INFINITELY DIVISIBLE PROBABILITIES
ON THE HYPERBOLIC PLANE

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1. **Introduction.** This paper may be regarded from two points of view. First of all it presents the theory of infinitely divisible radially symmetric probability measures on the hyperbolic plane and the naturally associated limit theorems. This point of view provided the motivation for the present paper and is explained in some detail in § 2 and 3. However, just as the analagous theory in the Euclidean case may be viewed as a chapter in the theory of Fourier transforms, so may the present theory be viewed as a chapter in the theory of Legendre transforms. That is, by using the harmonic analysis described in §§ 2 and 3 one can set up a one-to-one correspondence between the radially symmetric probability measures, μ, on the hyperbolic plane and certain functions of a complex variable $\phi(z)$ in such a way that the convolution of μ_1 and μ_2 on the hyperbolic plane corresponds to the pointwise product of their "transforms" ϕ_1 and ϕ_2. Since μ is radially symmetric it is completely specified by a distribution function $F(\lambda)$ on $\lambda \geqq 0$ and the correspondence between ϕ and μ (or F) is given by

$$(1.1) \qquad \phi(z) = \int_0^\infty K(z, \lambda) dF(\lambda)$$

where $K(z, \lambda)$ is a certain Legendre function given by (4.9). The convolution of μ_1 and μ_2, at least in the case where F_1 and F_2 have densities, is written down explicitly in (3.9).

This second point of view is adopted for the most part beginning in § 4 and so the majority of the paper (sections 4–10) deals with certain problems in the theory of the Legendre transform (1.1). The tools we use are those of classical analysis, but the problems treated are motivated by probability theory. The main results of the paper are contained in §§ 7 and 8. In § 10 Gaussian and stable distributions are defined within the present context. Finally in § 11 we indicate the extensions of these ideas to a wider class of Legendre transforms which includes the theory of radially symmetric probability measures on the higher dimensional hyperbolic spaces as special cases.

We would like to thank Professor H. P. McKean who first introduced us to the material in § 2, and who expressed interest and encouragement when the present paper was in its formative stages.

2. **General Remarks.** The present section is devoted to a general situation that we will specialize to the hyperbolic plane in the next section. We follow, more or less, the expository article of Godement [4]. Let G be a locally compact second countable topological group and let K be a compact subgroup. Let x, y, z denote elements of G and u, v elements of K. We define two equivalence relations on G as follows:

(2.1) $$x \sim y \Longleftrightarrow xy^{-1} \in K$$

(2.2) $\quad x \approx y \Longleftrightarrow$ there exist $u, v \in K$ such that $x = uyv$.

Thus $H = G/\sim$ is the space of right cosets and $R = G/\approx$ is the space of double cosets. We give H and R the usual quotient space topology. Let dx be *right invariant* Haar measure on G, then dx induces measures dh and dr on H and R which are invariant under the (right) action of G. In order to avoid notational complications it will be convenient to regard all functions as being defined on G. Thus the statement $f \in L_1(R)$ will mean $f(x) = f(y)$ if $x \approx y$ and $\int |f(x)| dx < \infty$ with the obvious conventions for functions defined almost everywhere. Thus we have

(2.3) $$L_p(R) \subset L_p(H) \subset L_p(G)$$

for each $p > 0$.

If f and g are in $L_1(G)$ we define their convolution

(2.4) $$f * g . (x) = \int f(xy^{-1})g(y)dy .$$

It is well known that $L_1(G)$ is a Banach algebra and it is immediate that $L_1(R)$ and $L_1(H)$ are closed sub-algebras. The basic assumption of [4] is that $L_1(R)$ be *commutative*. Selberg [6] has shown that if H is a symmetric (or more generally, weakly symmetric) space then $L_1(R)$ is commutative. For us the following simple sufficient condition (whose proof is a routine calculation and is therefore omitted) will suffice.

THEOREM 2.1[1]. *If* $x \approx x^{-1}$ *for all* x *in* G, *then* $L_1(R)$ *is commutative.*

In the remainder of this section we will assume that $L_1(R)$ is commutative. If α is a multiplicative linear functional on $L_1(R)$ then

(2.5) $$\alpha(f) = \int p_\alpha(x)f(x)dx$$

where p_α is in $L_\infty(R)$. It is easy to see that α defined by (2.5) is a multiplicative linear functional if and only if

(2.6) $$p_\alpha(x)p_\alpha(y) = \int_K p_\alpha(xuy)du$$

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[1] This remark is due to H. P. McKean.

for almost all x, y. Here du is *normalized* Haar measure on K. Moreover (2.6) implies that p_α is equal almost everywhere to a continuous function and thus the multiplicative linear functionals on $L_1(R)$ may be identified with the bounded continuous functions on R satisfying (2.6). Such functions are called *spherical* functions on R.

Let C_∞ be the continuous functions on G with compact support. If $f \in C_\infty$ we define $\tilde{f}(x) = \bar{f}(x^{-1})$. A (signed) Radon measure μ on G is said to be of *positive type* if

$$(2.7) \qquad\qquad \mu(\tilde{f} * f) \geqq 0$$

for all $f \in C_\infty$. A continuous function p on G is of positive type if the measure $p(x)dx$ is. Let $\hat{R}$ be the totality of all spherical functions on R which are of positive type. For $f \in L_1(R)$ we define

$$(2.8) \qquad\qquad \hat{f}(p) = \int f(x)p(x)dx \quad \text{for} \quad p \in \hat{R} \, ,$$

then, at least if G is unimodular, one can develop a complete theory of harmonic analysis including a Plancherel theorem. For details see [4]. Since we won't need this general theory we will terminate our general discussion at this point.

3. The Hyperbolic Plane.

Let D be the interior of the unit disc in the complex plane, i.e., $D = \{z : z \text{ complex}, |z| < 1\}$. The set D furnished with the Riemannian metric

$$(3.1) \qquad\qquad ds^2 = 4(1 - r^2)^{-2}[dx^2 + dy^2]$$

where $r^2 = |z|^2 = x^2 + y^2$ will be called the hyperbolic plane. The geodesic joining z_1 and z_2 is the unique circle through them cutting the circle $|z| = 1$ orthogonally. The hyperbolic distance $\zeta(z_1, z_2)$ between z_1 and z_2 is given by

$$(3.2) \qquad\qquad th\zeta/2 = |z_1 - z_2| \cdot |1 - \bar{z}_1 z_2|^{-1}$$

where "*th*" denotes the hyperbolic tangent, similarly "*ch*" and "*sh*" will denote the hyperbolic cosine and sine. See [2] for a discussion of the hyperbolic plane including the above facts.

If u and b are complex numbers with $|u| = 1$ and $|b| < 1$ we define the hyperbolic motion (u, b) as follows:

$$(3.3) \qquad\qquad (u, b) : z \to u(z - b)(1 - \bar{b}z)^{-1} \, .$$

It is easy to check that (3.3) maps D onto D and preserves the hyperbolic distance (3.2). Let G be the totality of all such motions with the obvious topology. Clearly G is a topological group satisfying the hypotheses of § 2. The multiplication in G is composition, i.e.,

$$(3.4) \qquad (u_1, b_1)(u_2, b_2) = \left(\frac{u_1(u_2 + b_1\bar{b}_2)}{u_2(\bar{u}_2 + \bar{b}_1 b_2)}, \frac{u_2 b_2 + b_1}{u_2 + b_1\bar{b}_2} \right)$$

and

$$(3.5) \qquad (u, b)^{-1} = (\bar{u}, -ub) .$$

Let K be the compact subgroup consisting of motions $(u, 0)$, then K is just the circle group. It is straight forward to check that the equivalence relations defined in (2.1) and (2.2) become

$$(3.6) \qquad (u_1, b_1) \sim (u_2, b_2) \Longleftrightarrow b_1 = b_2$$

$$(3.7) \qquad (u_1, b_1) \approx (u_2, b_2) \Longleftrightarrow |b_1| = |b_2| .$$

Thus H can be identified with D and R with the half-open segment $0 \leqq r < 1$. Moreover Haar measure in G can be chosen so that the invariant measure induced in $H = D$ is just that associated with the Riemannian metric (3.1).

It is convenient to introduce geodesic polar coordinates (ζ, θ) in D with pole at $z = 0$. Here ζ is the hyperbolic distance and the coordinates of the point $z = re^{i\theta}$ with $0 \leqq \theta < 2\pi$ are $(\log 1 + r/1 - r, \theta)$. In terms of these coordinates we have $ds^2 = d\zeta^2 + sh^2\zeta d\theta^2$ and the corresponding volume element is given by

$$(3.8) \qquad sh\zeta \, d\zeta \, d\theta .$$

We now regard R as the half-line $0 \leqq \zeta < \infty$ and write $f(ch\zeta)$ for the generic function defined on R. Theorem 2.1, (3.5), and (3.7) imply that $L_1(R)$ is commutative and routine calculations show that if $f, g \in L_1(R)$, then their convolution is given by

$$(3.9) \qquad f*g.(ch\zeta) = \int_0^\infty \int_0^{2\pi} f(ch\lambda)g[ch\zeta ch\lambda - sh\zeta sh\lambda \cos \theta]sh\lambda d\theta d\lambda$$

$$= \int_0^\infty \int_0^{2\pi} f[ch\zeta ch\lambda - sh\zeta sh\lambda \cos \theta]g(ch\lambda)sh\lambda d\theta d\lambda .$$

Moreover the defining equation (2.6) for spherical functions becomes

$$(3.10) \qquad p(ch\lambda)p(ch\zeta) = \frac{1}{2\pi} \int_0^{2\pi} p[ch\lambda ch\zeta - sh\lambda sh\zeta \cos \theta]d\theta .$$

From (3.10) one can show that p is a solution of the Legendre differential equation and since $p(1) = 1$, it follows that the solutions of (3.10) are $P_\nu(ch\zeta)$ where P_ν denotes the Legendre function of the first kind. See [3]. Equation (3.10) is then a simple consequence of the usual addition theorem for Legendre functions [3].

Since the spherical functions are bounded we must have $-1 \leqq Re(\nu) \leqq 0$. Finally it is not too difficult to see that $P_\nu(ch\zeta)$ is of positive

type if and only if $Re(\nu) = -1/2$. Thus we have identified the spherical functions and the spherical functions of positive type for the hyperbolic plane. See also [4].

The fact that the two integrals in (3.9) are equal is, of course, a consequence of the general theory of § 2. However, one can see this directly as follows. In the first integral in (3.9) we regard (λ, θ) as geodesic polar coordinates for the hyperbolic plane D with pole at 0. Let z_0 be the point whose coordinates are $(\zeta, 0)$ and let (μ, φ) be geodesic polar coordinates for D with pole at z_0 with the same polar axis. Using (3.2) it is not difficult to see that relationship between the coordinates (λ, θ) and (μ, φ) of a point z is

$$(3.11) \qquad ch\mu = ch\lambda ch\zeta - sh\lambda sh\zeta \cos \theta$$
$$ch\lambda = ch\mu ch\zeta + sh\mu sh\zeta \cos \varphi .$$

Thus regarding the first integral in (3.9) as an integral over the hyperbolic plane we see that it is equal to (the volume elements are $sh\lambda d\lambda d\theta$ and $sh\mu d\mu d\varphi$, see (3.8))

$$\int_0^\infty \int_0^{2\pi} f[ch\mu ch\zeta + sh\mu sh\zeta \cos \varphi]g(ch\mu)sh\mu d\varphi d\mu$$

and this is obviously equal to the second integral in (3.9). The relation expressed in (3.11) is just the law of cosines for hyperbolic trigonometry.

4. Preliminaries on Legendre Functions. We intend to study integral transforms of the form

$$(4.1) \qquad \varphi(\nu) = \int_0^\infty P_\nu(ch\lambda)dF(\lambda)$$

where F is a bounded monotone nondecreasing function and $P_\nu(ch\lambda)$ is the usual Legendre function. In this section we gather together some facts about the kernels $P_\nu(ch\lambda)$ that we will need in the sequel.

Combining formula (3) of § 3.2 and formula (22) of § 2.1 of [3] we see that

$$(4.2) \qquad P_\nu(ch\lambda) = \left(\frac{1 + ch\lambda}{2}\right)^\nu F\left(-\nu, -\nu; 1; \frac{ch\lambda - 1}{ch\lambda + 1}\right)$$

for all complex ν provided $0 \leqq \lambda < \infty$, where F is the usual hypergeometric function. In all statements to follow λ is a nonnegative real number. It is immediate from (4.2) that for each fixed λ the function $P_\nu(ch\lambda)$ is an entire function of the complex variable ν. Also

$$(4.3) \qquad P_\nu(1) = 1$$
$$P_0(ch\lambda) = P_{-1}(ch\lambda) = 1 .$$

From formula (9) of § 3.7 of [3] we have

$$(4.4) \qquad P_\nu(ch\lambda) = -\frac{\sqrt{2}}{\pi} \sin \nu\pi \int_0^\infty \frac{ch\left(\nu + \frac{1}{2}\right)t\, dt}{(ch\lambda + cht)^{1/2}}$$

provided $-1 < Re(\nu) < 0$ and in particular for real x we have

$$(4.5) \qquad P_{-\frac{1}{2}+ix}(ch\lambda) = \frac{\sqrt{2}}{\pi} ch\pi x \int_0^\infty \frac{\cos xt\, dt}{(ch\lambda + cht)^{1/2}}\ .$$

Using (4.2) above and the standard integral representation for the hypergeometric function (§ 2.1.3 of [3]) we find that

$$(4.6) \qquad P_\nu(ch\lambda) = \left(\frac{1 + ch\lambda}{2}\right)^\nu [\Gamma(-\nu)\Gamma(1 + \nu)]^{-1}$$

$$\int_0^1 t^{-\nu-1}(1 - t)^\nu \left(1 - \frac{ch\lambda - 1}{ch\lambda + 1}t\right)^\nu dt$$

provided $-1 < Re\nu < 0$. For ν real with $-1 < \nu < 0$ it is immediate from (4.6) that

$$(4.7) \qquad\qquad 0 \leqq P_\nu(ch\lambda) \leqq 1$$

and since $P_\nu(ch\lambda)$ is a continuous function of ν the inequality (4.7) must hold for all ν in the interval $-1 \leqq \nu \leqq 0$. On the other hand using formula (14) of § 3.7 of [3] we have

$$|P_\nu(ch\lambda)| \leqq \frac{1}{\pi} \int_0^\pi [ch\lambda + sh\lambda \cos t]^{Re\nu} dt = P_{Re\nu}(ch\lambda)\ ,$$

and combining this with (4.7) we obtain

$$(4.8) \qquad\qquad |P_\nu(ch\lambda)| \leqq 1$$

provided $-1 \leqq Re\nu \leqq 0$.

Let $z = x + iy$ be a complex variable and define the function

$$(4.9) \qquad\qquad K(z, \lambda) = P_{-\frac{1}{2}+iz}(ch\lambda)$$

for $\lambda \geqq 0$ and $-1/2 \leqq y \leqq 1/2$. For each fixed λ, $K(z, \lambda)$ is an analytic function of z in the strip $-1/2 < y < 1/2$ and is continuous in the closed strip $-1/2 \leqq y \leqq 1/2$. The properties (4.3) and (4.8) become

$$(4.10) \qquad K(z, 0) = 1,\ K\left(-\frac{i}{2}, \lambda\right) = K\left(\frac{i}{2}, \lambda\right) = 1$$

$$|K(z, \lambda)| \leqq 1\ .$$

Moreover $K(x, \lambda)$ is given by the right hand side of (4.5).

5. Uniqueness and Continuity. Let $\mathscr{F}$ be the collection of all bounded monotone nondecreasing real valued functions defined on $0 \leqq \lambda < \infty$ and normalized so that

(5.1) (i) $$F(0) = 0$$

(ii) $\qquad F(\lambda) \to F(\mu)$ as $\lambda \downarrow \mu$ for all $\mu > 0$.

Note that $F(0+)$ need not be zero. Let $\mathscr{F}_0$ be those $F \in \mathscr{F}$ which satisfy

(5.2) $$F(\infty) = \lim_{\lambda \to \infty} F(\lambda) = 1 .$$

All integrals are to be in the Lebesgue-Stieltjes sense. Integrals over $0 \leqq \lambda < \infty$ will be written $\int_0^\infty$, while integrals over $0 < \lambda < \infty$ will be written $\int_{0+}^\infty$.

It $\{F_n\}$ is a sequence in $\mathscr{F}$ and $F \in \mathscr{F}$, then we say that F_n converges weakly to F (written $F_n \to F$) provided

(5.3) $$\int_0^\infty f dF_n \to \int_0^\infty f dF$$

for all continuous f with compact support. We say that F_n is *Bernoulli convergent* to F provided (5.3) holds for all bounded continuous f. It is obvious that if each $F_n \in \mathscr{F}_0$ and $F_n \Rightarrow F$, then $F \in \mathscr{F}_0$. ($F_n \Rightarrow F$ means F_n is Bernoulli convergent to F.)

If $K(z, \lambda)$ is the kernel defined in (4.9) we define the transform of $F \in \mathscr{F}$ by

(5.4) $$\varphi(z) = \int_0^\infty K(z, \lambda) dF(\lambda) .$$

It is immediate that φ is bounded in absolute value and continuous in the strip $-1/2 \leqq y \leqq 1/2$, and is analytic in $-1/2 < y < 1/2$. In particular $\varphi(x)$ is a real valued even function of x, and $F \in \mathscr{F}_0$ if and only if $\varphi(-i/2) = 1$. Of course, the values of φ on the real axis completely determine φ in the strip $-1/2 \leqq y \leqq 1/2$. We now show that F is uniquely determined by φ.

THEOREM 5.1 *If φ is the transform of F, then $\varphi(x)$ uniquely determines F.*

Proof. It suffices to prove that if F is of bounded variation (not necessarily monotone) and if the integral in (5.4) vanishes for all x then F is identically zero. Using the representation (4.5) for $K(x, \lambda)$ we have

(5.5) $$\int_0^\infty h(t) \cos xt\, dt = 0 \text{ for all } x$$

where

(5.6) $$h(t) = H(cht) = \int_0^\infty [ch\lambda + cht]^{-\frac{1}{2}} dF(\lambda) .$$

The interchange of order of integration is justified since

$$|h(t)| \leq Var(F)[1 + cht]^{-\frac{1}{2}} .$$

Moreover h is continuous and the above inequality implies $h \in L_1$, hence (5.5) coupled with the uniqueness theorem for Fourier integrals yields $h(t) = 0$ for all $t \geq 0$. Thus if we define $G(u) = F(ch^{-1}(u))$ for $u \geq 1$ and $G(u) = 0$ for $0 \leq u < 1$ we have

$$H(t) = \int_0^\infty (u + t)^{-\frac{1}{2}} dG(u) = 0$$

for all $t \geq 1$. But

$$H(t) = \pi^{-\frac{1}{2}} \int_0^\infty \int_0^\infty e^{-\sigma(u+t)} \sigma^{-\frac{1}{2}} d\sigma dG(u)$$

$$= \pi^{-\frac{1}{2}} \int_0^\infty e^{-\sigma t} \sigma^{-\frac{1}{2}} g(\sigma) d\sigma$$

where $g(\sigma)$ is the Laplace-Stieltjes transform of G. Since $H(t) = 0$ for $t \geq 1$ we see that $\sigma^{-\frac{1}{2}} g(\sigma) = 0$ for almost all σ which in turn implies that G, and hence F, is zero. Here we have used the uniqueness theorem [7] for Laplace transforms twice.

In the present work the following rather weak continuity theorem will suffice.

THEOREM 5.2.   (i)   *If $F_n \to F$ and $F_n(\infty) \leq M$ then $\varphi_n(x) \to \varphi(x)$.*

(ii)   *Let φ_n be the transform of F_n and suppose $\varphi_n(-i/2) = F_n(\infty) \leq M$, then if $\varphi_n(x) \to \varphi(x)$ there exists an $F \in \mathscr{F}$ such that $F_n \to F$ and φ is the trasform of F.*

Proof.   (i)   For each x the function $K(x, \cdot)$ is continuous and (4.5) implies that it vanishes at infinity. Thus it is immediate that for each fixed x we have $\varphi_n(x) \to \varphi(x)$.

(ii)   Since $F_n(\infty) \leq M$ the Helly theorem implies the existence of a subsequence $\{F'_n\}$ weakly convergent to F'. If φ' is the transform of F' then (i) implies that $\varphi'(x) = \varphi(x)$ for all x. If the entire sequence $\{F_n\}$ does not converge to F', then there exists another subsequence $\{F''_n\}$ converging to $F'' \neq F'$. But as before $\varphi''(x) = \varphi(x) = \varphi'(x)$ which contradicts the uniqueness theorem. Thus if we let $F = F'$ the proof of (ii) is complete.

REMARK 1. Since the limit φ in (ii) above is a transform it follows that $\varphi(z)$ may be defined as a continuous function on the strip $-1/2 \leqq y \leqq 1/2$ which is analytic in the open strip. If $\varphi(-i/2) = \lim_{n \to \infty} \varphi_n(-i/2)$, then $F_n(\infty) \to F(\infty)$ which implies that $F_n \Rightarrow F$.

REMARK 2. If $F_n \Rightarrow F$ then since $K(z, \lambda)$ is continuous and bounded for $\lambda \geqq 0$ and z in the strip $-1/2 \leqq y \leqq 1/2$, it follows that $\varphi_n(z) \to \varphi(z)$ uniformly on each compact sub strip $-1/2 \leqq y \leqq 1/2$ and $0 \leqq x \leqq x_0$. See, for example, Lemma 1.5.2 (iv) of [1].

6. **Closure Properties.** Let $\mathscr{P}(\mathscr{P}_0)$ be the class of all transforms of functions in $\mathscr{F}(\mathscr{F}_0)$. It is then immediate that if φ and ψ are in $\mathscr{P}$ and $\alpha, \beta \geqq 0$ then $\alpha\varphi + \beta\psi \in \mathscr{P}$, while if $\varphi, \psi \in \mathscr{P}_0$ and $\alpha + \beta = 1$, $\alpha \geqq 0, \beta \geqq 0$, then $\alpha\varphi + \beta\psi \in \mathscr{P}_0$. Moreover Theorem 5.2 (ii) implies that if $\{\varphi_n\}$ is a sequence in $\mathscr{P}_0$ and $\varphi_n(x) \to \varphi(x)$, then $\varphi \in \mathscr{P}$.

The main result of this section is that $\mathscr{P}$ and $\mathscr{P}_0$ are closed under pointwise products. The proof of the following theorem is, of course, motivated by the general discussion of §§ 2 and 3.

THEOREM 6.1. *If φ_1 and φ_2 are in $\mathscr{P}(\mathscr{P}_0)$ then $\varphi_1\varphi_2$ is in $\mathscr{P}(\mathscr{P}_0)$.*

Proof. Let φ_1 and φ_2 be the transforms of F_1 and F_2. We first consider the case in which F_1 and F_2 have continuous densities $f_1(ch\lambda)$ and $f_2(ch\lambda)$ with respect to the measure $sh\lambda d\lambda$. Of course, $f_j(ch\lambda)$ may be unbounded near $\lambda = 0$. Thus

$$(6.1) \qquad \varphi_j(x) = \int_0^\infty K(x, \lambda) f_j(ch\lambda) sh\lambda d\lambda \quad j = 1, 2$$

$$\int_0^\infty f_j(ch\lambda) sh\lambda d\lambda < \infty \; ; f_j(ch\lambda) \geqq 0 \; ; j = 1, 2.$$

For the purposes of the present proof it will be convenient to write

$$(6.2) \qquad p(z, ch\lambda) = K(z, \lambda) = P_{-\frac{1}{2}+iz}(ch\lambda) \; .$$

An immediate consequence of the addition theorem for Legendre functions [3] is that

$$(6.3) \qquad p(z, ch\lambda)p(z, ch\mu) = \frac{1}{2\pi} \int_0^{2\pi} p(z, a(\theta)) d\theta$$

where

$$(6.4) \qquad a(\theta) = ch\lambda ch\mu - sh\lambda sh\mu \cos \theta \; .$$

Therefore

$$\varphi_1(z)\varphi_2(z) = \frac{1}{2\pi}\int_0^\infty f_1(ch\lambda)sh\lambda d\lambda \int_0^\infty\int_0^{2\pi} f_2(ch\mu)p(z, a(\theta))sh\mu d\theta d\mu \; .$$

If in the inner integrals we make the change of variable described in the last paragraph of § 3 we find (the manipulations are justified since $f_2(ch\mu) \in L_1(sh\mu d\mu)$ and p is bounded)

$$\varphi_1(z)\varphi_2(z) = \int_0^\infty p(z, ch\mu)h(ch\mu)sh\mu d\mu$$

where

$$h(ch\mu) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f_1(ch\lambda)f_2(a(\theta))sh\lambda d\theta d\lambda \; .$$

clearly $h \geqq 0$ and

$$F_1(\infty)F_2(\infty) = \varphi_1\left(-\frac{i}{2}\right)\varphi_2\left(-\frac{i}{2}\right) = \int_0^\infty h(ch\mu)sh\mu d\mu \; .$$

Thus $\varphi_1\varphi_2$ is in $\mathscr{P}$, and if φ_1, φ_2 are in $\mathscr{P}_0$, then $\varphi_1\varphi_2$ is in $\mathscr{P}_0$.

We now turn to the general case. Let $k(\lambda) = e^{-\lambda}$ if $\lambda \geqq 0$ and $k(\lambda) = 0$ if $\lambda < 0$, and put $k_n(\lambda) = nk(n\lambda)$. Defining $F_1(\lambda)$ and $F_2(\lambda)$ to be zero for $\lambda < 0$, it is clear that

$$_nf_i(\lambda) = \int_{-\infty}^\infty k_n(\lambda - \mu)dF_i(\mu) = \int_0^\lambda k_n(\lambda - \mu)dF_i(\mu)$$

are continuous functions of $\lambda \geqq 0$. Moreover if we define $_nF_i(\lambda) = \int_0^\lambda {_nf_i(\mu)}d\mu$, it follows that $_nF_i \in \mathscr{F}$, and $_nF_i \in \mathscr{F}_0$ if $F_i \in \mathscr{F}_0$. Here $i = 1, 2$. It is well known [1, Th. 1.5.3] that $_nF_i \Rightarrow F_i$ as $n \to \infty$. If $_ng_i(ch\lambda) = {_nf_i(\lambda)}$ then $_ng_i(ch\lambda)(sh\lambda)^{-1}$ is the density of $_nF_i$ with respect to $sh\lambda d\lambda$. Thus if $_n\varphi_i$ is the transform of $_nF_i$, it follows from what we proved above that $\psi_n(z) = {_n\varphi_1(z)}{_n\varphi_2(z)}$ is in $\mathscr{P}$ (or $\mathscr{P}_0$). But by the second remark following theorem 5.2 we have $_n\varphi_i(z) \to \varphi_i(z)$ everywhere on the strip $-1/2 \leqq y \leqq 1/2$. Thus $\varphi_1(z)\varphi_2(z) = \lim_{n\to\infty}\psi_n(z)$ is in $\mathscr{P}$, and $\varphi_1\varphi_2 \in \mathscr{P}_0$ if both φ_1 and φ_2 are since $\psi_n(-i/2) \to \varphi_1(-i/2)\varphi_2(-i/2)$. This completes the proof of Theorem 6.1.

The following theorem gives another interesting closure property of $\mathscr{P}$.

THEOREM 6.2.   *If* $\varphi \in \mathscr{P}$, *then* $\psi(z) = \exp(t[\varphi(z) - \varphi(-i/2)])$ *is in* $\mathscr{P}_0$ *for all* $t > 0$.

Proof.   If $\alpha = \varphi(-i/2) = F(\infty) \geqq 0$, then

$$\psi(z) = e^{-\alpha t}\lim_{n\to\infty}\sum_{k=0}^n \frac{t^k\varphi(z)^k}{k!} \; .$$

Using Theorem 6.1, the fact that $\mathscr{P}$ is closed under positive linear combinations, and Theorem 5.2 it follows that $\psi \in \mathscr{P}$. (Theorem 5.2 is applicable since

$$\sum_{k=0}^{n} \frac{t^k \varphi \left(-\dfrac{i}{2} \right)^k}{k!} \leqq e^{\alpha t}$$

for all n.)   Moreover $\psi(-i/2) = 1$ and so $\psi \in \mathscr{P}_0$.

7. Limit Theorems.

In this section we will consider only the class $\mathscr{P}_0$. We begin by making the following definitions suggested by probability theory.

DEFINITION 1. $\varphi \in \mathscr{P}_0$ is *infinitely divisible* if for each positive integer n there exists a $\psi_n \in \mathscr{P}_0$ such that $\varphi = (\psi_n)^n$.

DEFINITION 2. $\varphi \in \mathscr{P}_0$ is a *generalized limit* if there exist $\varphi_{nk} \in \mathscr{P}_n$ for $n = 1, 2, \cdots$, and $k = 1, 2, \cdots, k_n$ with $k_n \to \infty$ as $n \to \infty$ such that

(7.1) $$\max_k | \varphi_{nk}(x) - 1 | \to 0$$

uniformly on each bounded interval $0 \leqq x \leqq x_0 < \infty$, and

(7.2) $$\varphi_n(x) = \prod_{k=1}^{k_n} \varphi_{nk}(x) \to \varphi(x)$$

for all x.

In this section we will show that φ is infinitely divisible if and only if φ is a generalized limit, and at the same time obtain a canonical form for such φ. In the course of our discussion we will need the following two lemmas which we state here for convenience. The proofs of these lemmas will be given in §9.

LEMMA 7.1. (i) *There exist constants* $\lambda_0 > 0$ *and* $M_R < \infty$ *such that* $\lambda^{-2} | 1 - K(z, \lambda) | \leqq M_R$ *provided* $0 \leqq \lambda \leqq \lambda_0$ *and* $|z| \leqq R$ *(of course z is in the strip* $-1/2 \leqq y \leqq 1/2$*).*

(ii) $\lambda^{-2}[1 - K(x, \lambda)] \to 1/4(1/4 + x^2)$ *as* $\lambda \to 0$.

LEMMA 7.2. *Let* $H_T(\lambda) = \dfrac{1}{T} \int_0^T [1 - K(x, \lambda)]dx$, *then*

(i) $0 \leqq H_T \leqq 2$ *and* $H_T(\lambda) > 0$ *for* $\lambda > 0$;

(ii) $H_T(\lambda) \to 1$ *as* $\lambda \to \infty$;

(iii) $\lambda^{-2} H_T(\lambda) \to C(T) > 0$ *as* $\lambda \to 0$.

We begin with the simple half of our main result.

THEOREM 7.1. *If φ is infinitely divisible, then φ is a generalized limit.*

Proof. Letting $k_n = n$ and $\varphi_{nk} = \psi_n$ for all k we see that φ_n (defined in (7.2)) is identical with φ for all n. Thus we need only verify (7.1). But (4.10) implies that $|\varphi(x)| \leq 1$, and, of course, $\varphi(x)$ and $\psi_n(x)$ are real. Therefore

$$\psi_n(x)^2 = [\varphi(x)^2]^{\frac{1}{n}} \longrightarrow \psi(x)$$

where $\psi(x) = 0$ or 1 according as $\varphi(x) = 0$ or $\varphi(x) \neq 0$. But $\varphi(0) > 0$ and hence $\psi(0) = 1$. Moreover $\psi_n^2 \in \mathscr{S}_0$ and thus Theorem 5.2 implies that $\psi \in \mathscr{P}$. In particular $\psi(x)$ is continuous and since ψ can only take on the values 0 and 1 it follows that $\psi(x) = 1$ for all x. Hence $\varphi(x)$ never vanishes and since $\psi_n(x) = \exp[1/n \log \varphi(x)]$ in a neighborhood (depending on n) of 0 we must have $\psi_n(x) = \exp[1/n \log \varphi(x)]$ for all $x \geq 0$. Therefore $\psi_n(x) \to 1$ as $n \to \infty$. This completes the proof of Theorem 7.1.

We now turn our attention to the converse of Theorem 7.1. This will not be established in full generality until § 8. In working with Definition 2 we will adopt the convention that F_{nk} is the element in $\mathscr{S}_0$ whose transform is φ_{nk}, similarly φ_n is the transform of F_n, φ of F. We begin with the following result.

THEOREM 7.2. *Condition (7.1) of definition (2) is equivalent to*

$$(7.3) \qquad \max_k \int_\varepsilon^\infty dF_{nk}(\lambda) \to 0$$

as $n \to \infty$ for all $\varepsilon > 0$.

Proof. Suppose (7.3) holds then

$$\max_k |\varphi_{nk}(x) - 1| \leq \max_k \int_0^\varepsilon [1 - K(x, \lambda)] dF_{nk}(\lambda)$$

$$+ \max_k \int_\varepsilon^\infty [1 - K(x, \lambda)] dF_{nk}(\lambda)$$

$$\leq \varepsilon^2 M_R + 2 \max_k \int_\varepsilon^\infty dF_{nk}(\lambda)$$

provided $\varepsilon < \lambda_0$ and $0 \leq x \leq R$ where λ_0 and M_R are defined in Lemma 7.1. Thus (7.1) follows.

Suppose (7.1) holds and (7.3) does not hold, then there exists an $\varepsilon > 0$ and a subsequence n_j such that

$$(7.4) \qquad \max_k \int_\varepsilon^\infty dF_{n_jk} \geq \eta > 0 .$$

Let k_j be the value of k at which the maximum in (7.4) is attained $(0 \leqq k_j \leqq k_{n_j})$ and let $G_j = F_{n_j k_j}$. Let ψ_j be the transform of G_j, then (7.1) implies that $\psi_j(x) \to 1$ which is the transform of E (unit mass at the origin). Thus $G_j \to E$ weakly, but this is clearly a contradiction since $\int_\varepsilon^\infty dG_j \geqq \eta > 0$.

THEOREM 7.3. *If φ is generalized limit then* $\varphi(x) = \exp\left[-\psi(x)\right]$ *where*

$$(7.5) \qquad \psi(x) = \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} d\Omega(\lambda)$$

with $\Omega \in \mathscr{F}$.

Proof. Theorem 6.1 implies that each $\varphi_n \in \mathscr{P}_0$. Hence $F_n(\infty) = 1 = F(\infty)$ and since $\varphi_n(x) \to \varphi(x)$ we see that $F_n \to F$. Combining these facts yields $F_n \Rightarrow F$. Thus $\varphi_n(z) \to \varphi(z)$ uniformly on each strip $-1/2 \leqq y \leqq 1/2$ and $0 \leqq x \leqq x_0$. (Remarks following Theorem 5.2). Also $\varphi(x)$ can not vanish near $x = 0$ since $\varphi \in \mathscr{P}_0$. Let x_0 be the first zero of φ, then $\varphi_n(x) \to \varphi(x)$ uniformly on $0 \leqq x \leqq x_0$. Condition (7.1) implies that $\varphi_{nk}(x)$ doesn't vanish for $0 \leqq x \leqq x_0$ and all k provided that n is sufficiently large (how large depending only on x_0). Thus for $0 \leqq x < x_0$ and n sufficiently large we can write

$$-\log \varphi_{nk}(x) = -\log \left\{ 1 - \int_0^\infty [1 - K(x, \lambda)] dF_{nk}(\lambda) \right\} .$$

Letting

$$a_{nk}(x) = 1 - \varphi_{nk}(x) = \int_0^\infty [1 - K(x, \lambda)] dF_{nk}(\lambda) \geqq 0 ,$$

it follows from (7.1) that $a_{nk}(x) \to 0$ uniformly on $0 \leqq x \leqq x_0$ uniformly in k as $n \to \infty$. Hence

$$(7.6) \qquad -\log \varphi_n(x) = -\sum_{k=1}^{k_n} \log \varphi_{nk}(x) = \sum_{k=1}^{k_n} \sum_{j=1}^\infty (j)^{-1}(a_{nk})^j ,$$

and letting $n \to \infty$ we obtain

$$(7.7) \qquad -\log \varphi(x) = \lim_{n \to \infty} \left\{ \sum_{k=1}^{k_n} a_{nk} + \sum_{k=1}^{k_n} \sum_{j=2}^\infty (j)^{-1}(a_{nk})^j \right\}$$

provided $0 \leqq x < x_0$.

Since all the terms involved are nonnegative we have for $0 \leqq x < x_0$

$$(7.8) \qquad 0 \leqq \sum_{k=1}^{k_n} \sum_{j=2}^\infty (j)^{-1}(a_{nk})^j \leqq \max (a_{nk}) \sum_{k=1}^{k_n} \sum_{j=1}^\infty (j)^{-1}(a_{nk})^j$$

$$\to 0(-\log \varphi(x)) = 0$$

as $n \to \infty$. If we define

$$G_n(\lambda) = \sum_{k=1}^{k_n} F_{nk}(\lambda)$$

and use (7.7) and (7.8) we obtain

(7.9) $$-\log\varphi(x) = \lim_{n\to\infty} \int_0^\infty [1 - K(x, \lambda)dG_n(\lambda)$$

provided $0 \leq x < x_0$.

We now investigate the behavior of the functions G_n as $n \to \infty$. It is an immediate consequence of Lemma 7.2 that for each $T > 0$ there exists a constant $A(T) > 0$ such that

$$\frac{1}{T}\int_0^T [1 - K(x, \lambda)]dx \geq A(T)\frac{\lambda^2}{1 + \lambda^2}$$

for all $\lambda \geq 0$. Also from (7.6) and the definition of G_n it is clear that

$$-\log\varphi_n(x) \geq \int_0^\infty [1 - K(x, \lambda)]dG_n(\lambda)$$

for $0 \leq x < x_0$ and n sufficiently large. Moreover since $\varphi_n(x) \to \varphi(x)$ uniformly on $0 \leq x \leq x_0$ and $\varphi(x)$ is continuous and bounded away from zero on $0 \leq x \leq 1/2x_0$ it follows that $\log\varphi_n(x) \to \log\varphi(x)$ uniformly on $0 \leq x \leq 1/2x_0$. Thus if $0 < T < 1/2x_0$ and n large enough we have

$$\int_0^\infty \frac{\lambda^2}{1+\lambda^2}dG_n(\lambda) \leq [TA(T)]^{-1}\int_0^T -\log\varphi_n(x)dx$$

$$\to [TA(T)]^{-1}\int_0^T -\log\varphi(x)dx < \infty .$$

Hence there exists a constant $M < \infty$ such that

(7.10) $$\int_0^\infty \frac{\lambda^2}{1+\lambda^2}dG_n(\lambda) \leq M .$$

Next we will show that given $\varepsilon > 0$ there exists an R (independent of n) such that

(7.11) $$\int_R^\infty dG_n(\lambda) \leq \varepsilon .$$

To this end we first note that $\varphi_n(iy) \to \varphi(iy)$ uniformly an $-1/2 \leq y \leq 1/2$. Also (4.2) and (4.9) yield $0 < K(iy, \lambda) \leq 1$ for $\lambda \geq 0$ and $-1/2 \leq y \leq 1/2$ with $K(-i/2, \lambda) = K(i/2, \lambda) = 1$. Thus $\varphi_n(iy)$ and $\varphi(iy)$ are strictly positive on $-1/2 \leq y \leq 1/2$. Since $a_{nk}(iy) = 1 - \varphi_{nk}(iy) < 1$ an argument similar to the one leading to (7.6) yields

$$(7.12) \qquad -\log \varphi_n(iy) \geqq \int_0^\infty [1 - K(iy, \lambda)] dG_n(\lambda)$$

Let $\eta > 0$ be given, then since $\varphi(-i/2) = 1$ we can choose T such that $-1/2 < T < 1/2$ and

$$\frac{-2}{2T + 1} \int_{-\frac{1}{2}}^T \log \varphi(iy) dy < \eta \; .$$

Moreover (4.4) and (4.9) imply $K(iy, \lambda) \to 0$ as $\lambda \to \infty$ for each fixed y with $-1/2 < y < 1/2$. Thus we can choose R_0 so that

$$\frac{2}{2T + 1} \int_{-\frac{1}{2}}^T [1 - K(iy, \lambda)] dy > 1 - \eta$$

for all $\lambda \geqq R_0$. Since $\varphi_n(iy) \to \varphi(iy)$ uniformly on $-1/2 \leqq y \leqq T$ and φ is bounded away from zero there we can choose n_0 so that for $n > n_0$

$$\frac{-2}{2T + 1} \int_{-\frac{1}{2}}^T \log \varphi_n(iy) dy \leqq \frac{-2}{2T + 1} \int_{-\frac{1}{2}}^T \log \varphi(iy) dy + \eta \leqq 2\eta \; .$$

Thus for $n > n_0$ we have

$$\int_{R_0}^\infty (1 - \eta) dG_n(\lambda) \leqq \frac{2}{2T + 1} \int_{-\frac{1}{2}}^T \int_0^\infty [1 - K(iy, \lambda)] dG_n(\lambda) dy$$

$$\leqq \frac{-2}{2T + 1} \int_{-\frac{1}{2}}^T \log \varphi_n(iy) dy \leqq 2\eta \; ,$$

or $\int_{R_0}^\infty dG_n \leqq \dfrac{2\eta}{1 - \eta}$ if $n > n_0$. It is now evident that given $\varepsilon > 0$ one can choose an R so that (7.11) holds (each G_n being monotone nondecreasing and bounded).

Define

$$(7.13) \qquad \Omega_n(\lambda) = \int_0^\lambda \frac{t^2}{1 + t^2} dG_n(t) \; .$$

Then each Ω_n is in $\mathscr{F}$. Using (7.10), the Helly theorem implies the existence of a subsequence (call it Ω_n again) such that $\Omega_n \to \Omega$ with $\Omega \in \mathscr{F}$ and $\Omega(\infty) \leqq M$. Moreover from (7.11)

$$\int_R^\infty d\Omega_n(\lambda) \leqq \int_R^\infty dG_n(\lambda) \leqq \varepsilon$$

uniformly in n for R sufficiently large and this easily implies that $\Omega_n \Rightarrow \Omega$. Therefore

$$(7.14) \quad \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} d\Omega_n(\lambda) \to \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} d\Omega(\lambda)$$

where the integrand is defined by continuity, using Lemma 7.1, to be $1/4(1/4 + x^2)$ at $\lambda = 0$. Combining (7.9), (7.13), and (7.14) we obtain

$$(7.15) \qquad -\log \varphi(x) = \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} \, d\Omega(\lambda)$$

provided $0 \leqq x < x_0$. But x_0 was the smallest zero of $\varphi(x)$ and thus $-\log \varphi(x) \to \infty$ as $x \uparrow x_0$ while the integral in (7.15) remains bounded. Hence $\varphi(x)$ never vanishes and (7.15) must hold for all x. Finally defining $\psi(x) = -\log \varphi(x)$ we obtain Theorem 7.3.

THEOREM 7.4. *A function ψ has the representation* (7.5) *if and only if it can be written in the form*

$$(7.16) \qquad \psi(x) = \frac{c}{4} \left(\frac{1}{4} + x^2 \right) + \int_{0+}^\infty [1 - K(x, \lambda)] dG(\lambda)$$

where $c \geqq 0$ and G is monotone nondecreasing, right continuous for $\lambda > 0$, $G(\infty) = 0$, and satisfying.

$$(7.17) \qquad \int_{0+}^1 \lambda^2 dG(\lambda) < \infty \ .$$

Proof. If ψ has the representation (7.5) define $c = \Omega(0+) \geqq 0$ and $G(\lambda) = -\int_{\lambda+}^\infty t^{-2}(1 + t^2) d\Omega(t)$ for $\lambda > 0$, then using Lemma 7.1 (ii) it is clear that (7.16) holds and that G has the required properties. Conversely if (7.16) hold define $\Omega(0) = 0$ and $\Omega(\lambda) = c + \int_{0+}^\lambda t^2(1 + t^2)^{-1} dG(t)$, then clearly $\Omega \in \mathscr{F}$ and (7.5) holds.

8. Uniqueness and Simple Consequences of (7.5). In order that our theory be reasonably complete the following uniqueness theorem is required.

THEOREM 8.1. *The representation* (7.5), *and hence* (7.16) *also, is unique.*

Proof. It is sufficient to prove that *if Ω is of bounded variation and*

$$(8.1) \qquad \psi(x) = \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} \, d\Omega(\lambda) = 0$$

for all x, then $\psi = 0$. We will use the following formula [5, p. 168]

(8.2) $$(ch\lambda + ch\mu)^{-1} = \pi \int_0^\infty \frac{xsh\pi x}{ch^2\pi x} K(x, \lambda)K(x, \mu)dx$$

which holds for all $\lambda, \mu \geq 0$. (Robin's $K_x(ch\lambda)$ is our $K(x, \lambda)$.) Since $K(0, \lambda) = 1$, if we multiply (8.1) by $\pi x sh\pi x (ch\pi x)^{-2} K(x, \mu)$ and integrate from 0 to ∞ we obtain, using (8.2),

(8.3) $$\int_0^\infty \frac{ch\lambda - 1}{ch\lambda + ch\mu} \frac{1 + \lambda^2}{\lambda^2} d\Omega(\lambda) = 0 \ .$$

The interchange of order of integration is justified since ($|K(x, \mu)| \leq 1$ and $K(0, \mu) = 1$)

$$\pi \int_0^\infty \frac{xsh\pi x}{ch^2\pi x} |K(x, \mu)| \int_0^\infty [1 - K(x, \lambda)] \frac{1 + \lambda^2}{\lambda^2} d|\Omega|(\lambda)dx$$

$$\leq \int_0^\infty \frac{ch\lambda - 1}{2(1 + ch\lambda)} \frac{1 + \lambda^2}{\lambda^2} d|\Omega|(\lambda) < \infty \ ,$$

where $|\Omega|$ stands for the total variation of Ω. But (8.3) may be written

(8.4) $$\int_1^\infty (t + s)^{-1} d\Omega^*(t) = 0$$

for all $s \geq 1$, where for $t \geq 1$

(8.5) $$\Omega^*(t) = \int_0^{ch^{-1}(t)} \lambda^{-2}(1 + \lambda^2)(ch\lambda - 1)d\Omega(\lambda) \ .$$

Noting that Ω^* is of bounded variation on each *finite* interval $1 \leq t \leq T$, we can apply the uniqueness theorem for Stieltjes transform [7, p. 336]. This leads to the conclusion that Ω^*, and hence Ω, is identically zero.

THEOREM 8.2. *Given a ψ of the form (7.5) then $\varphi = e^{-\psi}$ is an infinitely divisible element of $\mathscr{F}_0$.*

Proof. Since $K(\cdot, \lambda)$ is in $\mathscr{F}_0$ for each $\lambda \geq 0$, it follows from Theorem 6.2 that $\exp\{-b[1 - K(\cdot, \lambda)]\}$ is in $\mathscr{F}_0$ for all $b, \lambda \geq 0$. Thus if we approximate the integral defining $\psi(x)$ by a Riemann sum and use the above fact and Theorem 5.2, we find that $\varphi(x) = \exp[-\psi(x)]$ is in $\mathscr{F}$. Since φ is in $\mathscr{F}$ it can be extended to a function $\varphi(z)$ which is continuous on the strip $-1/2 \leq y \leq 1/2$ and analytic on $-1/2 < y < 1/2$.

Using Lemma 7.1 (i) it is immediate that the integral in (7.5) converges for z in the strip $-1/2 \leq y \leq 1/2$ to a function which we denote by $\psi(z)$. It is also clear that ψ is continuous on $-1/2 \leq y \leq 1/2$ and analytic on $-1/2 < y < 1/2$. It now follows that $\varphi(z) = \exp[-\psi(z)]$ for all z in the strip $-1/2 \leq y \leq 1/2$. Since $\psi(-i/2) = 0$ it follows that $\varphi \in \mathscr{F}_0$. Similarly $\exp[-1/n\psi]$ is in $\mathscr{F}_0$ for each $n > 0$ and thus $\varphi = \exp(-\psi)$ is infinitely divisible.

COROLLARY 8.1. φ *is a generalized limit if and only if* φ *is infinitely divisible.*

Proof.   This results from Theorems 7.1, 7.3, and 8.2.

Actually contained in the proofs of Theorem 7.3, 8.1, and 8.2 is the following result which we state explicitly for completeness.

THEOREM 8.3.   *Let* $\varphi_{nK} \in \mathscr{P}_0$ *for* $n = 1, 2, \cdots$ *and* $k = 1, 2, \cdots, k_n$ *with* $k_n \to \infty$ *and satisfy* (7.1). *Let* φ_n *be defined by* (7.2), *then a necessary and sufficient condition* $\varphi_n \to \varphi \in \mathscr{P}_0$ *is that*

$$\Omega_n(\lambda) = \sum_{k=1}^{k_n} \int_0^\lambda \frac{t^2}{1 + t^2} \, dF_{nk}(t)$$

be Bernoulli convergent to $\Omega \in \mathscr{F}$. *In this case* $\varphi = \exp(-\psi)$ *where* ψ *is defined by* (7.5).

9. Proofs of the Lemmas.   We begin with Lemma 7.1 (i). In view of the definition (4.9) of $K(z, \lambda)$ it will suffice to show $\lambda^{-2} |1 - P_\nu(ch\lambda)| \leqq M_r$ for $0 \leqq \lambda \leqq \lambda_0$ and $|\nu| \leqq r$. Choose λ_0 such that the inequalities.

(9.1) $ch\lambda - 1 < 1 \,; \lambda^{-2}(ch\lambda - 1) < 1$

hold for $0 \leqq \lambda \leqq \lambda_0$. Since [3, p. 122]

(9.2) $P_\nu(ch\lambda) = F\left(-\nu, \nu + 1; 1; \frac{1}{2}(1 - ch\lambda)\right)$

provided $|1 - ch\lambda| < 2$, we easily find that for $0 \leqq \lambda \leqq \lambda_0$

$$\lambda^{-2} |1 - P_\nu(ch\lambda)| \leqq F\left(|\nu|, |\nu| + 1; 1; \frac{1}{2}\right).$$

Taking $M_r = F(r, r + 1; 1; 1/2)$ we obtain the desired conclusion.

Let λ_0 be as above, then from (9.2) and (4.9) we have for $0 \leqq \lambda \leqq \lambda_0$

$$K(x, \lambda) = F\left[\frac{1}{2} - ix, \frac{1}{2} + ix; 1; \frac{1}{2}(1 - ch\lambda)\right].$$

Expanding the hypergeometric function we find that

$$\lambda^{-2}[1 - K(x, \lambda)] + \frac{1}{2}\lambda^{-2}\left[\left(\frac{1}{4} + x^2\right)(1 - ch\lambda)\right] = G(x, \lambda)$$

where $G(x, \lambda) \to 0$ as $\lambda \to 0$ for each fixed x. The second conclusion of Lemma 7.1 is now immediate.

Finally all of the conclusions of Lemma 7.2 with the possible exception of $H_T(\lambda) > 0$ for $\lambda > 0$ are easy consequences of Lemma 7.1 and

(4.5). But if $H_r(\lambda_0) = 0$ for $\lambda_0 > 0$ we would have $K(x, \lambda_0) = 1$ for all x in $[0, T]$. In particular $K(0, \lambda_0) = 1$. Using (4.5) and the fact that $K(0, 0) = 1$ this leads to the conclusion that

$$\int_0^\infty \frac{dt}{[ch\lambda_0 + cht]^{1/2}} = \int_0^\infty \frac{dt}{[1 + cht]^{1/2}}$$

which is clearly impossible if $\lambda_0 > 0$. This then completes the proofs of our lemmas.

10. Gaussian and Stable Distributions. It is an immediate consequence of Theorems 8.2 and 7.4 that for each $t > 0$ the function

(10.1)
$$\varphi(x) = \exp\left[-t\left(\frac{1}{4} + x^2\right)\right]$$

is an infinitely divisible element of $\mathscr{F}_0$. Following Bochner [1] a φ of the form (10.1) will be called *Gaussian (or normal)*. Let $U_t(\lambda)$ be the element in $\mathscr{F}_0$ corresponding to $\exp[-t(1/4 + x^2)]$. If one uses the inversion formula of Fock [5, p. 165] and the fact that [5, p. 154]

$$K(x, \lambda) = \frac{\sqrt{2}}{\pi} \coth \pi x \int_\lambda^\infty \frac{\sin xt \, dt}{(cht - ch\lambda)^{1/2}}$$

one finds that

$$U_t(\lambda) = \int_0^\lambda u(t, \mu) sh\mu \, d\mu$$

where

(10.2)
$$u(t, \lambda) = \frac{e^{-t/4}}{(2t)^{3/2}} \int_\lambda^\infty \frac{se^{-s^2/4t} ds}{(chs - ch\lambda)^{1/2}} \, .$$

The function $u(t, \lambda)$ defined in (10.2) is therefore the density (with respect to $sh\lambda d\lambda$) of the rotationally invariant Gaussian distribution on the hyperbolic plane. It is not difficult to check directly that

$$\int_0^\infty u(t, \lambda) sh\lambda \, d\lambda = 1 \quad \text{for all} \quad t > 0 \, ,$$

although it is not necessary for us to do so since we know that $U_t \in \mathscr{F}_0$. Finally it is interesting, but not unexpected, to note that

(10.3)
$$\frac{\partial u}{\partial t} = (sh\lambda)^{-1} \frac{\partial}{\partial \lambda}\left(sh\lambda \frac{\partial u}{\partial \lambda}\right) \, .$$

The differential operator on the right side of (10.3) is the radial part of the Laplace-Beltrami operator in geodesic polar coordinates (λ, θ) for the hyperbolic plane.

Let $0 < \alpha < 1$ and let $g_\alpha(u)$ be the positive continuous function defined on $u \geqq 0$ by the relation

$$(10.4) \qquad e^{-s^\alpha} = \int_0^\infty e^{-su} g_\alpha(u) du$$

for $s \geqq 0$. See § 4.3 of [1]. (The function g_α is the density of a one-sided stable law of index α on the real line.) For $t > 0$ we define

$$(10.5) \qquad u_\alpha(t, \lambda) = \int_0^\infty u(t^{\frac{1}{\alpha}} s, \lambda) g_\alpha(s) ds$$

where $u(t, \lambda)$ is the normal density (10.2). Clearly $u_\alpha(t, \cdot)$ is a probability density with respect to $sh\lambda d\lambda$. Moreover

$$(10.6) \qquad \int_0^\infty K(x, \lambda) u_\alpha(t, \lambda) sh(\lambda) d\lambda = \exp\left[-t(1/4 + x^2)^\alpha\right]$$

Thus for each $t > 0$ the function $\exp[-t(1/4 + x^2)^\alpha]$ is an infinitely divisible element of $\mathscr{P}_0$. By analogy with the Euclidean case one might call the densities (10.5) or the transforms (10.6) *stable*. We will investigate the properties of these distributions in a future paper, in particular, we will give a fuller justification of the name stable.

11. Higher Dimensional Hyperbolic Spaces. All that has gone before can be easily extended to a more general class of integral transforms that are related to the higher dimensional hyperbolic spaces. Let μ be a real parameter with $\mu > -1/2$ and define the kernels

$$(11.1) \qquad K_\mu(x, \lambda) = 2^\mu \Gamma(\mu + 1)(sh\lambda)^{-\mu} P_{-1/2+ix}^{(\mu)}(ch\lambda)$$
$$= \left(\frac{1 + ch\lambda}{2}\right)^{-\mu} F\left(1/2 - ix, 1/2 + ix; \mu + 1; \frac{1 - ch\lambda}{2}\right),$$

where $P_\nu^{(\mu)}$ is the usual associated Legendre function [3]. Similarly we define $K_\mu(z, \lambda)$ for complex z by replacing x by z in (11.1). Clearly $K_\mu(z, \lambda)$ is analytic in z and it is not difficult to verify that

$$(11.2) \qquad |K_\mu(z, \lambda)| \leqq 1 \quad \text{if} \quad -\mu - 1/2 \leqq y \leqq 1/2$$

and

$$(11.3) \qquad K_\mu(z, 0) = 1, \qquad K_\mu(-i(\mu + 1/2), \lambda) = 1 .$$

Moreover it follows from 3.7(8) of [3] that

$$(11.4) \qquad K_\mu(x, \lambda) = \sqrt{\frac{2}{\pi}} \frac{2^\mu \Gamma(\mu + 1)}{\Gamma(\mu + 1/2)(sh\lambda)^{2\mu}} \int_0^\lambda (ch\lambda - cht)^{\mu-1/2} \cos xt\, dt ,$$

and combining this with (11.2) results in

(11.5) $-1 \leq K_\mu(x, \lambda) \leq 1$.

If $\mu = 0$ then K_μ reduces to the kernel K considered in the previous sections.

Let $\mu = N/2 - 1$ where $N \geq 2$ is an integer, then the functions $K_\mu(z, \cdot)$ are the spherical functions on N-dimensional hyperbolic space and the functions $K_\mu(x, \cdot)$ are the spherical functions of positive type.

If for arbitrary $\mu > -1/2$ we define the μ-transform of an element $F \in \mathscr{F}$ by

(11.6) $$\varphi(x) = \int_0^\infty K_\mu(x, \lambda) dF(\lambda)$$

then all of the results of the preceding sections can be carried over to μ-transforms with only minor changes. In particular, in (7.16) one must replace $c/4(1/4 + x^2)$ by $c/[4(\mu + 1)][(\mu + 1/2)^2 + x^2]$ and then the Gaussian elements have the form $\exp(-t[(\mu + 1/2)^2 + x^2])$. The proofs require only minor technical changes with the exception of Theorem 6.1.

In order to prove the analog of Theorem 6.1 for μ-transforms one needs the following formula

(11.7) $$K_\mu(x, \lambda) K_\mu(x, t) = \frac{2^{2\mu} \Gamma(1 + \mu)^2}{\pi \Gamma(2\mu + 1)} \int_0^\pi K_\mu(x, w) \sin^{2\mu}\theta \, d\theta$$

where

$$w = ch\lambda cht + sh\lambda sht \cos\theta .$$

Formula (11.7) is a simple consequence of the addition theorem for associated Legendre functions (formula 80 of Peter Henrici, Addition Theorems for General Legendre and Gegenbauer Functions, Journ. of Rat. Mech. and Anal. (4) 1955; note the misprint in this formula, namely $-\mu - n$ should be $-\nu - n$) and the orthogonality relations for Gegenbauer polynomials. Using (11.7) the fact that the product of two μ-transforms is again a μ-transform is proved by an argument similar to the one used in § 6.

Note added in proof. Some results which are similar to a part of ours appeared in a paper by F. I. Karpelevitch, V. N. Tutubalin, and M. G. Šur entitled "Limit theorems for convolutions of distributions on Lobachevsky's plane and space", Theory of Probability and its Applications, **4**, (1959), 432-436. These authors were particularly interested in convergence to the normal distribution.

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