eta < \pi$, then $\{B_n(x)\}$ is summable (T) to the jump $l(x)$ at x.*

3. Let us consider

$$\begin{aligned}
\sigma_n(x) &= \sum_{\nu=0}^{\infty} \tau_{n\nu} B_\nu(x) \\
&= \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} \int_0^\pi \psi_x(t) \nu \sin \nu t dt \\
&= l(x) \sum_{\nu=0}^{\infty} \tau_{n\nu} + \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} \int_0^\pi \cos \nu t d\psi_x(t) \\
&= l(x) + 0(1) + \frac{1}{\pi} \sum_{\nu=0}^{\infty} \tau_{n\nu} I_\nu .
\end{aligned}$$

We are going to prove that $\sum \tau_{n\nu} I_\nu = 0(1)$ as $n \to \infty$. Since $\psi_x(t)$ is of bounded variation in the neighborhood of $t = 0$, for a given $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$\int_0^\delta | d\psi_x(t) | < \varepsilon .$$

Write

$$I_\nu = \left(\int_0^\delta + \int_\delta^\pi\right) \cos \nu t d\psi_x(t)$$
$$= I'_\nu + II''_\nu ,$$

say. Then

$$\left|\sum_{\nu=0}^{\infty} \tau_{n\nu} I'_\nu\right| \leqq \sum_{\nu=0}^{\infty} |\tau_{n\nu}| \int_0^\delta |d\psi_x(t)|$$
$$< \varepsilon \sum_{\nu=0}^{\infty} |\tau_{n\nu}|$$
$$\leqq K\varepsilon .$$

Remembering that $\psi_x(t)$ is absolutely continuous in (δ, π), we have

$$\int_\delta^\pi \cos \nu t d\psi_x(t) = \int_\delta^\pi \cos \nu t \psi'_x(t) dt .$$

For the given $\varepsilon > 0$, we can find ν_0 such that

$$\left|\int_\delta^\pi \cos \nu t \psi'_x(t) dt\right| < \varepsilon$$

for $\nu < \nu_0$ by Riemann-Lebesgue's theorem. Fixing ν_0, we can take a positive integer n_0 making $|\tau_{n\nu}| < \varepsilon/(\nu_0 + 1)$ $0 \leqq \nu \leqq \nu_0$, $n < n_0$. If we write

$$\sum_{\nu=0}^{\infty} \tau_{n\nu} I''_\nu = \left(\sum_{\nu=0}^{\nu_0} + \sum_{\nu_0+1}^{\infty}\right) \tau_{n\nu} \int_\delta^\pi \cos \nu t \psi'_x(t) dt$$
$$= I_1 + I_2 ,$$

say, then

$$|I_1| \leqq \sum_{\nu=0}^{\nu_0} |\tau_{n\nu}| \int_\delta^\pi |\psi'_x(t)| dt$$
$$\leqq M \sum_{\nu=0}^{\nu_0} |\tau_{n\nu}|$$
$$< M(\nu_0 + 1)/(\nu_0 + 1)$$
$$= M\varepsilon .$$

for $n > n_0$, where

$$M = \int_0^\pi |\psi'_x(t)| dt .$$

$$|I_2| = \left|\sum_{\nu=\nu_0+1}^{\infty} \tau_{n\nu} \int_\delta^\pi \cos \nu t \psi'_x(t) dt\right|$$
$$< \varepsilon \sum_{\nu=\nu_0+1}^{\infty} |\tau_{n\nu}|$$

$$\leqq \varepsilon \sum_{\nu=0}^{\infty} |\tau_{n\nu}|$$
$$\leqq K\varepsilon$$

by (iii). From the above analysis, it follows that

$$\left|\sum_{\nu=0}^{\infty} \tau_{n\nu} I_\nu\right| < (M + 2K)\varepsilon$$

for $n > n_0$. Since ε is an arbitrary quantity, we obtain $\sum \tau_{n\nu} I_\nu = 0(1)$ as $n \to \infty$. This proves the theorem.

REFERENCES

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