ON THE FIELD OF RATIONAL FUNCTIONS OF ALGEBRAIC GROUPS

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0. Introduction. Let K be an algebraically closed field of characteristic 0, let k be a subfield of K and suppose that G is a (k, K) algebraic group, i.e., an algebraic group defined over k and composed of K-rational points. Let $k(G)$ denote the fields of k-rational functions on G. G_k denotes the subgroup of G composed of all k-rational points of G. If $g \in G_k$ then the regular mapping $L_g(R_g)$ of G onto G defined by $L_g x = gx$ $(R_g x = xg)$ induces an automorphism of $k(G)$ denoted by $g_l(g_r)$. Let D_k denote the Lie algebra of all k-derivations of $k(G)$ (i.e., of all derivations of $k(G)$ that are trivial on k) which commute with g_r, for every $g \in G_k$.

For any subset A of $k(G)$ let $G(A)$ denote the subgroup of G composed of all elements g such that $g_r(f) = f$, for every $f \in A$. In the sequel we shall always assume that G_k is dense in G.

The main result of this paper is the following theorem:

THEOREM 1. *Let F be a subfield of $k(G)$ containing k. Then the following three conditions are equivalent:*

(1) *F is $(G_k)_l$ — stable*

(2) *F is D_k — stable*

(3) *$F = k(G/G(F))$ and so F coincides with the field of all elements of $k(G)$ that are fixed under $G(F)_r$.*

By means of the theorem, we establish a Galois correspondence between a family of subgroups of G and the family of $(G_k)_l$-stable subalgebras of the algebra of representative functions of G.

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1. Let K be an algebraically closed field of characteristic 0, let k be a subfield of K and suppose that V, W are (k, K) — algebraic varieties. Let $k(V)$, $k(W)$ denote the fields of k-rational functions on V and W, respectively. If A is a subset of $k(V)$ then $k(A)$ denotes the fields generated by k and A.

The following result is known[1]:

(1) Let F be a rational mapping of V onto a dense subset of W and let φ be the cohomomorphism corresponding to F. Then there exists

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[1] See e.g. [2],

an open subset $W_1 \subset W$ such that $F^{-1}(x)$ contains exactly $[k(V): \varphi(k(W))]$ elements, for every $x \in W_1$.

LEMMA 1. *Let A be a subset of $k(V)$ and suppose that there exists a dense set $V_1 \subset V$ and an open subset $V_2 \subset V$ such that for any two distinct points x_1, x_2, where $x_1 \in V_1, x_2 \in V_2$, there exists a function $f \in A$ which is defined at x_1, x_2 and $f(x_1) \neq f(x_2)$. Then $k(A) = k(V)$.*

Proof. Let B be a finite subset of A, say $B = \{f_1, \cdots, f_n\}$. Then F_B denotes the rational mapping $F_B: V \rightarrow K^n$ defined by $F_B(x) = (f_1(x), \cdots, f_n(x))$ and $W_B = (F_B(V)^- \subset K^n$. Let $\Delta(W_B)$ be the diagonal of $W_B \times W_B$ and $V_B = ((F_B \times F_B)^{-1}\Delta(W_B))^- \subset V \times V$. Then there exists a finite subset $B_0 \subset A$ such that $V_{B_0} \subset V_B$, for every finite subset $B \subset A$ (since $V \times V$ satisfies the minimal condition for closed sets). Let V_0 be an open subset of V such that F_{B_0} is regular on V_0. We may assume that $V_0 = V_2 = V$, since we may replace V by $V_0 \cap V_2$. If $x_1 \in V_1$, $x_2 \in V$ and $x_1 \neq x_2$ then there exists $f \in A$ such that f is defined at x_1, x_2 and $f(x_1) \neq f(x_2)$. Hence $(x_1, x_2) \notin V_{\{f\}}$ and so $(x_1, x_2) \notin V_{B_0}$. Thus $F_{B_0}(x_1) \neq F_{B_0}(x_2)$. Therefore, for every $x \in F_{B_0}(V_1)$, $F_{B_0}^{-1}(x)$ contains exactly one element. But $F_{B_0}(V_1)$ is dense in W_{B_0}. Hence it follows from (i) that $[k(V): k(B_0)] = 1$, i.e., $k(V) = k(B_0)$. Thus $k(V) = k(A)$.

Let G be a (k, K) – algebraic group. Suppose that G_k is dense in G. Let D be the Lie algebra of all derivations of $K(G)$ commuting with g_r, for every $g \in G$, and let D_k denote the Lie algebra consisting of all derivations from D that map $k(G)$ into $k(G)$. Let $k[D]$ ($K[D]$) denote the k-algebra (K – algebra) of transformations generated by the identity map and $D_k(D)$.

If $d \in D_k$ then d restricted to $k(G)$ is a k-derivation commuting with g_r, for every $g \in G_k$. On the other hand if d_1 is a k-derivation of $k(G)$ commuting with g_r, for every $g \in G_k$, then there exists a unique extension d of d_1 to a K-derivation of $K(G)$, and the extension belongs to D_k. Hence we may identify D_k and the Lie algebra of all k-derivations of $k(G)$ that commute with g_r, for every $g \in G_k$.

(ii)[2] If $f \in K(G)$ and f is defined at a point $g \in G$ then df is defined at g, for any $d \in K[D]$.

LEMMA 2. *Let $f \in K(G)$ and suppose that f is defined at $g \in G_k$. If $f \neq 0$ then there exists $d \in k[D]$ such that $(df)(g) \neq 0$.*

Proof. Suppose that $f \neq 0$. If $f(g) \neq 0$ then the identity element of $k[D]$ satisfies the desired condition. Hence we may assume that $f(g) = 0$, Let $\mathcal{O}_k(\mathcal{O}_K)$ denote the local ring of g in $k(G)$ $(K(G))$ and let $m_k(m_K)$ be the maximal ideal of $\mathcal{O}_k(\mathcal{O}_K)$. Then $f \in m_K$. Let

[2] See [4] p.51,

$x_1, \cdots, x_m$ be elements of m_k such that $x_1 + m_k^2, \cdots, x_m + m_k^2$ is a k-basis of m_k/m_k^2. The $x_1 + m_K^2, \cdots, x_m + m_K^2$ is a K-basis of m_K/m_K^2. Hence every mapping $(x_1, \cdots, x_m) \to k$ can be extended to a derivation $\partial: \mathcal{O}_K \to K$. On the other hand $f \neq 0$ and so there exists an integer t such the $f \in m_K^t - m_K^{t+1}$. Hence $f = \sum_{i_1+\cdots+i_m=t} a_{i_1,\ldots,i_m} x_1^{i_1}, \cdots, x_m^{i_m} + f_1$, where $f_1 \in m_K^{t+1}$, $a_{i_1,\ldots,i_m} \in K$ and at least one $a_{i_1,\ldots,i_m}$ is different from zero. Let ∂_i be the derivation of $\mathcal{O}_K$ into K such than $\partial_i\, x_j = \delta_{ij}$, where $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j. \end{cases}$ It is known[3], that there exist $d_i \in D_k$ such that $(d_i f)(g) = \partial_i f$ for every $f \in \mathcal{O}_K$. Then $(d_1^{i_1} \cdots d_m^{i_m}) f(g) = i_1! \cdots i_m! a_{i_1,\ldots,i_m} \neq 0$ if $a_{i_1,\ldots,i_m} \neq 0$. Hence the lemma is proved.

If A is a subset of $k(G)$ then $G(A)$ denotes the subgroup of G composed of all elements g such that g_r leaves the elements of A fixed. For any $A \subset k(G)$, $G(A)$ is a k-closed subgroup of G.

(iii)[4] Let G_1 be a k-closed subgroup of G. Then G/G_1 is defined over k. Let φ be the cohomomorphism of $k(G/G_1)$ into $k(G)$ corresponding to the canonical mapping $G \to G/G_1$. Then $\varphi(k(G/G_1))$ coincides with the subfield of all elements of $k(G)$ which are fixed under g_r, for every $g \in G_1$. In the sequel we shall identify $k(G/G_1)$ and $\varphi(k(G/G_1))$.

Proof of the theorem.

Implications $(3) \Rightarrow (1)$ and $(3) \Rightarrow (2)$ are obvious.

$(1) \Rightarrow (3)$[5]. Let $g_1 \in G_k$, $g_2 \in G$ and $G(F)g_1 \neq G(F)g_2$. Then $g_2 g_1^{-1} \notin G(F)$. Hence there exists $f_0 \in F$ such that $(g_2 g_1^{-1})_r f_0 \neq f_0$. Therefore there exists an element $g \in G_k$ such that $(g_2 g_1^{-1})_r f_0$ and f_0 are defined at g and $(g_2 g_1^{-1})_r f_0(g) \neq f_0(g)$, i.e., $f_0(g_2 g_1^{-1} g) \neq f_0(g)$, $(g_1^{-1} g)_l f_0(g_2) \neq (g_1^{-1} g)_l f_0(g_1)$. Let $f = (g_1^{-1} g)_l f_0$. Then $f \in F$ since $g_1^{-1} g \in G_k$; f is defined at g_1 and g_2, and $f(g_1) \neq f(g_2)$. Thus it follows from Lemma 1 that $F = k(G/G(F))$, because $G(F) \cdot G_k / G(F)$ is dense in $G/G(F)$.

$(2) \Rightarrow (3)$. Let $f_1, \cdots, f_n$ be a set of generators of F over k, and let V_1 be an open subset of G such that $f_1, \cdots, f_n$ are regular on V_1. We may assume that $V_1 = G(F) V_1$. Let $g_1 \in V_1 \cap G_k$, $g_2 \in V_1$, $G(F)g_1 \neq G(F)g_2$. Then $g_2 g_1^{-1} \notin G(F)$ and so there exists f_i such that $(g_2 g_1^{-1})_r f_i \neq f_i$. We know that $(g_2 g_1^{-1})_r f_i$ and f_i are defined at g_1. Hence it follows from Lemma 2 that there exists an element $d \in k[D]$ such that

$$d((g_2 g_1^{-1})_r f_i)(g) \neq (d f_i)(g), \text{ i.e., } (d f_i)(g_1) \neq (d f_i)(g_2)\ .$$

Therefore, for any pair of distinct elements $G(F)g_1$, $G(F)g_2$ such that

$$G(F)g_1 \in G(F) \cdot G_k \cap V_1/G(F) \text{ and } G(F)g_2 \in V_1/G(F)\ ,$$

[3] See [4] p. 51,

[4] See Proposition 2, p. 495 in [5].

[5] This part of the proof is modeled after the proof of Lemma 5.3 p. 515 in [3].

there exists an element $f \in F$ which is defined at $G(F)g_1$, $G(F)g_2$ and such that $f(G(F)g_1) \neq f(G(F)g_2)$. But $V_1/G(F)$ is an open subset of $G/G(F)$, and $G(F)G_k \cap V_1/G(F)$ is dense in $G/G(F)$. Hence it follows from Lemma 1 that $F = k(G/G(F))$.

This completes the proof of the theorem.

2. Applications. As a consequence of Lemma 2 one can get the following corollary:

COROLLARY. *If α is an automorphism of $k(G)$ commuting with D_k and leaving the elements of k fixed then there exists $h \in G_k$ such that $\alpha = h_r$.*

Proof. α induces a rational map $F_\alpha: G \to G$. Let $g \in G_k$ be a point such that F_α is defined at g and let $F_\alpha(g) = h^{-1}g$ Then $h \in G_k$ and $f(g) = (\alpha f)(h^{-1}g)$, for every $f \in k(G)$ that is defined at g. Hence $(df)g = (\alpha(df))(h^{-1}g)$, for every $d \in k[D]$. But $(\alpha(df))(h^{-1}g) = (h_r^{-1}(\alpha(df)))(g)$ and d commutes with α and h_r^{-1}. Therefore $(df)(g) = (d(h_r^{-1}(\alpha f)(g)))$. Hence it follows from Lemma 2 that $f = h_r^{-1}(\alpha f)$. Thus $h_r f = \alpha f$, for every f that is defined at g. Therefore $h_r f = \alpha f$, for every $f \in k(G)$.

It follows from the corollary that if F is a D_k — stable subfield of $k(G)$ containing k then every D_k — automorphism of $k(G)$ leaving the elements of F fixed belongs to $G(F)_r$, i.e., the D_k — Galois group of $k(G)$ over F coincides with $G(F)_r$. Combining this result and the above theorem we obtain that there exists the usual one to one Galois correspondence between D_k — stable subfields of $k(G)$ containing k and k-closed subgroups of G.

Let $k[G]$ denote the ring of regular (i.e., representative) functions on G. Let $\mathscr{R}$ be the family of all $(G_k)_l$ — stable (or, equivalently, D_k — stable) subrings R of $k[G]$ containing k and satisfying the following condition if $f \in R, g \in R$ and $f/g \in k[G]$ then $f/g \in R$. Let $\mathscr{G}$ denote the family of all k-closed subgroups H of G such that G/H is isomorphic to an open subset of an affine variety.

THEOREM 2. *The mappings $H \to k[G] \cap k(G/H)$ and $R \to G(R)$ establish a Golois correspondence between $\mathscr{G}$ and $\mathscr{R}$*[6].

Proof. $H \in \mathscr{G}$ then $k[G] \cap k(G/H) \in \mathscr{R}$ and $G(k[G] \cap k(G/H)) = H$, since $k(G/H)$ is generated by $k[G] \cap k(G/H)$.

Now, if $R \in \mathscr{R}$ then $G(R) \in \mathscr{G}$. In fact, if $R \in \mathscr{R}$, then $k(R)$ is $(G_k)_l$ — stable and so $k(R) = k(G/G(R))$. For every $f \in R$, $(G_k)_l f$ generates a finite dimensional k-vector space, Hence there exists a finitely generated over k $(G_k)_l$ — stable subring R_0 of R such that $k(R_0) = k(R)$. Let W denote

[6] C.f. [1] p. 324.

the affine variety that has R_0 as its coordinate ring. One can define a structure of a G-homogeneaus space on W, since $K[R_0]$ is G_ι — stable. Let η be the canonical mapping of $G/G(R)$ into W. Then η commutes with the action of G and is birational. Hence η is an isomorphism of $G/G(R)$ onto an open subset $\eta(G/G(R))$ of W.

Moreover, $R = k[G] \cap k(G/G(R))$, since $R \in \mathscr{R}$ and $k(R) = k(G/G(R))$. This completes the proof of the theorem.

Added in Proof. The equivalence (1) $\Longleftrightarrow$ (2) of Theorem 1 in the case where k is algebraically closed has been proved by E. Abe and T. Kanno (Tohoku Math. Jour. 2nd series **11** (1959), 376–384).

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