UPPER BOUNDS FOR THE EIGENVALUES OF SOME VIBRATING SYSTEMS

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1. Introduction. Let $p(x) \geqq 0$, $x \in [0, a]$, be the density of a string fixed at the points $x = 0$ and $x = a$ under unit tension. The natural frequencies of the string are determined by the eigenvalues of the differential system

$$u'' + \lambda p(x)u = 0,\ u(0) = u(a) = 0 . \tag{1}$$

We note that these eigenvalues depend on the density function $p(x)$ and denote them accordingly by

$$0 < \lambda_1(p) < \lambda_2(p) < \lambda_3(p) < \cdots .$$

M. G. Krein [5] has found the sharp bounds

$$\frac{4Hn^2}{M^2}X\left(\frac{M}{aH}\right) \leqq \lambda_n(p) \leqq \frac{\pi^2 n^2 H}{M^2} \qquad (n = 1, 2, \cdots)$$

where $X(t)$ is the least positive root of the equation

$$\sqrt{X}\tan X = \frac{t}{1-t}$$

and where $p(x)$ is such that $\int_0^a p(x)dx = M$ and $0 \leqq p(x) \leqq H$.

Sharp lower bounds are found in [1] when instead of the condition $p(x) \leqq H$, we have $p(x)$ either monotone, $p(x)$ convex, or $p(x)$ concave. The precise definitions of convex and concave are given below.

In this paper, we find sharp upper bounds for $\lambda_n(p)$ $(n = 1, 2, 3, \cdots)$ whenever $p(x)$ belongs to any one of the following sets of functions:

(a) $E_1(M, H, a)$, the set of monotone increasing functions where

$$\int_0^a p(x)dx = M \text{ and } 0 \leqq p(x) \leqq H,\ x \in [0, a] .$$

(b) $E_2(M, H, a)$, the set of continuous convex functions, i.e., continuous functions $p(x)$ such that

$$p(x) \leqq \frac{x_2 - x}{x_2 - x_1}p(x_1) + \frac{x - x_1}{x_2 - x_1}p(x_2),\ 0 \leqq x_1 \leqq x_2 \leqq a ,$$

with $\int_0^a p(x)dx = M$ and $0 \leqq p(x) \leqq H$, $x \in [0, a]$.

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(c) $E_3(M, a)$, the set of continuous concave functions, i.e., $-p(x)$ convex, such that $\int_0^a p(x)dx = M$, $x \in [0, a]$.

In general, the values of the maxima appear as the roots of a transcendental system of equations and are not obtained explicitly. However, explicit bounds are given in some special cases.

The methods used generalize to give bounds for the eigenvalues of a vibrating rod. Upper bounds are also found for the lowest eigenvalue of a vibrating membrane over a circular domain when the density is bounded and convex and also when the density is concave.

We make use of the following lemmas.

LEMMA 1. *Let $p(x)$ and $q(x)$ be nonnegative integrable functions defined for $x \in [a, b]$ and let $f(x)$ be nonnegative, continuous and monotone increasing in $[a, b]$. If $c \in (a, b)$ is such that $p(x) \geqq q(x)$ for $x \in (a, c)$ and $p(x) \leqq q(x)$ for $x \in (c, b)$, then*

$$\int_a^b p(x)dx = \int_a^b q(x)dx$$

implies that

$$\int_a^b p(x)f(x)dx \leqq \int_a^b q(x)f(x)dx .$$

If $f(x)$ is monotone decreasing, the inequality sign is reversed.

A proof of this lemma is given in [1].

LEMMA 2. *Let E_k be one of the classes of functions defined above. There exists a function $\rho(x) \in E_k$ such that*

$$\lambda_n(\rho) = \sup_{p(x) \in E_k} \lambda_n(p) .$$

Let $p(x) \in E_k$ for some $k = 1, 2,$ or 3. By the definition of E_k, there is a number H such that $0 \leqq p(x) \leqq H$, $x \in [0, a]$. (When $k = 3$, that is when $p(x)$ is concave, we take $H = \frac{2M}{a}$.) It follows that

$$\lambda_n(p) \leqq \frac{n^2\pi^2}{Ha^2} .$$

Hence, there is a number μ such that

$$\mu = \sup_{p(x) \in E_k} \lambda_n(p) .$$

Let $E(M, H, a)$ be the set of all functions $p(x)$, $x \in [0, a]$ such that

$0 \leqq p(x) \leqq H < \infty$ and $\int_0^a p(x)dx = M$. Krein [5] has shown that there exists a subset $\{p_\nu(x)\}$ of $E(M, H, a)$ and a function $\rho(x) \in E(M, H, a)$ such that

$$\lim_{\nu\to\infty}\left(\int_0^x p_\nu(x)dx\right) = \int_0^x \rho(x)dx \ .$$

The convergence is uniform for $x \in [0, a]$ and furthermore

$$\lim_{\nu\to\infty} \lambda_n(p_\nu) = \lambda_n(\rho) \ .$$

In particular if $p(x) \in E_k$, then the functions $p_\nu(x)$ also belong to E_k. We now show that in each of the cases $k = 1, 2, 3$, $\rho(x) \in E_k$ also.

We first consider $E_1(M, H, a)$, that is, the family of all monotone increasing bounded functions $p(x)$ such that $\int_0^a p(x)dx = M$. Then $p_\nu(x) \in E_1(M, H, a)$, $(\nu = 1, 2, \cdots)$. Let

$$\sigma_\nu(x) = \int_0^x p_\nu(x)dx \ .$$

Since $p_\nu(x)$ is increasing, $\sigma_\nu(x)$ must be convex. Hence, $\lim_{\nu\to\infty} \sigma_\nu(x) = \sigma_0(x) = \int_0^a \rho(x)dx$ must also be convex. For if

$$\sigma_\nu(x) \leqq \frac{x - x_1}{x_2 - x_1}\sigma_\nu(x_2) + \frac{x_2 - x}{x_2 - x_1}\sigma_\nu(x_1) \ ,$$

$(x_1 < x < x_2)$, then the same inequality must hold in the limit. It then follows that $\rho(x)$ is increasing.

For the family $E_2(M, H, a)$, that is for convex $p(x)$, we first note that the functions $p_\nu(x)$ $(\nu = 1, 2, \cdots)$ are also convex. We now consider these functions while restricting x to lie in the interval $[\delta, a-\delta]$ where $0 < \delta < a/2$. From the convexity of $p_\nu(x)$, it follows that

$$\left|\frac{p_\nu(x + h) - p_\nu(x)}{h}\right| < H/\delta \ , \qquad (x \in [\delta, a - \delta],\ \nu = 1, 2, \cdots).$$

Hence $\{p_\nu(x)\}$ is an equicontinuous family of functions in this interval. We now consider

$$| p_\nu(x) - \rho(x) | \leqq \left| p_\nu(x) - \frac{\sigma_\nu(x + h) - \sigma_\nu(x)}{h}\right|$$
$$+ \left|\frac{\sigma_\nu(x + h) - \sigma_0(x)}{h} - \frac{\sigma_0(x + h) - \sigma_0(x)}{h}\right| + \left|\frac{\sigma_0(x + h) - \sigma_0(x)}{h} + \rho(x)\right|$$

where x, $x + h \in [\delta, a - \delta]$. Since $\dfrac{\rho_\nu(x + h) - \sigma_\nu(x)}{h} = p_\nu(x + \theta h)$ for some $0 < \theta < 1$, it follows from the equicontinuity that the first term on the right may be made small by choosing h small. The last may be made small by choosing h small since $\sigma_0'(x) = \rho(x)$. Then for fixed h, the middle term may be made small by choosing ν sufficiently large. Thus $p_\nu(x) \to \rho(x)$ as $\nu \to \infty$ in any closed interval properly contained in $(0, a)$. Hence we must have point wise convergence and $\rho(x)$ must be convex, $x \in (0, a)$.

The corresponding result for the family of functions $E_3(M, a)$, that is when $p(x)$ is concave, follows directly from the convex case by considering $\{-p_\nu(x)\}$.

LEMMA 3. *The first variation of* $\lambda_n(p)$ *with the condition* $\int_0^a p(x)dx = M$ *is*

$$\delta\lambda_n(p) = -\lambda_n(p)\int_0^a \delta p(x) u_n^2(x)dx \tag{2}$$

where $u_n(x)$ *is the normalized eigenfunction corresponding to* $\lambda_n(p)$ *and* $\int_0^a (\delta p)dx = 0$.

Consider the differential system associated with a vibrating string of linear density $p(x) + \varepsilon q(x) \geqq 0$, namely

$$(u + \varepsilon v)'' + (\lambda + \varepsilon\mu)(p(x) + \varepsilon q(x))(u + \varepsilon v) = 0\ ,$$
$$u(0) + \varepsilon v(0) = u(a) + \varepsilon v(a) = 0\ ,$$

where $\int_0^a [p(x) + \varepsilon q(x)]dx = M$. We denote the nth eigenvalue of this system by $\lambda_n(p) + \varepsilon\mu_n$ and the corresponding eigenfunction by $u_n(x) + \varepsilon v_n(x)$ where $u_n(x)$ is the eigenfunction corresponding to $\lambda_n(p)$. $u_n + \varepsilon v_n(x)$ then satisfies the equation

$$u_n'' + v_n'' + (\lambda_n(p) + \varepsilon\mu_n)(p(x) + \varepsilon q(x))(u_n + \varepsilon v_n) = 0\ .$$

Multiplying this by $u_n(x)$ and integrating the resulting expression over the interval $(0, a)$, we get

$$-\lambda_n(p) + \varepsilon\int_0^a u_n v_n'' dx + (\lambda_n(p) + \varepsilon\mu_n)[1 + \varepsilon\int_0^a (pu_n v_n + qu_n^2)dx + 0(\varepsilon^2)] = 0\ .$$

We have used the relation $\int_0^a u_n'' u_n dx = -\lambda_n(p)$ and taken $\int_0^a pu_n^2 dx = 1$. Solving for μ_n, we find

$$\mu_n = \frac{-\lambda_n(p)\int_0^a q(x)u_n^2(x)dx - \lambda_n\int_0^a (v_n'' u_n - v_n u_n'')dx + 0(\varepsilon)}{1 + 0(\varepsilon)}\ .$$

Integrating the second integral by parts, we find that it vanishes so that letting $\varepsilon \to 0$, we get

$$\mu_n = -\lambda_n(p)\int_0^a q(x)u_n^2(x)dx .$$

Hence

$$\delta\lambda_n(p)\, \varepsilon\mu_n = -\lambda_n(p)\int_0^a \delta p(x)u_n^2(x)dx$$

where we have taken $\delta p(x) = \varepsilon q(x)$. Since $\int_0^a [p(x) + \varepsilon q(x)]dx = M$ and $\int_0^a p(x)dx = M$, it necessarily follows that $\int_0^a \delta p(x)dx = 0$.

2. Monotone density functions. We first consider the case where $p(x)$ is a monotone increasing function such that $0 \leqq p(x) \leqq H < \infty$, that is when $p(x)\varepsilon E_1(M, H, a)$.

THEOREM 1. *Let $\lambda_n(p)$ be the nth eigenvalue of a vibrating string with fixed boundary values and with a monotone increasing density function $p(x)\varepsilon E_1(M, H, a)$. Then*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x)\varepsilon E_1(M, H, a)$ is a step function with at least one and at most n discontinuities in the open interval $(0, a)$.

By Lemma 2 there exists a monotone bounded function $\rho(x) \in E_1(M, H, a)$ such that $\lambda_n(\rho) = \max_{p\in E_1} \lambda_n(p)$. Hence, letting $p(x) = \rho(x)$ in the variational formula (2), we have $\delta\lambda_n(\rho) \leqq 0$. We now show that unless $p(x) \in E_1(M, H, \text{a})$ is a step function with at most n discontinuities $\delta\lambda_n(p) > 0$ for some $\delta p = \varepsilon q$ where $p(x) + \delta p(x) \in E_1(M, H, a)$. Hence, $\rho(x)$ must be a step function with at most n discontinuities.[1]

Let $u_n(x)$ be the eigenfunction corresponding to $\lambda_n(p)$. Denote the nodal points of $u_n(x)$ by x_k $(k = 0, 1, \cdots, n)$ where $x_0 = 0$ and $x_n = a$. Since $u_n(x)$ has only one extremum point in each of the intervals (x_{k-1}, x_k) $(k = 1, 2, \cdots, n) u_n^2(x)$ has only one maximum there. Let that point in (x_k, x_{k+1}) be $\bar{x}_k$ $(k = 1, 2, \cdots, n)$. For $k = 1, 2, \cdots, n$, we let

$$r(x) = a_k = \int_{x_{k-1}}^{\bar{x}_k} p(x)dx/(\bar{x}_k - x_{k-1}),\ x \in [x_{k-1}, \bar{x}_k] .$$

Since a_k is the mean value of $p(x)$ in $(x_{k-1}, \bar{x}_k)$ and $p(x)$ is monotone increasing, it follows that $a_{k+1} \geqq p(x)$ if $x \in [\bar{x}_k, x_k]$ $(k = 1, 2, \cdots, n-1)$ and that $a_k \leqq p(x)$ if $x \in [\bar{x}_k, x_k]$ $(k = 1, 2, \cdots, n)$. Hence, it is possible

[1] The author is indebted Z. Nehari for suggesting the variational approach used in this paper.

to find a point $\xi_k \in (\bar{x}_k, x_k)$ such that

$$r(x) = \begin{cases} a_k & \text{if } x \in [\bar{x}_k, \xi_k) \\ a_{k+1} & \text{if } x \in [\xi_k, x_k] \end{cases}, \qquad (k = 1, 2, \cdots, n)$$

satisfies the relation

$$\int_{\bar{x}_k}^{x_k} r(x)dx = \int_{\bar{x}_k}^{x_k} p(x)dx$$

$(k = 1, 2, \cdots, n)$. We have taken $a_{n+1} = H$, the upper bound of $p(x)$. In each of the intervals $(x_{k-1}, \bar{x}_k)$ and $(\bar{x}_k, x_k)$ $(k = 1, 2, \cdots, n)$, $r(x)$ and $p(x)$ satisfy the hypothesis of Lemma 1.1 relative to $u_n^2(x)$. Hence, we have

$$\int_{x_{k-1}}^{\bar{x}_k} p(x)u_n^2(x)dx \geqq \int_{x_{k-1}}^{\bar{x}_k} r(x)u_n^2(x)dx$$

and

$$\int_{\bar{x}_k}^{x_k} p(x)u_n^2(x)dx \geqq \int_{\bar{x}_k}^{x_k} r(x)u_n^2(x)dx$$

$(k = 1, 2, \cdots, n)$. Summing on k, we find that

$$\int_0^a [p(x) - r(x)]u_n^2(x)dx \geqq 0 .$$

The equality sign will hold if and only if $p(x) = r(x)$, i.e., $p(x)$ is constant or is a step function with precisely one jump in each of the intervals (x_{k-1}, x_k) $(k = 1, 2, \cdots, n)$. If we let $q(x) = r(x) - p(x)$, then for small $\varepsilon > 0$ Lemma 3 gives the result

$$\begin{aligned} \delta\lambda_n(p) &= -\lambda_n(p) \int_0^a \in q(x)u_n^2(x)dx \\ &= -\lambda_n(p)\int_0^a \delta p(x)u_n^2(x)dx > 0 \end{aligned}$$

unless $p(x) = r(x)$. Hence, $\rho(x) = r(x)$ if $\lambda_n(\rho)$ is a maximum. But $r(x)$ is a step function with at most n jumps in $(0, a)$.

Finally, we show that the maximizing density cannot be a constant so that there must be at least one jump. We first consider the lowest eigenvalue. We show that $\delta\lambda_1(p) > 0$ when $p(x) = M/a$ for a particular $\delta p = \varepsilon q$.

The eigenfunction corresponding to $\lambda_1(M/a)$ is

$$u_1(x) = \sqrt{2/a} \sin \frac{\pi x}{a} .$$

If we let

$$\delta p(x) = \varepsilon q(x) = \begin{cases} -\varepsilon & \text{if } x \in (0, a/2+\eta)\ , \\ \varepsilon \dfrac{a/2+\eta}{a/2-\eta} & \text{if } x \in (a/2+\eta, a)\ , \end{cases}$$

where $0 < \eta < a/2$ then $\int_0^a \delta p(x)dx = 0$ and

$$\delta\lambda_1(M/a) = -\lambda_1(M/a)\int_0^a \delta p(x)u_1^2(x)dx\ .$$

From the symmetry of $u_1(x)$ about the point $x = a/2$ and Lemma 1 it is easily seen that

$$\int_0^a \delta p(x)u_1^2(x)dx > 0\ .$$

Hence, $\delta\lambda_1(M/a) > 0$ so that $\lambda_1(M/a)$ cannot be a maximum value of $\lambda_1(p)$.

The corresponding result for the higher eigenvalues can be obtained by choosing

$$\delta p(x) = \varepsilon q(x) = \begin{cases} -\varepsilon & \text{if } x \in \left(0, \dfrac{a}{2n}+\eta\right)\ , \\ \dfrac{\varepsilon(a/2n+\eta)}{\dfrac{(2n-1)a}{2n}-\eta} & \text{if } x \in (a/2n+\eta, a)\ , \end{cases}$$

where $0 < \eta < a/2n$. It then follows from the periodicity of

$$u_n(x) = \sqrt{2/a}\sin\frac{n\pi x}{a}$$

and the argument used for $\lambda_1(M/a)$ that $\lambda_n(M/a)$ cannot be a maximum value of $\lambda_n(p)$, $p \in E_1(M, H, a)$.

The upper bound of $\lambda_1(p)$, $p \in E_1(M, H, a)$ is thus given as the maximum of the lowest eigenvalue of the system.

$$u'' + \lambda p_\theta(x)u = 0,\ u(0) = u(a) = 0 \tag{3}$$

where

$$p_\theta(x) = \begin{cases} \theta H \text{ if } x \in [0, \xi a)\ , \\ H \quad \text{if } x \in [\xi a, a]\ ; \end{cases}$$

$0 < \theta < 1$ and $\xi = \dfrac{1 - M/Ha}{1-\theta}$. That $\theta = 0$ may be excluded from consideration follows easily from the derivation of the form of $\rho(x)$ and the fact that the maximum of $u_1(x)$ in this case must occur in the open interval $(\xi a, a)$. For we would have $a_1 = \int_0^{\bar{x}_1} p_\theta(x)dx \neq 0$.

The eigenfunctions of (3), are [2]

$$u_n(x) = \begin{cases} \sin \sqrt{\lambda_n H}\,(1-\xi)a \cdot \sin \sqrt{\lambda_n \theta H}\, x\ , & x \in [0, \xi a)\ , \\ \sin \sqrt{\lambda_n \theta H}\, \xi a \cdot \sin \sqrt{\lambda_n H}\,(a - x)\ , & x \in [\xi a, a] \end{cases}$$

where $\lambda_n(p_\theta)$ is the nth positive root of

$$\tan(\xi a\sqrt{\lambda\theta H}) + \sqrt{\theta}\tan a(1-\xi)\sqrt{\lambda H} = 0\ .$$

We could now compute $\dfrac{d\lambda_1(p_\theta)}{d\theta}$ and determine the value which maximizes $\lambda_1(p_\theta)$.

The determination of the bounds for the higher eigenvalues is also seen to be a problem in ordinary calculus since the jumps of the step function which give the maximum must occur in the open interval $(0, a)$.

3. Convex density functions. Let $p(x)$, $x \in [0, a]$ be a continuous convex function such that $\int_0^a p(x)dx = M$ and $0 \leqq p(x) \leqq H$, that is, let $p(x) \in E_2(M, H, a)$.

THEOREM 2. *Let $\lambda_1(p)$ be the lowest eigenvalue of a string with fixed end points and with density $p(x) \in E_2(M, H, a)$. Then*

$$aM\lambda_1(p) \leqq \mu\left(\frac{aH}{M}\right)$$

where $\mu(h) = [6(h-1)t_1]^2/h^3$ and t_1 is the least positive root of

$$J_{1/3}(t)J_{2/3}\left(\frac{(2-h)^3 t}{h^3}\right) - J_{-1/3}(t)J_{-2/3}\left(\frac{(2-h)^3 t}{h^3}\right) = 0$$

if $1 < h < 2$ and $\mu(h) = h(3t_1/2)^2$ and t_1 is the least positive root of $J_{-2/3}(t) = 0$ if $h \geqq 2$. The minimum is uniquely attained for the function

$$\rho(x) = \begin{cases} \dfrac{4}{a^2}(M - aH)x + H\ , & x \in (0, a/2)\ , \\ \rho(a - x) & , \quad x \in (a/2, a)\ , \end{cases} \tag{5}$$

if $1 < h = \dfrac{aH}{M} < 2$ and

$$\rho(x) = \begin{cases} [H/M](M - Hx)\ , & x \in (0, M/H)\ , \\ 0 & , \quad x \in (M/H, a/2)\ , \\ \rho(a - x) & , \quad x \in (a/2, a)\ , \end{cases} \tag{6}$$

if $h = \dfrac{aH}{M} \geqq 2$.

It is well known that $\lambda_1(p)$ is the minimum of

$$J(u) = \frac{\int_0^a u'^2(x)dx}{\int_0^a p(x)u^2(x)dx}$$

where the minimum is taken over all functions $u \in C'$ which vanish at $x = 0$ and $x = a$. If we let

$$\bar{p}(x) = \frac{1}{2}[p(x) + p(a - x)]$$

then

$$\begin{aligned}\lambda_1^{-1}(\bar{p}) &= \max_{u \in C'} \frac{\int_0^a \bar{p}(x)u^2(x)dx}{\int_0^a u'^2(x)dx} \\ &\leqq \max_{u \in C'} \frac{\int_0^a p(x)u^2(x)dx}{2\int_0^a u'^2(x)dx} + \max_{u \in C'} \frac{\int_0^a p(a - x)u^2(x)dx}{2\int_0^a u'^2(x)dx} \\ &= \lambda_1^{-1}(p)\end{aligned}$$

since the eigenvalues of a string with density $p(a - x)$ are the same as those of a string with density $p(x)$. Hence any upper bound of $\lambda_1(\bar{p})$ is also an upper bound of $\lambda_1(p)$.

The differential system (1) with $p(x)$ replaced by $\bar{p}(x)$ has the same lowest eigenvalue as the system

$$u'' + \lambda\bar{p}(x)u = 0,\ u(0) = u'(a/2) = 0,\ x \in [0, a/2]. \tag{7}$$

Furthermore, since $p(x)$ is convex, so is $\bar{p}(x)$, $x \in [0, a]$, and the bound H is also a bound of $\bar{p}(x)$.

We now compare the lowest eigenvalue of the system (7) with that of the same system when $\bar{p}(x)$ is replaced by

$$\rho_1(x) = [4/a^2](M - aH)x + H,\ x \in [0, a/2]\ ,$$

if $1 < \dfrac{aH}{M} < 2$ and

$$\rho_1(x) = \begin{cases} \dfrac{H}{M}(M - Hx)\ , & x \in [0, M/H]\ , \\ 0\ , & x \in [M/H, a/2]\ , \end{cases}$$

if $\frac{aH}{M} \geqq 2$. In either case, since $\rho_1(0) = H \geqq \bar{p}(0)$ and $\int_0^{a/2} \rho_1(x)dx = \int_0^{a/2} \bar{p}(x)dx$, it follows from the convexity of $\bar{p}(x)$ that there is a point $\xi \in (0, a/2)$ such that $\rho_1(x) \geqq \bar{p}(x)$ if $x \in (0, \xi)$ and $\rho_1(x) \leqq \bar{p}(x)$ if $x \in (\xi, a/2)$. There will be strict inequality in each of these open intervals unless $\bar{p}(x) = \rho_1(x)$, $x \in [0, a/2]$. If $u(x)$ is monotone increasing in $[0, a/2]$ with $u(0) = u'(a/2) = 0$, we have by Lemma 1

$$\int_0^{a/2} \rho_1(x)u^2(x)dx \leqq \int_0^{a/2} \bar{p}(x)u^2(x)dx \ . \tag{8}$$

Since the first eigenfunction of the system (7) is a monotone increasing function, it follows from the comparison theorem [2] that

$$\lambda_1(\bar{p}) \leqq \lambda_1(\rho_1) \ .$$

There will be equality if and only if $\bar{p}(x) = \rho_1(x)$, for if $u(x)$ is the eigenfunction corresponding to the lowest eigenvalue of (7) with $\bar{p}(x)$ replaced by $\rho_1(x) \neq \bar{p}(x)$ then (8) will be a strict inequality and hence

$$\lambda_1(\rho_1) = \frac{\int_0^{a/2} u'^2(x)dx}{\int_0^{a/2} \rho_1(x)u^2(x)dx} > \frac{\int_0^{a/2} u'^2(x)dx}{\int_0^{a/2} \bar{p}(x)u^2(x)dx} \geqq \lambda_1(p) \ .$$

But $\lambda_1(\rho_1)$ is also the lowest eigenvalue of the system (1) with $p(x)$ replaced by

$$\rho(x) = \begin{cases} \rho_1(x) \ , & x \in [0, a/2] \ , \\ \rho_1(a - x) \ , & x \in [a/2, a] \ . \end{cases}$$

This is just the function (5) if $1 < \frac{aH}{M} < 2$ and the function (6) if $\frac{aH}{M} \geqq 2$. Hence we see that $\lambda_1(\rho) \geqq \lambda_1(p)$ for any bounded convex $p(x)$.

When $\rho(x)$ is defined by (5) we find that

$$\lambda_1(\rho) = \frac{\mu\left(\frac{aH}{M}\right)}{a\int_0^a p(x)dx}$$

where $\mu(h) = [6(h-1)t_1]^2/h^3$ and t_1 is the least positive root of

$$J_{1/3}(t)J_{2/3}(kt) - J_{-1/3}(t)J_{-2/3}(kt) = 0 \ ,$$

$k = \frac{(2-h)^3}{h^3}$ [4]. When $\rho(x)$ is defined by (6) we have

$$\lambda_1(\rho) = \frac{\mu\left(\frac{aH}{M}\right)}{a\int_0^a p(x)dx}$$

where $\mu(h) = h(3t_1/2)^2$ and t_1 is the least positive root of $J_{-2/3}(t) = 0$ [4].

A better bound is obtained if, instead of the bound H, we use $\bar{H} = \frac{1}{2}[p(0) + p(a)]$ for the bound of $\bar{p}(x)$. This results in a smaller value of $\mu(aH/M)$ whenever $p(0) \neq p(a)$.

For the larger eigenvalues we prove the following.

THEOREM 3. *Let $\lambda_n(p)$ be the nth eigenvalue of a vibrating string with fixed boundary values and with a convex density $p(x) \in E_2(M, H, a)$. Then*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x) \in E_2(M, H, a)$ is a piecewise linear convex function with at most $(n + 2)$ pieces.

The existence of a bounded convex function $\rho(x)$ such that $\max_{p \in E_2} \lambda_n(p) = \lambda_n(\rho)$ follows from Lemma 2. It then follows by Lemma 3 that

$$\delta\lambda_n(\rho) = -\lambda_n(\rho)\int_0^a \delta\rho(x)u_n^2(x)dx \leqq 0 .$$

We now show that either $p(x)$ is a convex piecewise linear function with at most $(n + 2)$ pieces or there exists a function $q(x)$ such that $\delta\lambda_n(p) > 0$ when $\delta p = \varepsilon q$ where $p(x) + \delta p(x) \in E_2(M, H, a)$. Let $u_n(x)$ be the eigenfunction corresponding to $\lambda_n(p)$. We first find a convex function $r(x)$ such that

$$\int_0^a r(x)u_n^2(x)dx \leqq \int_0^a p(x)u_n^2(x)dx .$$

Instead of trying to find $r(x)$ directly, we carry out a preliminary construction. As in Theorem 1, we denote the minimum points of $u_n^2(x)$ by x_k $(k = 0, 1, \cdots, n)$ and the maximum points by $\bar{x}_k$ $(k = 1, 2, \cdots, n)$. We first consider each of the intervals $(\bar{x}_k, \bar{x}_{k+1})$ $(k = 1, 2, \cdots, n - 1)$ separately.

Let $L(x)$ be any linear function such that $L(x) \leqq p(x)$, $x \in (x_k, \bar{x}_{k+1})$ for some fixed integer k $(1 \leqq k \leqq n - 1)$. Then $m(x) = \max\ \{L(x), 0\}$ satisfies the inequality $0 \leqq m(x) \leqq p(x)$. Now let c_k be any number such that $c_k \geqq p(x_k)$. Then there is a number a_k such that

$$\int_{x_k}^{\bar{x}_{k+1}} [a_k(x - x_k) + c_k]dx = \int_{x_k}^{\bar{x}_{k+1}} p(x)dx . \tag{9}$$

If $a_k(x - x_k) + c_k \geqq m(x)$, $x \in (x_k, \bar{x}_{k+1})$, then we let

$$g_k(x, c_k) = a_k(x - x_k) + c_k \ , \quad x \in (x_k, \bar{x}_{k+1}) \ .$$

If $a_k(x - x_k) + c_k < m(x)$ for some $x \in (x_k, \bar{x}_{k+1})$, then we redefine a_k by the condition

$$\int_{x_k}^{\xi_k} [a_k(x - x_k) + c_k]dx + \int_{\xi_k}^{\bar{x}_{k+1}} m(x)dx = \int_{x_k}^{\bar{x}_{k+1}} p(x)dx \tag{10}$$

where ξ_k satisfies the equation $a_k(\xi_k - x_k) + c_k = m(\xi_k)$. In this case, we define $g_k(x, c_k)$ by

$$g_k(x, c_k) = \begin{cases} a_k(x - x_k) + c_k \ , & x \in (x_k, \xi_k) \ , \\ m(x) \ , & x \in [\xi_k, \bar{x}_{k+1}) \ . \end{cases}$$

Now consider the interval $(\bar{x}_k, x_k)$. Let $m(x) = \max\{L(x), 0\}$ where $L(x)$ is any linear function such that $L(x) \leqq p(x)$ if $x \in (\bar{x}_k, x_k)$. There is a number b_k such that

$$\int_{\bar{x}_k}^{x_k} [b_k(x - x_k) + c_k]dx = \int_{\bar{x}_k}^{x_k} p(x)dx \ . \tag{11}$$

If $b_k(x - x_k) + c_k \geqq m(x)$ for $x \in (\bar{x}_k, x_k)$, we let

$$h_k(x, c_k) = b_k(x - x_k) + c_k \ , \quad x \in (\bar{x}_k, x_k) \ .$$

If $b_k(x - x_k) + c_k < m(x)$ for some $x \in (\bar{x}_k, x_k)$, we redefine b_k by the condition

$$\int_{\bar{x}_k}^{\eta_k} m(x)dx + \int_{\eta_k}^{x_k} [b_k(x - x_k) + c_k]dx = \int_{\bar{x}_k}^{x_k} p(x)dx \tag{12}$$

where η_k satisfies the equation $b_k(\eta_k - x_k) + c_k = m(\eta_k)$. We then define $h_k(x, c_k)$ by

$$h_k(x, c_k) = \begin{cases} m(x) \ , & x \in (\bar{x}_k, \eta_k) \ , \\ b_k(x - x_k) + c_k \ , & x \in (\eta_k, x_k) \ . \end{cases}$$

We may consider a_k and b_k to be functions of c_k. They are continuous functions as is easily seen from the defining relations of a_k and b_k. It follows that there is a number $\gamma_k \geqq p(x_k)$ such that $a_k = b_k$ if $c_k = \gamma_k$. For if $c_k = p(x_k)$, the convexity of $p(x)$ implies that $a_k - b_k \geqq 0$. On the other hand, if c_k is sufficiently large, $a_k - b_k < 0$. Hence, by the continuity, the value γ_k exists such that $c_k = \gamma_k$ implies $a_k = b_k$.

In the interval $[x_0, \bar{x}_1]$, we define $g_0(x, c_0)$, in the same way that $g_k(x, c_k)$ was defined except that we specify $c_0 = p(0) = \gamma_0$. Similarly in $[\bar{x}_n, a]$ we define $h_n(x, c_n)$ as above except that we take $c_n = p(a) = \gamma_n$.

We now let

$$r_1(x) = g_0(x, \gamma_0) \ , \quad x \in [0, \bar{x}_1] \ ,$$

$$r_k(x) = \begin{cases} h_k(x, \gamma_k) , & x \in [\bar{x}_k, x_k] , \\ g_k(x, \gamma_k) , & x \in [x_k, \bar{x}_{k+1}] , \end{cases}$$
$$r_n(x) = h_n(x, \gamma_n) , \quad x \in [\bar{x}_n, x_n] ,$$

$(k = 1, 2, \cdots, n-1)$. From (9) or (10), which ever applies, we have

$$\int_{x_{k-1}}^{\bar{x}_k} r_k(x)dx = \int_{x_{k-1}}^{\bar{x}_k} p(x)dx .$$

The convexity of $p(x)$ and the definition of $r_k(x)$ imply by Lemma 1 that

$$\int_{x_{k-1}}^{\bar{x}_k} r_k(x)u_n^2(x)dx \leqq \int_{x_{k-1}}^{\bar{x}_k} p(x)u_n^2(x)dx . \tag{13}$$

Similarly from (11) or (12) we have

$$\int_{\bar{x}_k}^{x_k} r_k(x)u_n^2(x)dx \leqq \int_{\bar{x}_k}^{x_k} p(x)u_n^2(x)dx . \tag{14}$$

Furthermore, we have strict inequality unless $r_k(x) = p(x)$ in each case.

We are now able to define the function $r(x)$ by induction. We carry out the process only for $n = 3$ to avoid unnecessary detail. In $(x_0, \bar{x}_1)$, we let $m(x) = 0$ and define $r_1(x)$ as above. In $(\bar{x}_1, \bar{x}_2)$, we also define $r_2(x)$ with $m(x) = 0$. Then, comparing $r_1(\bar{x}_1)$ and $r_2(\bar{x}_1)$ we have the following alternatives:

(i) If $r_1(\bar{x}_1) > r_2(\bar{x}_1)$, we define a new function $r_2(x)$ with $m(x) = \max\{r_1(x), 0\}$, $x \in [\bar{x}_1, \bar{x}_2]$ where we define $r_1(x)$ in this interval by extrapolation.

(ii) If $r_1(\bar{x}_1) < r_2(\bar{x}_1)$, we define a new function $r_1(x)$ with $m(x) = \max\{r_2(x), 0\}$, $x \in [x_0, \bar{x}_1]$, where $r_2(x)$ is defined in this interval by extrapolation.

(iii) If $r_1(\bar{x}_1) = r_2(\bar{x}_1)$ we leave $r_1(x)$ and $r_2(x)$ as they are.

Using whichever alternative applies, we define

$$r^{(1)}(x) = \begin{cases} r_1(x) , & x \in [x_0, \bar{x}_1] , \\ r_2(x) , & x \in [\bar{x}_1, \bar{x}_2] . \end{cases}$$

Now, define $r_3(x)$, $x \in [\bar{x}_2, \bar{x}_3]$ with $m(x) = 0$ and compare $r^{(1)}(\bar{x}_2)$ and $r_3(\bar{x}_2)$. We use the same alternatives as above, the only difference being that if $r^{(1)}(\bar{x}_2) < r_3(\bar{x}_2)$ we must redefine $r^{(1)}(x)$ with $m(x) = \max\{r_3(x), 0\}$, $x \in [x_0, \bar{x}_2]$ where as above we define $r_3(x)$ by extrapolation.

It is clear that the above process can be completed for any integer n. The function which we obtain by this method we call $r(x)$. It will be a convex function since any two adjacent segments of the graph of $r(x)$ can only have a point of intersection which lies on or below the graph of $p(x)$. Since there is possibly a subinterval of $[0, a]$ where $r(x)$ may be zero, $r(x)$ may have up to $n + 2$ linear pieces.

If we sum the inequalities (13) and (14) we find that

$$\int_0^a r(x)u^2(x)dx \leqq \int_0^a p(x)u_n^2(x)dx$$

with strict inequality unless $r(x) = p(x)$ in $(0, a)$. Choosing $\delta p = \varepsilon q(x) = \varepsilon[r(x) - p(x)]$, we have that $p(x) + \delta p(x)$ is convex if $\varepsilon > 0$ is small and hence

$$\int_0^a \delta p u_n^2(x)dx < 0$$

or $\delta\lambda_n(p) > 0$ unless $p(x) \equiv r(x)$, $x \in [0, a]$. Since we must have $\delta\lambda_n(\rho) \leqq 0$, it follows that $\rho(x)$ is the same type of function as $r(x)$. From the method of determining $r(x)$, we see the $\rho(x)$ is a convex piecewise linear function with at most $n + 2$ linear segments.

We note from Theorem 2 that this is precisely the case when $n = 1$.

4. Concave density functions. We consider the case when $p(x)$, $x \in [0, a]$ is a continuous concave function such that $\int_0^a p(x)dx = M$, that is, when $p(x) \in E_3(M, a)$.

THEOREM 4. *Let $\lambda_n(p)$ be the* nth *eigenvalue of a string with fixed end points and with a concave density function $p(x) \in E_3(M, a)$. Then*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x) \in E_3(M, a)$ and is a piecewise linear concave function with at most n pieces.

The existence of a concave function $\rho(x)$ such that

$$\max_{p \in E_3} \lambda_n(p) = \lambda_n(\rho)$$

follows from Lemma 2. As in the previous cases, we must have $\delta\lambda_n(\rho) \leqq 0$. We show that it is always possible to find a function $q(x)$ such that

$$\delta\lambda_n(p) = -\lambda_n(p)\int_0^a \delta p(x)u_n^2(x)dx > 0$$

when $p(x) = \varepsilon q(x)$ where $p(x) + \delta p(x) \in E_3(M, a)$, unless $p(x) \in E_3(M, a)$ is a piecewise linear concave function with at most n pieces. Hence, it follows that $\rho(x)$ must be such a function.

We find the function $q(x)$ by the method used in the proof of Theorem 3. Thus, we seek a function $r(x)$ such that

$$\int_0^a r(x)u_n^2(x)dx \leqq \int_0^a p(x)u_n^2(x)dx .$$

Where $u_n(x)$ is the eigenfunction corresponding to $\lambda_n(p)$. To apply the method of Theorem 3, we consider

$$\int_0^a p(x)u_n^2(x)dx = \int_0^a [-p(x)][-u_n^2(x)]dx .$$

Then $-p(x)$ is convex and the zeros x_k $(k = 0, 1, 2, \cdots, n)$ of $u_n(x)$ are the maximum points of $-u_n^2(x)$. The maximum points $\bar{x}_k$ $(k = 1, 2, \cdots, n)$ of $u_n^2(x)$ are the minimum points of $-u_n^2(x)$.

Over each of the intervals (x_k, x_{k+1}) $(k = 0, 1, \cdots, n-1)$ we define $-r_k(x, c_k)$ where $-p(\bar{x}_k) \leqq c_k \leqq 0$. As in the convex case, there is a number γ_k such that $r_k(x, \gamma_k)$ is linear at $x = \bar{x}_k$. Using the inductive argument as before, we let $m(x) \equiv L(x)$ since $L(x)$ will be negative and form new functions $-r_k(x, \gamma_k)$. Finally we obtain $-r(x)$ which is convex and satisfies the inequality

$$\int_0^a p(x)u_n^2(x)dx \geqq \int_0^a r(x)u_n^2(x)dx .$$

Hence, choosing $q(x) = r(x) - p(x)$, we have

$$\int_0^a \delta p u_n^2(x)dx = \int_0^a \epsilon\, q(x)u_n^2(x)dx \leqq 0 ,$$

where for ε sufficiently small $p(x) + \delta p(x) \in E_3(M, a)$. Furthermore, there is strict inequality unless $p(x)$ is a concave piecewise linear function with at most n pieces. This proves the theorem.

It follows immediately from Theorem 4 that

$$\lambda_1(p) \leqq \frac{\pi^2}{aM}$$

when $p(x)$ is concave.[1] For in this case, $\rho(x)$ is a linear function. But, as was shown in the proof of Theorem 3, $\lambda_1(\bar{\rho}) \geqq \lambda_1(\rho)$ where $\bar{\rho}(x) = \frac{1}{2}[\rho(x) + \rho(a-x)]$. In this case, $\bar{\rho}(x) = M/a$ and $\lambda_1(M/a) = \pi^2/aM$.

5. The vibrating rod. The eigenvalue problem associated with a vibrating rod with clamped ends and density $p(x) \geqq 0$, $x \in [0, a]$ is

$$u^{iv} - \lambda p(x)u = 0 , \quad u(0) = u'(0) = u(a) = u'(a) = 0 . \tag{15}$$

As in the case of the string, we denote the ordered eigenvalues by

$$0 < \lambda_1(p) < \lambda_2(p) < \cdots .$$

That there should be strict inequalities in this expression has been

[1] This result has already been obtained by Z. Nehari. His proof is the one dimensional analog of that given in [7] where he shows that the lowest eigenvalue of a circular membrane with a superharmonic density $p(x, y)$ is bounded above by that of a homogeneous membrane of the same total mass.

shown in [6].

In this section, we consider the problem of finding upper bounds for these eigenvalues when $p(x)$ is restricted to be either monotone, convex or concave. In the first two cases, we require in addition that $p(x) \leqq H < \infty$. As in the case of the string, we denote the set of all functions $p(x) \geqq 0,\ x \in [0, a]$ with $\int_0^a p(x)dx = M$ where $p(x)$ is monotone increasing, convex and concave by $E_1(M, H, a)$, $E_2(M, H, a)$ and $E_3(M, a)$ respectively. The H in $E_1(M, H, a)$ and $E_2(M, H, a)$ indicates that in these cases $p(x) \leqq H$.

LEMMA 4. *Let E_k be one of the sets of functions defined above. There exists a function $\rho(x) \in E_k$ such that*

$$\lambda_n(\rho) = \sup_{p \in E_k} \lambda_n(p) .$$

This follows in exactly the same manner as the result of Lemma 2. We need only note that the result of Krein quoted in Lemma 2 may be generalized to this case. The generalization is trivial for the Green's function of the system (15) and its first partial derivatives are bounded. Krein's proof then applies word for word to this case and hence the proof of Lemma 4 follows as in the proof of Lemma 2.

LEMMA 5. *The first variation of $\lambda_n(p)$ with $\int_0^a p(x)dx = M$ is*

$$\delta\lambda_n(p) = -\lambda_n(p)\int_0^a \delta p(x) u_n^2(x)dx$$

where $u_n(x)$ is the normalized eigenfunction corresponding to $\lambda_n(p)$.

In particular we may choose $\delta p(x) = \varepsilon q(x)$ such that $\int_0^a \delta p(x) = 0$. The result is easily derived in the same way as the result of Lemma 3.

The results of Theorems 1, 3 and 4 will now generalize to the case of a vibrating rod with clamped ends. The only question which arises concerns the properties of the eigenfunction $u_n(x)$ corresponding to $\lambda_n(p)$. It must be true that $u_n(x)$ has the same general character as the nth eigenfunction of a vibrating string. In particular, it has been shown in [6] that $u_n(x)$ has exactly $n-1$ zeros in the open interval $(0, a)$. Furthermore $u_n^2(x)$ has exactly one maximum between any consecutive pairs of zeros. For suppose there are two or more maximum points between some consecutive pair of zeros. Then $u_n'(x)$ must have at least $n+4$ zeros in $[0, a]$. Hence $u_n''(x)$, $u_n'''(x)$ and $u_n^{(iv)}(x)$ must have at least $n+3$, $n+2$, and $n+1$ zeros respectively in the open interval $(0, a)$. This leads to a contradiction if $p(x) > 0$ since $u_n^{(iv)} = \lambda_n p(x) u_n(x)$ may have only $n-1$ zeros in $(0, a)$. If $p(x) \geqq 0$, we may apply the same argument with $p(x)$ replaced by $p(x) + \varepsilon$, $\varepsilon > 0$. Thus, if $u_{n\varepsilon}(x)$ is the nth eigenfunction, $u_{n\varepsilon}^2(x)$ has n maximum points in $(0, a)$. Letting $\varepsilon \to 0$, we see that the same must be true of the nth eigenfunction when the rod density is $p(x) \geqq 0$.

From these observations, Lemmas 4 and 5, and the arguments used in Theorems 1, 3 and 4, we have the following result.

THEOREM 5. *Let $\lambda_n(p)$ be the* nth *eigenvalue of a rod with clamped ends and density $p(x)$, $x \in [0, a]$, such that $\int_0^a p(x)dx = M$.*

(a) *If $p(x)$ is monotone increasing and bounded*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x)$, $x \in [0, a]$ is an increasing step function with at least one and at most n discontinuities in the open interval $(0, a)$ and $\int_0^a \rho(x)dx = M$.

(b) *If $p(x)$ is convex and bounded*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x)$, $x \in [0, a]$ is a bounded piecewise linear convex function with at most $n + 2$ linear pieces and $\int_0^a \rho(x)dx = M$.

(c) *If $p(x)$ is concave*

$$\lambda_n(p) \leqq \lambda_n(\rho)$$

where $\rho(x)$, $x \in [0, a]$ is a piecewise linear concave function with at most n linear pieces and $\int_0^a \rho(x)dx = M$.

In the case of the lowest eigenvalue, the density which gives the upper bound may be obtained precisely when $p(x)$ is convex or concave. It follows from the Rayleigh quotient as in Theorem 2 that for $\bar{p}(x) = \frac{1}{2}[p(x) + p(a - x)]$

$$\lambda_1(\bar{p}) \geqq \lambda_1(p) .$$

This and the above theorem thus show that when $p(x)$ is convex, $\rho(x)$ is symmetric and piece wise linear with at most three linear pieces and that when $p(x)$ is concave, $\rho(x)$ is a constant. This result may also be obtained by the method used in the proof of Theorem 2.

6. The membrane. We consider a vibrating membrane stretched with uniform unit tension over a disk $D = \{(x, y) \mid x^2 + y^2 < R^2\}$. We assume the areal density of the membrane is given by the measurable function $p(x, y)$ where

$$\iint_D p(x, y)dxdy = M .$$

For such a membrane with a fixed boundary, the eigenvalues and eigenfunctions are determined by the integral equation [8]

$$u(x, y) = \iint_D G(x, y, \xi, \eta)p(\xi, \eta)u(\xi, \eta)d\xi d\eta \tag{16}$$

where $G(x, y, \xi, \eta)$ is the Green's function of D. We denote the first eigenvalue by $\lambda_1(p)$ and the corresponding eigenfunction by $u_1(x, y)$.

We find upper bounds for $\lambda_1(p)$ by use of the following result.

LEMMA 6. *The lowest eigenvalue of a circular membrane with fixed boundary and integrable density $p(x, y)$ is always less than that of a circular membrane with fixed boundary and density.*

$$\bar{p}(x, y) = p(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r\cos\theta, r\sin\theta)d\theta .$$

Proof. We use the fact the first eigenvalue is given by the infimum of the Rayleigh quotient

$$R(u) = \frac{\iint_D (u_x^2 + u_y^2)dxdy}{\iint_D p(x, y)u^2(x, y)dxdy}$$

where the infimum is taken over all functions $u(x, y) \in C'$ such that $u(x, y)$ vanishes on the boundary D. In particular, the lowest eigenvalue of a circular membrane with density $p(r)$ is given by

$$\lambda_1(p(r)) = \inf_{u \in C'} \frac{\iint_D (u_x^2 + u_y^2)dxdy}{\iint_D p(r)u^2(x, y)dxdy} .$$

We note that

$$p(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r, \phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} p(r, \phi + \theta)d\phi = \bar{p}(r, \theta) .$$

Hence, it follows that

$$\begin{aligned}
\frac{1}{\lambda_1(p(r))} &= \sup_{u\in C'} \frac{\iint_D \bar{p}(r, \phi)u^2(r, \phi)rd\phi\, dr}{\iint_D (u_x^2 + u_y^2)dxdy} \\
&= \sup_{u\in C'} \frac{\iint_D \left(\frac{1}{2\pi}\int_0^{2\pi} p(r, \phi + \theta)d\theta\right)u^2(r, \phi)rd\phi dr}{\iint_D (u_x^2 + u_y^2)dxdy} \\
&\leqq \frac{1}{2\pi}\int_0^{2\pi} d\theta \cdot \sup_{u\in C'} \frac{\iint_D p(r, \phi + \theta)u^2(r, \phi)rd\phi dr}{\iint_D (u_x^2 + u_y^2)dxdy} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\lambda_1(p)} d\theta = \frac{1}{\lambda_1(p)}
\end{aligned}$$

i.e., $\lambda_1(p) \leqq \lambda_1(\bar{p})$ since $\lambda_1(p)$ does not depend on θ. We may now prove the following.

THEOREM 6. *The lowest eigenvalue of a circular membrane with fixed boundary and a bounded convex density $p(x, y)$ is less than the lowest eigenvalue of a circular membrane with density*

$$q(r) = \begin{cases} 0\,, & 0 < r \leqq R - H/\alpha\,, \\ \alpha(r - R) + H\,, & R - H/\alpha < r \leqq R\,, \end{cases}$$

if $R > H/\alpha$ and

$$q(r) = \alpha(r - R) + H$$

if $R < H/\alpha$ where α is such that $2\pi\int_0^R q(r)rdr = M$.

We first note that since $p(x, y)$ is convex, so is

$$p(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r\cos\phi, r\sin\phi)d\phi\ .$$

For suppose r_1 and r_2 are such that $-R \leqq r_1 < r_2 \leqq R$. By the convexity of $p(x, y)$ we have

$$\begin{aligned}
p\left(\frac{r_1 + r_2}{2}\cos\phi, \frac{r_1 + r_2}{2}\sin\phi\right) \leqq \tfrac{1}{2}[&p(r_1\cos\phi, r_1\sin\phi) \\
&+ p(r_2\cos\phi, r_2\sin\phi)]\ .
\end{aligned}$$

Integrating this with respect to ϕ, we have

$$p\left(\frac{r_1 + r_2}{2}\right) \leqq \tfrac{1}{2}[p(r_1) + p(r_2)]\ .$$

We now consider

$$\frac{1}{\lambda_1(p(r))} = \frac{\iint_D \bar{p}(x, y)u^2(x, y)dxdy}{\iint_D (u_x^2 + u_y^2)dxdy}$$

where $u(x, y)$ is the eigenfunction corresponding to $\lambda_1(p(r))$. For any function $u_1(x, y) \in C'$, we then have

$$\frac{1}{\lambda_1(p(r))} \geqq \frac{\iint_D \bar{p}(x, y)u_1^2(x, y)dxdy}{\iint_D (u_{1x}^2 + u_{1y}^2)dxdy}$$

In particular, if $u_1(x, y) = u_1(r)$ is the eigenfunction corresponding to the first eigenvalue $\lambda_1(q)$ of a membrane with density $q(r)$, it is a decreasing function of r. This is easily seen by considering the differential equation which is equivalent to the integral equation (16) [3]. By Lemma 1, we thus have

$$2\pi\int_0^R \bar{p}(r)u_1^2(r)rdr \geqq 2\pi\int_0^R q(r)u_1^2(r)rdr \; . \tag{17}$$

Hence,

$$\frac{1}{\lambda_1(p(r))} \geqq \frac{\iint_D q(r)u_1^2(r)rdrd}{\iint_D \text{grad}\, u_1^2\, rdrd} = \frac{1}{\lambda_1(q)} \; .$$

This same method yields a result if $p(x, y)$ is a concave function. For $p(r)$ is also concave and the inequality (17) holds if we choose $q(r) = \iint_D p(x, y)dxdy = M$. Hence we find that

$$\lambda_1(p) \leqq \frac{\pi j_0^2}{R^2} M$$

where j_0 is the least positive zero of $J_0(x) = 0$. As pointed out in [1], this result is a corollary to a theorem of Nehari [7] which says that if $p(x, y)$ is superharmonic in D, then $\lambda_1(p) \leqq \pi\frac{j_0^2}{R^2}M$. Since a concave function is superharmonic, this implies the above result.

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