BASES OF TENSOR PRODUCTS OF BANACH SPACES

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1. Introduction. In this note we use the conventions and notations of Schatten [4] with the exception that we use B' to indicate the dual (conjugate) space of a Banach space B and $\langle x, x' \rangle$ as the action of an element x and a functional x' on each other. Schatten defines the tensor product $B_1 \otimes_\alpha B_2$ as the completion of the algebraic tensor product $B_1 \otimes B_2$ of two Banach spaces B_1 and B_2, on which the cross norm α has been imposed. We discuss the proposition, "If B_1 and B_2 have Schauder bases, then $B_1 \otimes_\alpha B_2$ has a Schauder basis." We prove this for $\alpha = \gamma$ ($B_1 \otimes_\gamma B_2$ is the trace class of transformations of B_1' into B_2). We also prove it for $\alpha = \lambda$ ($B_1 \otimes_\lambda B_2$ is the class of all completely continuous linear transformations of B_1' into B_2) in the case in which the bases of B_1 and B_2 satisfy an "isometry condition". This condition is not very restrictive. We know of no instance in which it is not satisfied. Next we show that unconditional bases of B_1 and B_2 do not necessarily yield an unconditional basis for the tensor product, even in the nicest conceivable infinite dimensional case, that in which $B_1 = B_2 =$ Hilbert space, and the bases are orthonormal and identical.

We recall certain facts about Schauder bases, and set some general notation that we use throughout the paper. We usually work with a biorthogonal set $\Omega = \{x_i, x_i'\}_i$ associated with a Banach space B, so that $\chi = \{x_i\}_i$ is a basis for B with coefficients supplied by the corresponding sequence of functionals $\chi' = \{x_i'\}_i$. We will have to do with the closed linear manifold B^Ω of B' generated by the elements of χ'. Since B and B^Ω are in duality it is possible to embed B in $(B^\Omega)'$ by the same formula that effects the embedding of B in B''. We denote by ${}_nP_m$ the projection of B defined by ${}_nP_m x = \sum_{i=n}^m \langle x, x_i' \rangle x_i$. The double sequence $\{{}_nP_m\}_{n,m}$ is uniformly bounded. We denote by T' the transpose of any transformation T. The following lemma, given without proof, is but a trivial strengthening of [2, p. 18, Theorem 1].

LEMMA 1. *Let E be a dense vector subspace of B, Ω a biorthogonal set of B such that $\chi \subset E$, the vector space spanned by χ is dense in E and the sequence $\{{}_nP_m\}_{n,m}$ is uniformly bounded on E. Then Ω defines a basis for B.*

2. The tensor product of two biorthogonal sets. Let $\Omega_1 = \{x_i, x_i'\}_i$ be a biorthogonal set of B_1 and $\Omega_2 = \{y_i, y_i'\}_i$ a biorthogonal set of B_2.

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The elements $x'_i \otimes y'_j$ can be considered as belonging to $(B_1 \otimes {}_\alpha B_2)'$ for any cross norm α [4 p. 43], and $\{x_i \otimes y_j, x'_i \otimes y'_j\}_{i,j}$ is clearly a biorthogonal set. We enumerate it, not by the diagonal method, i.e., as in the usual proof that the rationals are denumerable, but as follows: In the table

$$\begin{array}{llll} x_1 \otimes y_1 & x_1 \otimes y_2 & x_1 \otimes y_3 & \cdots\cdots \\ x_2 \otimes y_1 & x_2 \otimes y_2 & x_2 \otimes y_3 & \cdots\cdots \\ x_3 \otimes y_1 & x_3 \otimes y_2 & x_3 \otimes y_3 & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}$$

we simply order the elements by listing the entries on the two inner sides of each successive upper left hand block to obtain $x_1 \otimes y_1, x_1 \otimes y_2, x_2 \otimes y_2, x_2 \otimes y_1, x_1 \otimes y_3, x_2 \otimes y_3, x_3 \otimes y_3, x_3 \otimes y_2, x_3 \otimes y_1, \cdots, x_1 \otimes y_k, x_2 \otimes y_k \cdots x_k \otimes y_k, x_k \otimes y_{k-1}, \cdots, x_k \otimes y_2, x_k \otimes y_1, \cdots$. This double sequence with the given order is called the tensor product of $\chi_1 = \{x_i\}_i$ and $\chi_2 = \{y_j\}_j$ and is denoted by $\chi_1 \otimes \chi_2$. Similarly $\chi'_1 \otimes \chi'_2$ denotes the set $\{x'_i \otimes y'_j\}_{i,j}$ with the corresponding order. The biorthogonal set formed by $\chi_1 \otimes \chi_2$ and $\chi'_1 \otimes \chi'_2$ is called the tensor product of Ω_1 and Ω_2 and denoted by $\Omega_1 \otimes \Omega_2$.

THEOREM 1. *If Ω_1 defines a basis for B_1 and Ω_2 defines a basis for B_2, then $\Omega_1 \otimes \Omega_2$ defines a basis for $B_1 \otimes {}_\gamma B_2$.*

Proof. We show that the vector space spanned by $\chi_1 \otimes \chi_2$ is dense in $B_1 \otimes B_2$. To see this let ${}_nP^i_m$ be the ${}_nP_m$ defined in § 1 for Ω_i, and define

$$(1)\quad \begin{aligned} A_m &= x \otimes y - \sum_{k,j=1}^{m} \langle x, x'_k\rangle\langle y, y'_j\rangle x_k \otimes y_j = x \otimes y - [{}_1P^1_m x] \otimes [{}_1P^2_m y] \\ &= x \otimes [y - {}_1P^2_m y] + [x - {}_1P^1_m x] \otimes {}_1P^2_m y\ . \end{aligned}$$

Then

$$(2)\qquad \gamma(A_m) \leqq ||x||\,||y - {}_1P^2_m y|| + ||x - {}_1P^1_m x||\,||{}_1P^2_m y||\ .$$

The right hand side of (2) tends to zero with m^{-1}. This argument extends by linearity to sums of elements of the form $x \otimes y$.

Let now T_q be the ${}_1P_q$ defined in § 1 corresponding to $\Omega_1 \otimes \Omega_2$. It remains to show that $\{T_q\}_q$ is uniformly bounded. It is easy to show that each T_q has one of the following three forms: ${}_1P^1_n \otimes {}_1P^2_n$, ${}_1P^1_n \otimes {}_1P^2_n + {}_{n+1}P^1_{n+1} \otimes {}_1P^2_m$, ${}_1P^1_m \otimes {}_{n+1}P^2_{n+1}$. Hence, it suffices to show that $\{{}_nP^1_m \otimes {}_qP^2_r\}_{q,r\,n,m}$ is uniformly bounded. Let M be a common bound for all ${}_nP^1_m$ and ${}_qP^2_r$. For $\Sigma x \otimes y \in B_1 \otimes B_2$

$$(3)\qquad \begin{aligned} \gamma[{}_nP^1_m \otimes {}_qP^2_r(\Sigma x \otimes y)] &= \gamma[\Sigma({}_nP^1_m x) \otimes ({}_qP^2_r y)] \\ &\leqq (\sum ||x||\,||y||)M^2\ . \end{aligned}$$

Since (3) holds for any representation $\Sigma x \otimes y$ of a given tensor product element, we may replace in it the sum $\Sigma \|x\| \|y\|$ by $\gamma(\Sigma x \otimes y)$, thereby proving our assertion. From Lemma 1, we can conclude that $\Omega_1 \otimes \Omega_2$ defines a basis for $B_1 \otimes_\gamma B_2$.

3. The space of completely continuous transformations. We recall that there is a canonical imbedding of B, with a biorthogonal set Ω defining a basis of B, into $(B^\Omega)'$. The norm of the image of an element $x \in B$ is less than or equal to $\|x\|$. We say that Ω satisfies the condition of isometry if the imbedding is actually an isometery. For such an Ω, $(B^\Omega)^\Omega = B$, isometrically. We state first the following corollary of Theorem 1.

COROLLARY 1. *If Ω_k is a biorthogonal set defining a basis for $B_k, k = 1, 2$, then $\Omega_1 \otimes \Omega_2$ defines a basis for $B_1^{\Omega_1} \otimes_\lambda B_2^{\Omega_2}$.*

Proof. Each $x'_i \otimes y'_j$ is an element of $B_1^{\Omega_1} \otimes B_2^{\Omega_2}$ which, as a subset of $B'_1 \otimes_\lambda B'_2$, can be imbedded isometrically in $(B_1 \otimes_\gamma B_2)'$ [4, *p*. 47, Theorem 3.2]. What is more, the vector space spanned by $\{x'_i \otimes y'_j\}_{i,j}$ is dense, with respect to λ, in $B_1^{\Omega_1} \otimes B_2^{\Omega_2}$, hence in $B_1^{\Omega_1} \otimes_\lambda B_2^{\Omega_2}$. This is true because

$$(4) \qquad \lambda[x' \otimes y' - (\sum_{i=1}^n \langle x_i, x'\rangle x'_i) \otimes (\sum_{i=1}^n \langle y_i, y'\rangle y'_i)] \leqq \gamma\, [x' \otimes y' - (\sum_{i=1}^n \langle x_i, x'\rangle x'_i) \otimes (\sum_{i=1}^n \langle y_i, y'\rangle y'_i)]$$

and the latter quantity tends to 0. Hence $B_1^{\Omega_1} \otimes_\lambda B_2^{\Omega_2} = (B_1 \otimes_\gamma B_2)^{\Omega_1 \otimes \Omega_2}$. Our result is a consequence of this.

The next theorem follows easily from this corollary.

THEOREM 2. *If both Ω_1 and Ω_2 satisfy the condition of isometry $\Omega_1 \otimes \Omega_2$ defines a basis for $B_1 \otimes_\lambda B_2$.*

Proof. If in Corollary 1 we replace B_1 by $B_1^{\Omega_1}$ and B_2 by $B_2^{\Omega_2}$, we conclude that $\Omega_1 \otimes \Omega_2$ defines a basis for $(B_1^{\Omega_1})^{\Omega_1} \otimes_\lambda (B_2^{\Omega_2})^{\Omega_2}$. When the condition of isometry is satisfied the last tensor product can be identified with $B_1 \otimes_\lambda B_2$, owing to the relations $B_k = (B_k^{\Omega_k})^{\Omega_k}$ for $k = 1, 2$, and the universal character of λ, [4, p. 35, Lemma 2.12].

Theorem 2 can be considered as a sharpening of the well known fact that if B_1 and B_2 have bases, then every completely continuous linear transformation of B'_1 into B_2 can be uniformly approximated by finite dimensional linear transformations. Our theorem goes further to state that if Ω_1 and Ω_2 satisfy the condition of isometry, the space of all completely continuous linear transformations of B'_1 into B_2 has a basis consisting of one-dimensional linear transformations.

The condition of isometry deserves some explanation. It is satisfied by a large class of bases, which includes every base for which

$$B^{\Omega} = B' \,.^{(1)} \tag{5}$$

The equation (5) holds always for reflexive spaces. It also holds for certain bases of non-reflexive spaces.

A non-reflexive example of (5) is exhibited in [2, p. 188, Example 1], involving the usual basis of c_0, $x_i = \{\delta^i_j\}_j$, with $x'_i = \{\delta^i_j\}_j \in l^1$. An example of the condition of isometry, in the absence (5), is obtained from this first example, by setting [2, p. 188, Example 2] $y_1 = x_1$, and $y_i = x_i - x_{i-1} + \cdots + (-1)^{i-1}x_1$, for $i > 1$, and $y'_i = x'_i + x'_{i+1}$. For $\Omega = \{y_i, y'_i\}_i$, $x'_1 \in B' \backslash B^{\Omega}$. Ω satisfies the condition of isometry for, if $x \in c_0$, then

$$\left\|\sum_{k=1}^{n} \langle x, y'_k\rangle y_k\right\| = \left\|\sum_{k=1}^{n} \langle x, x'_k\rangle x_k\right\| \leqq \|x\|.$$

The conclusion is now a consequence of the following theorem and its corollary.

THEOREM 3. *If for every $x' \in B'$, $\|{}_1P'_n x'\| \to \|x'\|$, then Ω satisfies the condition of isometry.*

Proof. Let $x_0 \in B$ and $x'_0 \in B'$ such that $\|x'_0\| = 1$ and $\langle x_0, x'_0\rangle = \|x_0\|$. Then

$$\lim_{n\to\infty} \frac{\langle x_0, {}_1P'_n x'_0\rangle}{\|{}_1P'_n x'_0\|} = \|x_0\| \,, \qquad \text{Q.E.D.}$$

COROLLARY 2. *If $\|{}_1P_n\| \leqq 1$ for every n, then Ω satisfies the condition of isometry.*

Proof. We show the above hypothesis implies the hypothesis of Theorem 3. To see this, let $x'_0 \in B'$, and $\varepsilon > 0$. There is $x_0 \in B$ so that $\|x_0\| = 1$ and $\langle x_0, x'_0\rangle > \|x_0\| - \varepsilon/2$ and an integer $N > 0$ so that

$$\begin{aligned}\|x'_0\| \geqq \|{}_1P'_n x'_0\| \geqq \langle x_0, {}_1P'_n x'_0\rangle = \langle {}_1P_n x_0, x'_0\rangle > \langle x_0, x'_0\rangle - \varepsilon/2 \\ > \|x'_0\| - \varepsilon \,, \qquad \text{Q.E.D.}\end{aligned}$$

As we have seen, the two biorthogonal sets described above for c_0 satisfy the hypothesis of Corollary 1.

An example of the isometry condition in which B' is not separable is furnished by Schauder's basis for $C([0, 1])$, given by the biorthogonal system $\Omega = \{x_i, x'_i\}_i$ described in [1, p. 69]. We consider [0, 1] imbedded

[1] This equation may be described by saying that $\{x'_i\}_i$ is a *retrobasis for B'*, [2, p. 188, Definition 1].

in B' and treat its points as functionals. The space B^{Ω} of this example contains the set D of all dyadic fractions. Consequently Ω satisfies the condition of isometry, since, for $f \in B$, $||f|| = \sup_{d \in D}|f(d)|$.

We know of no biorthogonal set defining a basis which does not satisfy the condition of isometry. Neither do we know if $B_1 \otimes_{\alpha} B_2$ has a basis for an arbitrary cross norm α, even if B_1 and B_2 have bases. It is clear that for any element of $B_1 \otimes B_2$, the formal expansion of Theorem 1 converges to that element with respect to α, since it does with respect to $\gamma \geqq \alpha$. The difficulty lies in establishing that the set $\{{}_pP^1_q \otimes {}_rP^2_s\}_{\substack{p,q\\ r,s}}$ is uniformly bounded with respect to α.

4. Hilbert spaces and unconditional bases. The problem of approximation of compact operators by finite dimensional operators in a Banach space, can, after elaborate rearrangement, lead to the following question: Can there exist a matrix $C = (c_{ij})^{\infty}_{i,j=1}$ satisfying the following conditions:

(a) For some $a_i \geqq 0$, $\sum_{i=1}^{\infty} a_i^2 < \infty$, $|c_{ij}| \leqq a_i a_j$;

(b) $C^2 = 0$;

(c) $\sum_{i=1}^{\infty} c_{ii} = 1$?

Of course, (b) and (c) are incompatible if C is in the trace class. Thus there arises the question: Does (a) imply that C is in the trace class? To this we can give a definite negative answer via the following theorems.

Therem 4. *Let* $\Omega = \{x_i, x'_i\}_i$, $x_i = \{\delta^i_j\}_j$, $x'_i = \{\delta^i_j\}_j$ *be the canonical orthonormal basis in* l_2. *Then* $\Omega \otimes \Omega$ *defines an unconditional basis in* $l_2 \otimes_{\gamma} l_2$ *if and only of condition* (a) *implies* C *is in the trace class.*

Proof. Let $\Omega \otimes \Omega$ define an unconditional basis for $l_2 \otimes_{\gamma} l_2$. Then we note that (a) may be rephrased by stating: $c_{ij} = \varepsilon_{ij} a_i a_j$, $|\varepsilon_{ij}| \leqq 1$. Since $l_2 \otimes_{\gamma} l_2$ is precisely the trace class of operators [4] it follows that $\sum_{i,j=1}^{\infty} \varepsilon_{ij}\, a_i a_j (x_i \otimes x_j)$ exists in $l_2 \otimes_{\gamma} l_2$ and is therefore in the trace class.

On the other hand, if (a) implies that C is in the trace class, then for $a \otimes a$ in $l_2 \otimes_{\gamma} l_2$ ($a = (a_1, a_2, \cdots)$), $a \otimes a = \sum_{i,j=1}^{\infty} a_i a_j (x_i \otimes x_j)$. If $B = (\varepsilon_{ij} a_i a_j)$ is in the trace class, then B has an expansion $\sum_{i,j=1}^{\infty} \varepsilon_{ij} a_i a_j (x_i \otimes x_j)$, which shows $\Omega \otimes \Omega$ defines an unconditional basis for $l_2 \otimes_{\gamma} l_2$.

THEOREM 5. *$\Omega \otimes \Omega$ does not define an unconditional basis for $l_2 \otimes_{\gamma} l_2$.*

Proof. Let $A_1 = (a_{ij})$ be a 2×2 matrix with $a_{11} = a_{12} = a_{22} = -a_{21} = 1$, and A_n the $2^n \times 2^n$ matrix (A_{ij}) $i, j = 1, 2$, with $A_{11} = A_{12} = A_{22} = -A_{21} = A_{n-1}$. Let B be the direct sum of the matrices $\{1/2^{n/2} A_n\}_n$. Then a direct computation reveals that B is unitary. Let $B = (b_{ij})$, and let

$C = (|b_{ij}|)$. If $\Omega \otimes \Omega$ were an unconditional basis for $l_2 \otimes_\gamma l_2$, then for B, regarded as a member of $(l_2 \otimes_\gamma l_2)'$ [4, p. 47, Theorem 3.2] and arbitrary $u \otimes v$ in $l_2 \otimes_\gamma l_2$, $\sum_{i,j=1}^\infty u_i v_j \langle x_i, Bx_j \rangle$ would converge unconditionally, i.e. $\sum_{i,j=1}^\infty u_k v_j |b_{ji}|$ would converge. In particular, let $u = v$, where u is given by the vector: $\sum_{n=1}^\infty (1/n) x_n$, $(\sqrt{2^n}) x_n = (\underbrace{0, 0, \cdots 0}_{2(2^{n-1}-1)}, \overbrace{1, 1, \cdots, 1}^{2^n}, 0, 0, \cdots)$. A simple verification shows that u exists in l_2. On the other hand, more calculation shows $\sum_{i,j=1}^\infty |b_{ij}| u_i u_j = \infty$. The contradiction implies the theorem.

Theorem 5 remains valid when γ is replaced by λ, since $l_2 \otimes_\gamma l_2 = (l_2 \otimes_\lambda l_2)'$, and unconditionality of $\Omega \otimes \Omega$ in $l_2 \otimes_\lambda l_2$ implies the same in $l_2 \otimes_\gamma l_2$.

NOTE. We owe to the referee the remark that a space B with a biorthogonal set Ω which defines a basis for B can always be renormed, preserving the topology of B [1, Theorem 1, p. 67], in such a way that Ω satisfies the condition of isometry (section 3) with respect to the resulting norm of B and the corresponding norm of B'. This makes possible the following completely general form of Theorem 2.

THEOREM 2'. *If Ω_i defines a basis for B_i, for $i = 1, 2$, then $\Omega_1 \otimes \Omega_2$ defines a basis for $B_1 \otimes_\lambda B_2$.*

Proof. Renorm B_1 and B_2 as indicated above. Then, if λ' denotes the operator norm with respect to the new norms of B_1 and B_2, $B_1 \otimes_{\lambda'} B_2$ has a basis defined by $\Omega_1 \otimes \Omega_2$ (Theorem 2). But $B_1 \otimes_{\lambda'} B_2 = B_1 \otimes_\lambda B_2$ both point-set-wise and topologically. Hence our conclusion.

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