THE GIBBS PHENOMENON FOR HAUSDORFF MEANS

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The existence of a Gibbs phenomenon for the Hausdorff summability method given by $dg(x)[dg$ any measure on $[0, 1]$ with total mass 1 and no mass point at 0] is equivalent to the statement

$$\int_0^1\int_0^{Ax}\frac{\sin t}{t}dt\,dg(x) > \frac{\pi}{2} \text{ for some } A > 0. \qquad \text{(See [3])}$$

Recently A. Livingston [1] has treated the case of a dg composed of finitely many mass points and has shown that the Gibbs phenomenon holds under certain additional restrictions. Our result, which follows, contains his and does not require these additional restrictions.

THEOREM 1. *Let dg have at least 1 mass point and satisfy $\int_0^1(|\,dg(x)\,|/x^2) < \infty$ then the Gibbs phenomenon occurs for dg.*

[In particular if dg consists of finitely many mass points, then we have the Gibbs phenomenon].

It seems peculiar that any condition at 0 is necessary, and L. Lorch had even made the conjecture that the Gibbs phenomenon persists for any dg with unbounded Lebesgue constants [and so certainly for any dg with at least one mass point]. Nevertheless we show that some condition at 0 is necessary.

THEOREM 2. *There exists a positive dg composed solely of mass points for which the Gibbs phenomenon does not hold.*

Thus Lorch's conjecture is definitely false and although our Theorem 1 is by no means best possible it is qualitatively the correct one.

Proof of Theorem 1. We are required to prove that, for a dg satisfying the hypotheses, there is an $A > 0$ for which

$$F(A) = \int_0^1\int_{Ax}^{\infty}\frac{\sin t}{t}dt \quad dg(x) \text{ becomes negative.}$$

This we accomplish by showing that

1. $\int_1^y F(A)dA$ remains bounded as $y \to \infty$

2. $F(A) \notin L^1(1, \infty)$.

Received May 29, 1961.

LEMMA. $F(A) = 1/A \int_0^1 ((\cos Ax)/x)\, dg(x) + 0(1/A^2)$, [*assuming, of course, that* $\int_0^1 (|\, dg(x)\,|)/x^2 < \infty$].

Proof. By two integrations by parts we have

$$\int_{Ax}^{\infty} \frac{\sin t}{t} dt = \frac{\cos Ax}{Ax} + \frac{\sin Ax}{A^2x^2} - 2\int_{Ax}^{\infty} \frac{\sin t}{t^3} dt$$
$$= \frac{\cos Ax}{Ax} + \frac{1}{A^2x^2} + 0\left(\int_{Ax}^{\infty} \frac{1}{t^3} dt\right)$$
$$= \frac{\cos Ax}{Ax} + 0\frac{1}{A^2x^2}$$

hence

$$F(A) = \int_0^1 \frac{\cos Ax}{Ax} dg(x) + \frac{1}{A^2}\, 0 \int_0^1 \frac{|\, dg(x)\,|}{x^2}$$

and the latter integral is finite, by hypothesis.

Proof of 1.

$$\int_1^y F(A)dA = \int_1^y \int_0^1 \frac{\cos Ax}{Ax} dg(x) + 0(1)$$

by the lemma. Invert the order of integration (this is valid by hypothesis) and we obtain

$$\int_1^y F(A)dA = \int_0^1 \int_1^y \frac{\cos Ax}{A} dA \frac{dg(x)}{x} + 0(1)$$
$$= \int_0^1 \int_x^{xy} \frac{\cos u}{u} du \frac{dg(x)}{x} + 0(1)\ .$$

But

$$\int_x^{xy} \frac{\cos u}{u}\, du = \int_x^1 \frac{\cos u}{u} du + \int_1^{xy} \frac{\cos u}{u} du$$

$0 \log 1/x + 0(1)$ since $\int_1^{\infty} ((\cos u)/u) du$ converges. Finally then

$$\int_1^y F(A)dA = \int_0^1 \log \frac{1}{x} \frac{|\, dg(x)\,|}{x} + 0(1) = 0(1)\ .$$

Proof of 2. Again by the lemma,

$$F(A) = \frac{1}{A}\int_0^1 \cos Ax \frac{dg(x)}{x} + 0\frac{1}{A^2}\ .$$

Now call

$$\int_0^1 \cos Ax \frac{dg(x)}{x} = G(A) .$$

If we split the measure $(dg(x))/x$ into its continuous part and its mass points, then we split $G(A)$ into $h(A) + j(A)$ where $j(A)$ is almost periodic. It is shown by Lorch and Newman [2] that

$$\frac{1}{T}\int_0^T |h(A)|\, dA \to 0$$

while, of course,

$$\frac{1}{T}\int_0^T |j(A)|\, dA \to M > 0 ,$$

M denoting the mean value of the *positive* almost periodic function $|j(A)|$. Hence $1/T\int_1^T |G(A)|\, dA \to M$. Integration by parts gives that $1/(\log T)\int_1^T (|G(A)|/A) dA \to M$ and so, since

$$\int_1^T |F(A)|\, dA = \int_1^T \frac{|G(A)|}{A} dA + 0(1) ,$$

we find that

$$\int_1^T |F(A)|\, dA \sim M \log T$$

and $F(A) \notin L_1(1, \infty)$. The proof is complete.

Proof of Theorem 2. Call $\phi(x) = \int_x^\infty ((\sin t)/t) dt$. We will find a sequence of α_n in $(0, 1)$ such that $\sum_{n=1}^\infty (1/2^n)\phi(\alpha_n A) \geqq 0$ for all $A \geqq 0$. This being so, the choice of dg with the masses $1/2^n$ at the points α_n will satisfy our requirements.

We will choose the α_n inductively so that $\sum_{n=1}^N (1/2^n)\phi(\alpha_n A) \geqq -1/2^N$ for all A. Clearly the choice $\alpha_1 = 1$ satisfies this requirement when $N = 1$. Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_N$ have been chosen so that this requirement is satisfied. Since $\phi(\infty) = 0$ we can determine A_0 such that $A \geqq A_0$ insures $\sum_{n=1}^N (1/2^n)\phi(\alpha_n A) \geqq -1/2^{N+2}$. Also since $\phi(0) = \pi/2$ we can determine an $\alpha > 0$ so small that $\phi(\alpha A) \geqq 1$ for $A \leqq A_0$. The claim is that this serves as our α_{N+1}.

We have

$$\sum_{n=1}^{N+1} \frac{1}{2^n} \phi(\alpha_n A) = \sum_{n=1}^{N} \frac{1}{2^n} \phi(\alpha_n A) + \frac{\phi(\alpha A)}{2^{N+1}}$$

$$\geqq -\frac{1}{2^N} + \frac{1}{2^{N+1}} + -\frac{1}{2^{N+1}} \quad \text{for } A \leqq A_0$$

while, since $\phi(x) \geqq -1/2$ always, we obtain

$$\sum_{n=1}^{N} \frac{1}{2^n}\phi(\alpha_n A) + \frac{\phi(\alpha A)}{2^{N+1}} \geqq -\frac{1}{2^{N+2}} + \frac{-1/2}{2^{N+1}} = -\frac{1}{2^{N+1}}$$

for $A \geqq A_0$. The construction of the α_n completes the proof.

It is interesting to note that the above construction, when carried out carefully, leads to a dg satisfying $\int |dg(x)|/x^{1/2} < \infty$, so that our Theorem 1 is false when the exponent 2 is replaced by 1/2. It would be interesting to find the correct exponent.

We wish to thank L. Lorch and A. Shields for their useful suggestions in regard to the above results.

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