ASYMPTOTIC DECAY OF SOLUTIONS OF DIFFERENTIAL INEQUALITIES

PAUL J. COHEN AND MILTON LEES†

1. Introduction. Let A be an operator in a Hilbert space H, and let $u(t)$, $0 \leqq t < \infty$ be a strongly continuously differentiable function of t with values in H such that $Au(t)$ is continuous. We say that $u(t)$ has property S if, as $t \to \infty$, it cannot vanish faster than *every* exponential, unless identically zero. A sufficient condition for all solutions of the abstract differential inequality

$$\left\| \frac{du}{dt} - Au \right\| \leqq \phi(t) \, \| u \| \, , \qquad 0 \leqq t < \infty \, , \tag{1.1}$$

to have property S was determined by P. D. Lax [1]. The required condition is that there exists an infinite sequence of lines parallel to the imaginary axis whose abcissae λ_n tend to $-\infty$ and on which the resolvent operator $(A - \lambda)^{-1}$ is uniformly bounded by some constant d^{-1}, and that $\sup \phi(t) < d$.

In this paper we give another sufficient condition for all of the solutions of (1.1) to have property S. We require that the operator A be symmetric, i.e., $(Au, v) = (u, Av)$, for all u and v in the domain of A, and that the function $\phi(t)$ be continuous and in $L^p(0, \infty)$, for some p in $1 \leqq p \leqq 2$. Actually, under these conditions, we prove a slightly stronger result; namely, that there exist constants $K > 0$ and μ such that the non-trivial solutions of (1.1) satisfy $\| u(t) \| \geqq Ke^{\mu t}$.

The restriction in Lax's result on the size of $\phi(t)$ cannot be lessened in general. For in the contrary case he constructed a solution of (1.1) that, as $t \to \infty$, behaves like $\exp(-bt^2)$, b being a positive linear function of $\sup \phi(t)$. It is therefore natural to ask whether there exist solutions of (1.1) which, as $t \to \infty$, tend to zero faster than $\exp(-\lambda t^2)$, for every $\lambda > 0$. We shall show that, at least for symmetric operators, this is only possible for the trivial solution. More generally, we obtain results that relate the rate of decay at infinity of the solutions of (1.1) to the asymptotic behavior of the function $\phi(t)$.

In the final portion of this paper we derive similar results for solutions of concrete parabolic differential inequalities. Results concerning the asymptotic behavior of solutions of parabolic partial differential ine-

Received October 1960. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF49(638)-253. Reproduction in whole or in part is permitted for any purpose of the United States Government.

† Now at New York University, the Institute for Mathematical Sciences.

qualities have been obtained recently by M. H. Protter [2].

2. The estimate from below. Throughout this paper A will denote a symmetric operator in a Hilbert space H, and $u(t)$ will denote a strongly continuously differentiable function defined for $0 \leqq t < \infty$ with values in H such that $Au(t)$ to continuous. We shall also assume that $\phi(t)$ is a positive continuous function belonging to $L^p(0, \infty)$, for some p in the interval $1 \leqq p \leqq 2$.

THEOREM 1. *If $u(t)$ is a solution of the abstract differential inequality*

$$\left\| \frac{du}{dt} - Au \right\| \leqq \phi(t) \| u \| , \qquad 0 \leqq t < \infty , \tag{2.1}$$

and $u(0) \neq 0$, then there exists $K > 0$ and μ such that

$$\| u(t) \| \geqq Ke^{\mu t} , \qquad 0 \leqq t < \infty . \tag{2.2}$$

The proof of Theorem 1 requires several lemmas concerning operators in finite-dimensional Hilbert spaces. Let D be a symmetric operator in a finite-dimensional Hilbert space F. Since F is finite-dimensional and D is symmetric, there is no loss of generality in assuming that D is in diagonal form.

For any real number λ and any vector v in F, denote by $P_\lambda v$ the projection of v onto the subspace of F spanned by those eigenvectors of D whose eigenvalues are not less than λ. Since D is in diagonal form, we have

$$(DP_\lambda v, P_\lambda v) \geqq \lambda \| P_\lambda v \|^2 . \tag{2.3}$$

Similarly, if we define $R_\lambda v = v - P_\lambda v$, then

$$(DR_\lambda v, R_\lambda v) \leqq \lambda \| R_\lambda v \|^2 . \tag{2.4}$$

Let ρ be an arbitrary positive number, and define a sequence $\{t_n\}$ as follows: $t_0 = 0$, and t_n, for positive integers n, is determined from the relation

$$\int_{t_n}^{t_{n+1}} \phi(\eta) d\eta = \rho , \tag{2.5}$$

where $t_{n+1} = \infty$ if

$$\int_{t_n}^{\infty} \phi(\eta) d\eta < \rho .$$

LEMMA 1. *Let $v(t)$, $0 \leqq t < \infty$, be a differentiable function of t with values in F such that*

$$\left\| \frac{dv}{dt} - Dv \right\| \leqq \phi(t) \| v \| , \qquad 0 \leqq t < \infty . \tag{2.6}$$

Assume that $v(0) \neq 0$ *and that*

$$\| P_\lambda v(t_n) \| \geqq \| R_\lambda v(t_n) \| . \tag{2.7}$$

Then there exists $\rho_0 > 0$ *such that for all* $\rho \leqq \rho_0$

$$2 \| P_\lambda v \| \geqq \| R_\lambda v \| , \qquad t_n \leqq t \leqq t_{n+1} . \tag{2.8}$$

Proof. The set $T = \{t\colon 2 \| R_\lambda v \| \leqq \| P_\lambda v \|\}$ is closed, and the inequality (2.8) obviously holds for each t in T. Thus it is sufficient to prove (2.8) for t in CT, the complement of T. Since CT is an open set of reals, it can be represented as a denumerable union of disjoint open intervals. Therefore it suffices to prove (2.8) for a generic open interval, $a < t < b$ say, forming this union.

We have

$$\| P_\lambda v(t) \| < 2 \| R_\lambda v(t) \| , \qquad a < t < b , \tag{2.9}$$

and

$$\| P_\lambda v(a) \| \geqq \| R_\lambda v(a) \| . \tag{2.10}$$

Since the space F is finite dimensional, D is a bounded operator (the bound for D may depend on the dimension of F), and this implies that the inequality (2.6) can have only one solution with prescribed initial value $v(0)$. Thus $v(t)$ can never vanish since $v(0) \neq 0$. It follows now from (2.9) that $R_\lambda v(t)$ is nonzero in $a < t < b$, so that we can form the function

$$f(t) = \frac{\| P_\lambda v(t) \|^2}{\| R_\lambda v(t) \|^2} . \tag{2.11}$$

Differentiating $f(t)$, we find that

$$\begin{aligned} \| R_\lambda v \|^4 \frac{df}{dt} &= 4 \| R_\lambda v \|^2 Re\left(P_\lambda v, P_\lambda \frac{dv}{dt} \right) \\ &\quad - 4 \| P_\lambda v \|^2 Re\left(R_\lambda v, R_\lambda \frac{dv}{dt} \right) . \end{aligned} \tag{2.12}$$

Since $v(t)$ satisfies the inequality (2.6) and P_λ and R_λ are projections, we can write

$$P_\lambda \frac{dv}{dt} = D(P_\lambda v) + Q_1 \tag{2.13}$$

and

$$P_\lambda \frac{dv}{dt} = D(R_\lambda v) + Q_2 \ , \tag{2.14}$$

where

$$\| Q_i \| \leqq \phi(t) \| v \| \ , \qquad (i = 1, 2) \ . \tag{2.15}$$

It follows from (2.13) and (2.15) that

$$Re\left(P_\lambda v,\ P_\lambda \frac{dv}{dt}\right) \geqq (P_\lambda v,\ DP_\lambda v) - \phi(t) \| v \|^2 \ .$$

Applying (2.3) to the first term on the right, we obtain

$$Re\left(P_\lambda v,\ P_\lambda \frac{dv}{dt}\right) \geqq \lambda \| P_\lambda v \|^2 - \phi(t) \| v \|^2 \ . \tag{2.16}$$

Similarly, we have

$$Re\left(R_\lambda v,\ R_\lambda \frac{dv}{dt}\right) \leqq \lambda \| R_\lambda v \|^2 + \phi(t) \| v \|^2 \ . \tag{2.17}$$

Entering the estimates (2.16) and (2.17) into the right side of (2.12), we find that

$$\| R_\lambda v \|^4 \frac{df}{dt} \geqq -8\phi(t) \| v \|^4 \ . \tag{2.18}$$

Here we have made use of the inequalities $\| P_\lambda v \| \leqq \| v \|$ and $\| R_\lambda v \| \leqq \| v \|$. It follows from (2.9) that

$$\| v \|^2 = \| P_\lambda v \|^2 + \| R_\lambda v \|^2 \leqq 5 \| R_\lambda v \| \ .$$

This inequality and (2.18) imply that

$$\frac{df}{dt} \geqq -200\phi(t) \ ,$$

and therefore

$$f(t) \geqq f(a) - 200\int_a^t \phi(\eta) d(\eta) \ . \tag{2.19}$$

Now, according to (2.10) and (2.11), $f(a) \geqq 1$. Therefore if we make use of (2.5), we conclude from (2.19) that

$$\frac{\| P_\lambda v(t) \|^2}{\| R_\lambda v(t) \|^2} \geqq 1 - 200\rho \geqq \tfrac{1}{4} \ ,$$

provided that $800\rho_0 = 3$. This completes the proof of the lemma.

LEMMA 2. *Let $v(t)$ satisfy the conditions of Lemma 1. If*

$$\pi = \lambda - 200\rho(t_{n+1} - t_n)^{-1}\,, \tag{2.20}$$

then, for all $\rho \leqq \rho_0$,

$$\| P_\pi v(t_{n+1}) \| \geqq \| R_\pi v(t_{n+1}) \|\,. \tag{2.21}$$

Proof. First, assume that

$$\| P_\lambda v(t) \| \leqq 2 \| R_\pi v(t) \|\,, \qquad t_n < t < t_{n+1}\,. \tag{2.22}$$

Setting

$$g(t) = \frac{\| P_\lambda v(t) \|^2}{\| R_\pi v(t) \|^2}\,, \tag{2.23}$$

we obtain

$$\begin{aligned} \| R_\pi v \|^4 \frac{dg}{dt} &= 4 \| R_\pi v \|^2 Re\Big(P_\lambda v,\, P_\lambda \frac{dv}{dt}\Big) \\ &\quad - 4 \| P_\pi v \|^2 Re\Big(R_\pi v,\, R_\pi \frac{dv}{dt}\Big)\,. \end{aligned} \tag{2.24}$$

As in the proof of Lemma 1, we have

$$Re\Big(R_\pi v,\, R_\pi \frac{dv}{dt}\Big) \leqq \pi \| R_\pi v \|^2 + \phi(t) \| v \|^2\,. \tag{2.25}$$

Inserting the estimates (2.16) and (2.25) into the right side of (2.24), we conclude that

$$\| R_\pi v \|^4 \frac{dg}{dt} \geqq 4(\lambda - \pi) \| R_\pi v \|^2 \| P_\lambda v \|^2 - 8\phi(t) \| v \|^2\,. \tag{2.26}$$

Since $\| P_\pi v \| \geqq \| P_\lambda v \|$, (2.22) implies that $\| v \| \leqq 5 \| R_\pi v \|^2$, which, when inserted into (2.21), yields

$$\frac{dg}{dt} \geqq 200\rho_0(t_{n+1} - t_n)^{-1} \frac{\| P_\lambda v \|^2}{\| R_\lambda v \|^2} - 200\phi(t)\,. \tag{2.27}$$

Here we have employed the inequality $\| R_\pi v \| \leqq \| R_\lambda v \|$. By Lemma 1, $4 \| P_\lambda v \|^2 \geqq \| R_\lambda v \|^2$, so that we obtain from (2.27)

$$\frac{dg}{dt} \geqq 200\rho_0(t_{n+1} - t_n)^{-1} - 200\phi(t)\,. \tag{2.28}$$

Finally, when we integrate (2.28) between t_n and t_{n+1} and apply (2.7) and (2.23) to the result, we obtain the desired inequality (2.21).

Now assume that there is a value of $t < t_{n+1}$ such that $\| P_\lambda v \| > 2 \| R_\pi v \|$. Let $\bar{t}$ be the last such value of t. If $\bar{t} = t_{n+1}$ there is nothing to prove, so we assume that $\bar{t} < t_{n+1}$. In this situation we have that

$$\| P_\lambda v(t) \| \leqq 2 \| R_\pi v(t) \| , \qquad \bar{t} < t < t_{n+1} ,$$

and

$$\| P_\pi v(\bar{t}) \| \geqq 2 \| R_\pi v(\bar{t}) \| .$$

The reasoning emplyed to prove Lemma 1 can now be used to establish the inequality

$$\| P_\pi v(t) \| \geqq \frac{\sqrt{11}}{2} \| R_\pi v(t) \| , \qquad \bar{t} \leqq t \leqq t_{n+1} ,$$

which certainly implies (2.21).

From the sequence $\{t_n\}$ we form the series

$$\sigma = \sum_{n=0}^{\infty} (t_{n+1} - t_n)^{-1} .$$

Our assumption that $\phi(t)$ belongs to $L^p(0, \infty)$, for some p in the interval $1 \leqq p \leqq 2$, implies that σ converges. This is clear when $p = 1$ since in this case the series has only a finite number of nonzero terms. Assume that $1 < p \leqq 2$. Applying Hölder's inequality to (2.5), we obtain the nequality

$$\rho \leqq \left(\int_{t_n}^{t_{n+1}} \phi^p(\eta) d\eta \right)^{1/p} (t_{n+1} - t_n)^{1/q} ,$$

where $p^{-1} + q^{-1} = 1$. Therefore

$$(t_{n+1} - t_n)^{-1} \leqq \rho^{-1} \left(\int_{t_n}^{t_{n+1}} \phi^p(\eta) d\eta \right)^{q/p} ,$$

which, since $q \geqq p$, implies that σ converges.

Also, we note here that our assumption that $\phi(t)$ belongs to $L^p(0, \infty)$, for some p in the interval $1 \leqq p \leqq 2$, implies that there exist constants C_1 and C_2 such that

$$\int_0^t \phi(\eta) d\eta \leqq C_1 t + C_2 . \tag{2.29}$$

From now on we shall assume that ρ has the fixed value ρ_0.

LEMMA 3. *Let $v(t)$ satisfy the conditions of Lemma* 1. *If*

$$\| P_\lambda v(0) \| \geqq \| R_\lambda v(0) \| , \tag{2.30}$$

then

$$\| v(t) \| \geqq \tfrac{1}{2} e^{-\sigma_2} \| v(0) \| e^{\mu t} , \qquad 0 \leqq t < \infty , \tag{2.31}$$

where $\mu = \lambda - 200\rho_0\sigma - 3C_1$.

Proof. Set $\lambda_0 = \lambda - 200\rho_0\sigma$. We assert that

$$2\,\|P_{\lambda_0}v(t)\| \geqq \|R_{\lambda_0}v(t)\|\,, \qquad 0 \leqq t < \infty\,. \tag{2.32}$$

Let t be arbitrary. Then for some n, $t_n \leqq t \leqq t_{n+1}$. It follows from (2.30), Lemma 1 and Lemma 2 that

$$2\,\|P_\pi v(t)\| \geqq \|R_\pi v(t)\|\,, \qquad t_n \leqq t \leqq t_{n+1}\,. \tag{2.33}$$

Hence the inequality

$$\|P_{\lambda_0}v\| \geqq \|P_\pi v\| \geqq \tfrac{1}{2}\|R_\pi v\| \geqq \tfrac{1}{2}\|R_{\lambda_0}v\|$$

implies (2.32) for this particular value of t.

It follows from (2.32) that

$$\|v(t)\| \leqq 3\,\|P_{\lambda_0}v(t)\|\,, \qquad 0 \leqq t < \infty\,. \tag{2.34}$$

Set $z(t) = P_{\lambda_0}v(t)$. Then by (2.34) $z(t)$ is a solution of the differential inequality

$$\left\|\frac{dz}{dt} - Dz\right\| \leqq 3\phi(t)\,\|z\|\,, \qquad 0 \leqq t < \infty\,. \tag{2.35}$$

Differentiating $\|z\|^2$, and taking (2.35) into account, we get

$$\frac{d}{dt}\|z\|^2 = 2Re\left(z, \frac{dz}{dt}\right) \geqq 2Re(z, Dz) - 6\phi(t)\,\|z\|^2\,. \tag{2.36}$$

Since $z(t) = P_{\lambda_0}v(t)$, it follows from (2.3) and (2.36) that

$$\frac{d}{dt}\|z\|^2 \geqq (2\lambda_0 - 6\phi(t))\,\|z\|^2\,. \tag{2.37}$$

Consequently, if we integrate (2.37), we obtain

$$\|v(t)\|^2 \geqq \|z(t)\|^2 \geqq \|z(0)\|^2 \exp\left[2\int_0^t(\lambda_0 - 3\phi(\eta)d\eta)\right] \geqq \tfrac{1}{4}e^{-C_2}\,\|v(0)\|^2 e^{2\mu t}\,,$$

which is equivalent to (2.31).

To pass from the finite to the infinite dimentional case, we have to show that the cut-off parameter λ can be selected independently of the dimension of the space F.

LEMMA 4. *Let $v(t)$ satisfy the conditions of Lemma 1. Then there exists a λ, depending only on $\|v(0)\|$, $\|v(1)\|$ and $\phi(t)$, such that*

$$\|P_\lambda v(1)\| \geqq \|R_\lambda v(1)\|\,. \tag{2.38}$$

Proof. Define $w(t) = v(1 - t)$. Then $w(t)$ is a solution of the differential inequality

$$\left\| \frac{dw}{dt} + Dw \right\| \leqq \phi(t) \| w \| , \qquad 0 \leqq t < \infty . \tag{2.39}$$

If for some λ

$$\| P_\lambda v(1) \| < \| R_\lambda v(1) \| ,$$

then

$$\| P_{-\lambda} w(0) \| > \| R_{-\lambda} w(0) \| . \tag{2.40}$$

Applying Lemma 3 to (2.39) and (2.40), we find that

$$\| v(0) \| = \| w(1) \| \geqq \tfrac{1}{2} e^{-C_2} \| w(0) \| e^m ,$$

where

$$m = -\lambda - 200\rho_0\sigma - 3C_1 .$$

Hence

$$\lambda \geqq \log \left[\frac{2 \| v(0) \|}{\| v(1) \|} e^{C_2} \right] - 200\rho_0 - 3C_1 . \tag{2.41}$$

Thus if λ is chosen smaller than the right side of (2.41), then the desired inequality (2.38) holds.

3. **Proof of Theorem 1.** Let k be an arbitrary positive integer. Using the continuity of the derivative of $v(t)$, one can show that for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, k) > 0$ such that

$$\left\| \frac{d}{dt} u(t) - \frac{u(t+h) - u(t)}{h} \right\| < \varepsilon , \tag{3.1}$$

for $|h| < \delta$ and $|t| \leqq k$.

We subdivide the interval $0 \leqq t \leqq k$ into equal subintervals of length Δ, where $\Delta < \delta$, and

$$\| Au(t+h) - Au(t) \| < \varepsilon , \tag{3.2}$$

for $|h| < \Delta$. We assume that the point $t = 1$ is included in the subdivision.

Let G be the subspace of H generated by $u(0), u(\Delta), u(2\Delta), \cdots, u(k)$. Let $A_0 = EA$, where E is the projection of H onto the subspace G. Clearly, the operator A_0 restricted to the subspace G is symmetric.

For any subdivision point $j\Delta$, we have

$$\left\| \frac{u((j+1)\Delta) - u(j\Delta)}{\Delta} - A_0 u(j\Delta) \right\| \leqq \left(1 + \frac{\varepsilon}{M}\right) \phi(j\Delta) \| u(j\Delta) \| , \tag{3.3}$$

where M is the infimum of $\phi(t) \| u(t) \|$ for $0 \leqq t \leqq k$. Let $u_0(t)$ be equal to $u(t)$ at the subdivision points and be linear in between. Note that $u_0(t)$ has its values in the finite-dimensional subspace G of H.

It follows from (3.2) and (3.3) that

$$\| D_+u_0(t) - A_0u_0(t) \| \leqq \left(1 + \frac{2\varepsilon}{M}\right)\phi(j\Delta) \| u(j\Delta) \| , \tag{3.4}$$

where D_+ denotes right differentiation, and $j\Delta \leqq t \leqq (j+1)\Delta$. By taking Δ sufficiently small and taking into account the continuity of $\phi(t)$, we obtain

$$\| D_+u_0 - A_0u_0 \| \leqq 2\phi(t) \| u_0 \| , \qquad 0 \leqq t \leqq k . \tag{3.5}$$

By Lemma 4, there is a $\bar{\lambda} = \bar{\lambda}(\| u(0) \|, \| u(1) \|, 2\phi(t))$ such that

$$\| P_{\bar{\lambda}}u_0(1) \| \geqq \| R_{\bar{\lambda}}u_0(1) \| . \tag{3.6}$$

Now we observe that the lemmas of the preceding section remain valid when $v(t)$ has a right derivative everywhere and is continuously differentiable, except at a finite number of points. Once this observation is made, we can conclude from (3.5), (3.6) and Lemma 3 that

$$\| u_0(t) \| \geqq \tfrac{1}{2} \| u(1) \| \exp\left[\int_1^t \psi(\eta)d\eta\right] , \qquad 1 \leqq t \leqq k ,$$

where

$$\psi(\eta) = \bar{\lambda} - 400\rho_0\sigma - 6\phi(\eta) .$$

Hence

$$\| u_0(t) \| \geqq \bar{K}e^{\bar{\mu}t} , \qquad 1 \leqq t \leqq k .$$

Letting $\Delta \to 0$, we conclude that

$$\| u(t) \| \geqq \bar{K}e^{\bar{\mu}t} , \qquad 1 \leqq t \leqq k ,$$

which is easily seen to imply the inequality (2.2) of Theorem 1.

In the proof of Theorem 1 we tacitly assumed that $u(t)$ never vanishes. The proof of this fact is easy. For let t_0 denote the first value of t for which $u(t)$ is zero. Since $u(0) \neq 0$, $t_0 > 0$. According to Theorem 1, we have $\| u(t) \| \geqq Ke^{\mu t}$, for $0 \leqq t < t_0$, which shows that $u(t)$ cannot possibly vanish at t_0.

4. An A priori inequality. In this section we derive an a priori inequality for a class of functions with a prescribed rate of decay at infinity.

LEMMA 5. *Let $\psi(t)$ belong to $L^2(0, a)$, for every $a > 0$, and define*

$$\beta(t) = \lambda\int_0^t (t - \eta)\psi^2(\eta)d\eta . \tag{4.1}$$

Let $U(t)$ be a strongly continuously differentiable mapping from $0 \leqq t < \infty$ with values in H such that $AU(t)$ is continuous. If the support of $U(t)$ is contained in $0 < \varepsilon \leqq t < \infty$ and

$$\lim_{t\to\infty} \| U(t) \| \exp \beta(t) = 0 \,, \tag{4.2}$$

for every $\lambda > 0$, then

$$\lambda\int_0^\infty e2^{\beta(t)}\psi^2(t) \, \| U(t) \|^2 dt \leqq \int_0^\infty e^{2\beta(t)} \left\| \frac{dU}{dt} - AU \right\|^2 dt \,, \tag{4.3}$$

provided that the left side is finite.

Proof. We may assume that $U(t)$ vanishes for all sufficiently large values of t. For in the general case we can approximate U by the sequence $U_n(t) = \zeta_n(t)U(t)$, $\zeta_n(t)$ being a C^∞ function equal to one for $t \leqq n$, zero for $t \geqq n+1$ and $0 \leqq \zeta_n \leqq 1$ in between. As $n \to \infty$, the inequality (4.3) for U_n goes over into (4.3) for U.

Now consider the integral

$$I = \int_0^\infty e^{2\beta(t)} \left\| \frac{dU}{dt} - AU \right\|^2 dt \,.$$

If we make the transformation $U(t) = e^{-\beta(t)} V(t)$, then

$$I = \int_0^\infty \left\| \frac{dV}{dt} - AV - \frac{d\beta}{dt} V \right\|^2 dt \,. \tag{4.4}$$

It follows from the elementary inequality

$$(a-b)^2 \geqq -2ab$$

and (4.4) that

$$I \geqq -2\int_0^\infty Re\left(\frac{dV}{dt}, \frac{d\beta}{dt} V + AV\right) dt \,. \tag{4.5}$$

We have

$$\begin{aligned} \int_0^\infty \left(\frac{dV}{dt}, \frac{d\beta}{dt} V\right) dt &= \int_0^\infty \frac{d}{dt}\left(V, \frac{d\beta}{dt} V\right) dt \\ &\quad - \int_0^\infty \left(V, \frac{d^2\beta}{dt^2} V + \frac{d\beta}{dt}\frac{dV}{dt}\right) dt \,. \end{aligned} \tag{4.6}$$

The first integral on the right vanishes since $V(t)$ has compact support. Hence

$$-2\int_0^\infty Re\left(\frac{dV}{dt}, \frac{d\beta}{dt} V\right) dt = \lambda\int_0^\infty e^{2\beta(t)}\psi^2(t) \, \| U(t) \|^2 dt \,. \tag{4.7}$$

In view of (4.5) and (4.7) it is sufficient to prove that

$$\int_0^\infty Re\left(\frac{dV}{dt}, AV\right)dt = 0 . \tag{4.8}$$

Taking into account the symmetry of A, we have that

$$\left(\frac{dV}{dt}, AV\right) = \frac{d}{dt}(V, AV) - \overline{\left(\frac{dV}{dt}, AV\right)} , \tag{4.9}$$

the bar denoting complex conjugation. Therefore

$$2\,Re\left(\frac{dV}{dt}, AV\right) = \frac{d}{dt}(V, AV) ,$$

from which (4.8) follows directly by integration. This completes the proof of the lemma.

5. A special instance of Theorem 1. As a first application of Lemma 5, we give a direct proof of a slightly weaker version of Theorem 1 in the case that $\phi(t)$ belongs to $L^2(0, \infty)$.

THEOREM 2. *Let $u(t)$ be a solution of the abstract differential inequality*

$$\left\|\frac{du}{dt} - Au\right\| \leqq \phi(t)\,\|u\| , \qquad 0 \leqq t < \infty , \tag{5.1}$$

where $\phi(t)$ belongs to $L^2(0, \infty)$. If

$$\lim_{t\to\infty} \|u(t)\|\, e^{\lambda t} = 0 \tag{5.2}$$

for every $\lambda > 0$, then u has property S, i.e., it vanishes identically for $0 \leqq t < \infty$.

Proof. Since $\phi \in L^2(0, \infty)$, it follows from (4.1) that (we take $\psi = \phi$)

$$\beta(t) \leqq \lambda t\int_0^\infty \phi^2(\eta)d\eta .$$

Therefore (5.2) implies that

$$\lim_{t\to\infty} \|u(t)\| \exp \beta(t) = 0 \tag{5.3}$$

for every $\lambda > 0$. Let $\zeta(t)$ be a C^∞ function equal to one for $0 < 2\varepsilon \leqq t$, equal to zero for $0 \leqq t \leqq \varepsilon$ and $0 \leqq \zeta \leqq 1$ in between. Set $U(t) = \zeta(t)u(t)$. Because of (5.3) and the fact that

$$\int_0^\infty \phi^2(t)e^{2\beta(t)}\,\|U(t)\|^2 dt < \infty ,$$

all of the conditions of Lemma 5 are met, and therefore

$$\lambda\int_{2\varepsilon}^{\infty}\phi^2(t)e^{2\beta(t)}\,||\,u(t)\,||^2 dt \leqq \int_{\varepsilon}^{2\varepsilon}e^{2\beta(t)}\left\|\frac{dU}{dt}-AU\right\|^2 dt + \int_{2\varepsilon}^{\infty}\phi^2(t)\,||\,u(t)\,||^2 dt\ .$$

If $\lambda \geqq 2$ then

$$\int_{3\varepsilon}^{\infty}\phi^2(t)e^{2\beta(t)}\,||\,u(t)\,||^2 dt \leqq \int_{\varepsilon}^{2\varepsilon}e^{2\beta(t)}\left\|\frac{dU}{dt}-AU\right\|^2 dt\ . \tag{5.4}$$

Using the monotonic character of $\beta(t)$, we get from (5.5) that

$$\int_{3\varepsilon}^{\infty}\phi^2(t)\,||\,u(t)\,||^2 dt \leqq \exp\left[\beta(2\varepsilon)-\beta(3\varepsilon)\right]\int_{\varepsilon}^{2\varepsilon}\left\|\frac{dU}{dt}-AU\right\|^2 dt\ . \tag{5.5}$$

Since $\beta(2\varepsilon)-\beta(3\varepsilon)\rightarrow -\infty$ as $\lambda\rightarrow\infty$, it follows from (5.5) that

$$\int_{3\varepsilon}^{\infty}\phi^2(t)\,||\,u(t)\,||^2 dt = 0\ .$$

Therefore $u(\mathrm{t}) = 0$ for $t \geqq 3\varepsilon$. Since ε is arbitrary, $u(t)$ vanishes identically for $0 \leqq t < \infty$.

In much the same way we can prove the following result for bounded ϕ.

THEOREM 3. *Let $u(t)$ be a solution of the abstract differential inequality* (5.1), *where $\phi(t) \leqq$ const. If*

$$\lim_{t\rightarrow\infty}||\,u(t)\,||\exp(\lambda t^2) = 0$$

for every $\lambda > 0$, then $u(t)$ vanishes identically.

More generally, we have the

THEOREM 4. *Let $u(t)$ be a solution of the abstract differential inequality* (5.1). *Assume that $\phi(t)$ belong to $L^2(0, a)$ for every $a > 0$, and*

$$\phi^2(t) \leqq \exp\left[\lambda\int_0^t (t-\eta)\phi^2(\eta)d\eta\right],$$

for all sufficiently large t and λ. If

$$\lim_{t\rightarrow\infty}||\,u(t)\,||\exp\beta(t) = 0\ ,$$

for every $\lambda > 0$, then $u(t)$ vanishes identically.

6. Parabolic differential inequalities. Let G be a bounded domain in the real Euclidean n-space R^n. For two real functions $u(x)$ and $v(x)$

belonging to $L^2(G)$ we denote by

$$(u, v) = \int_G u(x)v(x)dx$$

their real scalar product and by $\| u \|_0 = (u, u)^{1/2}$ the associated norm. Let $H_1^0(G)$ denote the closure of $C_0^\infty(G)$, the C^∞ functions on G with compact support in G, relative to the norm

$$\| u \|^2 = \int_G \left(| u(x) |^2 + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(x) \right|^2 \right) dx \ .$$

Consider the differential operator

$$L = \sum_{i,j=1} \frac{\partial}{\partial x_i}\left(a^{ij}(x) \frac{\partial}{\partial x_j} \right) , \tag{6.1}$$

where $a^{ij}(x) = a^{ji}(x)$. We assume that there exist positive constants m and M such that, for all x in $\bar{G}$ and all real vectors $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$,

$$m \sum_{i=1}^n \zeta_i^2 \leqq \sum_{i,j=1}^n a^{ij}(x)\zeta_i\zeta_j \leqq M \sum_{i=1}^n \zeta_i^2 \ . \tag{6.2}$$

Thus L is a real elliptic differential operator.

If $u \in H_1^0(G)$ we say that $Lu \in L^2(G)$ when $\{a^{ij}(x)\}(\partial u/\partial x_j)$ is differentiable with respect to x_i (in the sense of distributions) and

$$\frac{\partial}{\partial x_i}\left(a^{ij}(x) \frac{\partial u}{\partial x_j} \right) \in L^2(G) \ .$$

It is not difficult to show that

$$(Lu, v) = (u, Lv) \ , \tag{6.3}$$

for u and v in $H_1^0(G)$ and Lu and Lv in $L^2(G)$, the common value of **(6.3)** is

$$(Lu, v) = - \int_G \left(\sum_{i,j=1}^n a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx \ . \tag{6.4}$$

Thus the operator L is formally self-adjoint.

Let $\Gamma(t) = \exp\left[\gamma \int_0^t \psi^2(\eta) d\eta \right]$, and introduce the function

$$\sigma(t) = \lambda \int_0^t \Gamma(\eta) \int_0^\eta \Gamma^{-1}(\zeta)\phi^2(\zeta) d\zeta d\eta \ . \tag{6.5}$$

The function $\sigma(t)$ is non-decreasing provided that γ and λ are nonnegative. We also note that

$$\Gamma \frac{d}{dt}\left[\Gamma^{-1} \frac{d\sigma}{dt} \right] = \lambda \phi^2 \ . \tag{6.6}$$

If the functions $\phi(t)$ and $\psi(t) \in L^2(0, \infty)$, there exists a constant Γ_0, depending on γ, such that

$$\sigma(t) \leqq \lambda \Gamma_0 t \; . \tag{6.7}$$

We introduce the norm

$$\| u \|_1^2 = \int_G \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i}(x) \right|^2 dx \; ,$$

which is equivalent to the norm defined above for $H_1^0(G)$.

LEMMA 6. *Let $\phi(t)$ and $\psi(t)$ belong to $L^2(0, \infty)$. Let $Z(t)$ be a strongly continuously differentiable mapping from $0 \leqq t < \infty$ with values in $H_1^0(G)$ such that $LZ(t) \in L^2(G)$ is continuous. If the support of $Z(t)$ is contained in $0 < t_0 \leqq t < \infty$ and*

$$\lim_{t \to \infty} \| Z(t) \|_1 e^{\lambda t} = 0 \; , \tag{6.8}$$

for every $\lambda > 0$, then

$$\begin{aligned} \int_0^\infty \Gamma^{-1}(t) e^{2\sigma(t)} \left\| LZ - \frac{\partial Z}{\partial t} \right\|_0^2 dt \geqq \lambda \int_0^\infty \Gamma^{-1}(t) e^{2\sigma(t)} \phi^2(t) \| Z \|_0^2 dt \\ + m\gamma \int_0^\infty \Gamma^{-1}(t) e^{2\sigma(t)} \psi^2(t) \| Z \|_1^2 dt \; . \end{aligned} \tag{6.9}$$

Proof. The integrals on the right side of (6.9) are finite because of (6.7) and (6.8). As in Lemma 5, we may assume that Z(t) is identically zero for all sufficiently large values of t. Set $Z(t) - e^{-\sigma(t)} V(t)$. Then if J denotes the integral on the left side of (6.9), we have

$$J \geqq -2 \int_0^\infty \Gamma^{-1}(t) \left(\frac{dV}{dt}, LV + V \frac{d\sigma}{dt} \right) dt \; . \tag{6.10}$$

Integrating by parts and using the fact that $V(t)$ has compact support, we find that

$$-2 \int_0^\infty \Gamma^{-1}(t) \left(\frac{dV}{dt}, V \frac{d\sigma}{dt} \right) dt = \lambda \int_0^\infty \Gamma^{-1}(t) e^{2\sigma(t)} \phi^2(t) \| Z \|_0^2 dt \; . \tag{6.11}$$

In proving (6.11) we have made use of (6.6).

Since L is real and symmetric, we have

$$-2 \int_0^\infty \Gamma^{-1}(t) \left(\frac{dV}{dt}, LV \right) dt = - \int_0^\infty \Gamma^{-1}(t) \frac{d}{dt} (V, LV) dt \; . \tag{6.12}$$

Another integration by parts yields

$$-2 \int_0^\infty \Gamma^{-1}(t) \left(\frac{dV}{dt}, LV \right) dt = - \gamma \int_0^\infty \Gamma^{-1}(t) \psi^2(t) (V, LV) dt \; . \tag{6.13}$$

In view of (6.2) and (6.4) we have $(V, LV) \leqq -m \| V \|_1^2$, so that (6.13) implies that

$$(6.14) \qquad -2\int_0^\infty \Gamma^{-1}(t)\Big(\frac{dV}{dt}, LV\Big)dt \geqq \gamma m \int_0^\infty \Gamma^{-1}(t)e^{2\sigma(t)}\psi^2(t) \| Z \|_1^2 dt \, .$$

Combining (6.10), (6.11) and (6.14), we get (6.9).

THEOREM 5. *Let $\phi(t)$ and $\psi(t)$ belong to $L^2(0, \infty)$. Let $u(t)$ be a strongly continuously differentiable function from $0 \leqq t < \infty$ with values in $H_1^0(G)$ such that $Lu(t) \in L^2(G)$ is continuous. If $u(t)$ satisfies the differential inequality*

$$(6.5) \qquad \Big\| Lu - \frac{\partial u}{\partial t} \Big\|_0^2 \leqq \phi^2(t) \| u \|_0^2 + \psi^2(t) \| u \|_1^2 \, , \qquad 0 \leqq t < \infty \, ,$$

and

$$\lim_{t \to \infty} \| u(t) \|_1 e^{\lambda t} = 0 \, ,$$

for every $\lambda > 0$, then u vanishes identically.

Theorem 5 follows from Lemma 6 in much the same way that Theorem 2 follows from Lemma 5, and for this reason the proof will be omitted.

If in Theorem 5 we only assume that $\phi(t)$ is bounded, then we can deduce from Lemma 6 that only the trivial solution of (6.15) can vanish faster than $\exp(-\lambda t^2)$, for every $\lambda > 0$.

BIBLIOGRAPHY

1. P. D. Lax, *A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations*, Comm. Pure and Appl. Math., **9** (1956), 747-766.

2. M. H. Protter, *Properties of Solutions of Parabolic Equations and Inequalities*, Univ. of California Tech. Report, Prepared under Contract AF 49(638)-398 (1960).

STANFORD UNIVERSITY
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