A NOTE ON AUTOMORPHISMS OF LIE ALGEBRAS

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In a beautiful paper which appeared in 1939 ([4]), F. Gantmacher made a thorough study of automorphisms of semi-simple Lie algebras over the field of complex numbers. Among other things, he defined the index $n(G_i)$ of a connected component G_i of the automorphism group $G = G(\mathfrak{L})$ as the minimum multiplicity of the characteristic root 1 for elements of G_i. The main purpose of this note is the determination of these indices. It is somewhat surprising that this does not appear in Gantmacher's paper since all the methods for deriving the formula for index G_i are available in his paper. The secondary purpose of this note is to extend Gantmacher's theory to the case of Lie algebras over arbitrary algebraically closed base fields of characteristic 0. This can be done by using algebraic group concepts and techiques which are by now well known. Nevertheless, it seems worthwhile to carry out the program in detail since Gantmacher's results give a real insight into the action of an automorphism in a semi-simple Lie algebra. For example, as we indicate, they can be used to give a new derivation and sharpening of theorems on fixed points which are due to Borel and Mostow ([1]).

1. Generalities on automorphisms. Let $\mathfrak{L}$ be a finite dimensional Lie algebra over an algebraically closed field Φ of characteristic 0, G the group of automorphisms of $\mathfrak{L}$. G is an algebraic group and it has a decomposition $G = G_0 \cup G_1 \cup \cdots \cup G_{r-1}$ where G_i are the algebraic components and G_0 is the component of the identity element 1. This is an invariant subgroup of finite index r in G and is irreducible, which means that the intersection of any two nonvacuous open subsets in G_0 is nonvacuous. The topology referred to here is the Zariski topology in which open sets are the complements of algebraic subsets of G.

If η is a nonsingular linear transformation in $\mathfrak{L}$, η has a unique decomposition as $\eta = \eta_s\eta_u = \eta_u\eta_s$ where η_s and η_u are polynomials in η, η_s is semi-simple and η_u is unipotent, that is, $\eta_u - 1$ is nilpotent. If η is an automorphism then η_u and η_s are automorphisms. If η is a unipotent automorphism then

$$\log \eta = (\eta - 1) - \frac{1}{2}(\eta - 1)^2 + \frac{1}{3}(\eta - 1)^3 - \cdots$$

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is a nilpotent derivation. Conversely, if D is a nilpotent derivation then $\eta = \exp D$ is a unipotent automorphism. Any unipotent automorphism is contained in the algebraic component of 1. Hence an automorphism η and its semi-simple part η_s are contained in the same component.

If η is a linear transformation in $\mathfrak{L}$ we can decompose $\mathfrak{L} = \mathfrak{L}_\alpha \oplus \mathfrak{L}_\beta \oplus \cdots \oplus \mathfrak{L}_\rho$ where $\mathfrak{L}_\alpha$ is the characteristic space of η corresponding to the characteristic root α: $\mathfrak{L}_\alpha = \{x_\alpha \mid x_\alpha(\eta - \alpha 1)^k = 0 \text{ for some } k\}$. The semi-simple part η_s is the linear transformation which leaves each $\mathfrak{L}_\alpha$ invariant and coincideds with the scalar multiplication $\alpha 1$ in $\mathfrak{L}_\alpha$. If η is an automorphism then $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] = 0$ if $\alpha\beta$ is not a characteristic root and $[\mathfrak{L}_\alpha \mathfrak{L}_\beta] \subseteqq \mathfrak{L}_{\alpha\beta}$ if $\alpha\beta$ is a characteristic root. Here $[\mathfrak{L}_\alpha \mathfrak{L}_\beta]$ is the subspace spanned by the Lie products $[x_\alpha x_\beta]$, $x_\alpha \in \mathfrak{L}_\alpha$, $x_\beta \in \mathfrak{L}_\beta$. In particular, $\mathfrak{L}_1$ the space of the characteristic root 1 is a subalgebra and $[\mathfrak{L}_\alpha \mathfrak{L}_1] \subseteqq \mathfrak{L}_\alpha$. In most considerations the refined decomposition $\mathfrak{L} = \Sigma \mathfrak{L}_\alpha$ will be replaced by a coarser Fitting type decomposition: $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{N}$ where $\mathfrak{N} = \sum_{\alpha \neq 1} \mathfrak{L}_\alpha$. These two spaces are invariant under η, η is unipotent in $\mathfrak{L}_1$ and $\eta - 1$ is nonsingular in $\mathfrak{N}$. We have $[\mathfrak{N}\mathfrak{L}_1] \subseteqq \mathfrak{N}$. If η is semi-simple then $\mathfrak{L}_1$ is the set of fixed points under η and $\mathfrak{N} = \mathfrak{L}^{\eta - 1} = \{x^\eta - x \mid x \in \mathfrak{L}\}$.

As before, let $G = G_0 \cup G_1 \cup \cdots \cup G_{r-1}$ be the decomposition of G into its algebraic irreducible components. We define the *index* $n(G_i)$ of G_i as the minimum multiplicity of the characteristic root 1 for the $\eta \in G_i$. The multiplicity of the root is the same as the dimensionality of the corresponding characteristic space. An element $\eta \in G_i$ is called *regular* if $\dim \mathfrak{L}_1(\eta) = \operatorname{index} G_i$. Let $(u_1, \cdots, u_n)$ be a basis for $\mathfrak{L}$ and let (α) be a matrix of η relative to this basis. Write the characteristic polynomial of $(\alpha) - 1$ as

$$(1) \qquad f_\eta(\lambda) = \lambda^n - \rho_1(\eta)\lambda^{n-1} + \cdots + (-1)^n \rho_n(\eta) .$$

The mappings $\eta \to \rho_j(\eta)$ are polynomial functions. If the index of $G_i = l_i$, then $\rho_n(\eta) = \cdots = \rho_{n-l_i+1}(\eta) = 0$ for all $\eta \in G_i$ but $\rho_{n-l_i}(\eta) \not\equiv 0$ on G_i. The regular elements of G_i are those such that $\rho_{n-l_i}(\eta) \neq 0$. Hence they form an open set in G_i.

2. Lie Algebras of algebraic groups. We need to recall some notions on linear algebraic groups. The results we shall quote can be found in two books by Chevalley ([3]). We recall first that if V and W are finite dimensional vector spaces, a rational mapping R of V into W is a mapping of the form $x = \Sigma \xi_i e_i \to y = \Sigma \eta_j f_j$ where $\eta_j = R_j(\xi) \equiv R_j(\xi_1, \cdots, \xi_m)$ are rational functions of the ξ's. Here $(e_1, \cdots, e_m)$ is a basis for V and $(f_1, \cdots, f_n)$ is a basis for W. The rational mapping is defined on an open subset of V. An important special case is that in which $W = \Phi$. Then R is a rational function on V. In the general case, if a is a point at which R is defined, the differential D_a of R at a is the linear mapping $x \to (D_a R)(x)$ where

$$(D_a R)(x) = \sum_1^n \left(\sum_1^m \left(\frac{\partial R_j}{\partial \lambda_k} \right)_{\lambda_h = \alpha_h} \xi_k \right) f_j$$

for $a = \Sigma \alpha_i e_i$. Here $\partial R_j / \partial \lambda_k$ is the formal partial derivative of the rational expression $R_j(\lambda_1, \cdots, \lambda_m)$ in indeterminates λ with respect to the indeterminate λ_k. Let E be an irreducible set in V, that is, the ideal $\mathfrak{A}$ of polynomials $P(\lambda_1, \cdots, \lambda_m)$ which are 0 at every point of E is prime. Let a be a point of E. We define the tangent space to E at a to be the subspace of vectors x in V such that $(D_a P)(x) = 0$ for every P in the ideal $\mathfrak{A}$ corresponding to E. The condition on $x = \Sigma \xi_k e_k$ are: $\sum_k (\partial P / \partial \lambda_k)_{\lambda_h = \alpha_h} \xi_k = 0$ for all $P \in \mathfrak{A}$. It suffices to have these conditions satisfied for a set of generators $(P_1, \cdots, P_q)$ of $\mathfrak{A}$. If R is a rational mapping defined on E then $R(E)$ is irreducible and $D_a R$ maps the tangent space to E at $a \in E$ into a subspace of the tangent space at $R(a)$ of $R(E)$.

If G is an irreducible algebraic linear group the tangent space $\mathfrak{L}$ at 1 for G can be made a Lie algebra by identifying its elements with derivations in the algebra of polynomial functions on G. The dimensionality of $\mathfrak{L}$ is the same as that of G. If $\mathfrak{L}$ is any Lie algebra and G_0 is the component of 1 of the group of automorphisms then the Lie algebra of G_0 is the Lie algebra of derivations of $\mathfrak{L}$. If $\mathfrak{L}$ is semi-simple then all the derivations are inner and the Lie algebra of G_0 is $ad\,\mathfrak{L}$ the set of adjoint mappings $\mathrm{ad}\, a \colon x \to [xa]$ in $\mathfrak{L}$.

3. Regular automorphisms of semi-simple Lie algebras. We shall now derive the purely algebraic form of Gantmacher's results on automorphisms of semi-simple Lie algebras.

Let $\mathfrak{L}$ be semi-simple and let η be an automorphism of $\mathfrak{L}$, $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{N}$ the Fitting decomposition of $\mathfrak{L}$ relative to η Thus $\mathfrak{L}_1$ is the space of the characteristic root 1, $\mathfrak{N}$ the sum of the other characteristic spaces, $\mathfrak{L}_1$ is a subalgebra and $[\mathfrak{N}\mathfrak{L}_1] \subseteqq \mathfrak{N}$. We note first

THEOREM 1. *$\mathfrak{L}_1$ is a reductive subalgebra of $\mathfrak{L}$.*

Proof. The assertion is that the subalgebra $ad_{\mathfrak{L}}\mathfrak{L}_1$ of the Lie algebra $ad_{\mathfrak{L}}\mathfrak{L} = ad\mathfrak{L}$ is completely reducible. This property holds for $ad\mathfrak{L}$ since $\mathfrak{L}$ is semi-simple. If η_s is the semi-simple part of η then $\mathfrak{L}_1 = \{l_1 \mid l_1^{\eta_s} = l_1\}$. This is equivalent to $[\eta_s, ad\, l_1] = 0$ and this implies that $ad_{\mathfrak{L}}\mathfrak{L}_1$ is a splittable Lie algebra of linear transformations in the sense of Malcev. Also we have $ad_{\mathfrak{L}}\mathfrak{L} = ad_{\mathfrak{L}}\mathfrak{L}_1 \oplus ad_{\mathfrak{L}}\mathfrak{N}$ and $[ad_{\mathfrak{L}}\mathfrak{N}, ad_{\mathfrak{L}}\mathfrak{L}_1] \subseteqq ad_{\mathfrak{L}}\mathfrak{N}$. These two properties imply that $ad_{\mathfrak{L}}\mathfrak{L}_1$ is completely reducible ([6] p. 109).

Since $\mathfrak{L}_1$ is reductive we have $\mathfrak{L}_1 = \mathfrak{L}_1' \oplus \mathfrak{C}$ where $\mathfrak{L}_1'$ is the derived algebra of $\mathfrak{L}_1$ and is semi-simple; $\mathfrak{C}$ is the center of $\mathfrak{L}_1$ and $ad_{\mathfrak{L}}c$ is semi-simple for every c in $\mathfrak{C}$ ([6] p. 106).

THEOREM 2. *If η is regular $\mathfrak{L}_1$ is abelian and η is semi-simple.*

Proof. Suppose $\mathfrak{L}_1$ is not abelian so that $\mathfrak{L}_1'$ is a nonzero semi-simple Lie algebra. There exist elements z and w in $\mathfrak{L}_1'$ such that $ad_{\mathfrak{L}_1'}z$ and $ad_{\mathfrak{L}_1'}w$ are nilpotent derivations and $(\exp ad_{\mathfrak{L}_1'}z)(\exp ad_{\mathfrak{L}_1'}w)$ is an automorphism of $\mathfrak{L}_1'$ which is not unipotent.[1] Since $ad_{\mathfrak{L}_1'}z$ and $ad_{\mathfrak{L}_1'}w$ are nilpotent it is known that $adz \equiv ad_{\mathfrak{L}}z$ and adw are nilpotent. If $\alpha, \beta \in \Phi$ we can form the automorphism $(\exp \alpha\, ad\, z)(\exp \beta\, ad\, w)$ which is in the component G_0 of 1 of the automorphism group of $\mathfrak{L}$. Since $[\mathfrak{L}_1\mathfrak{L}_1] \subseteqq \mathfrak{L}_1$ and $[\mathfrak{R}\mathfrak{L}_1] \subseteqq \mathfrak{R}$ it follows that the automorphism

$$\zeta(\alpha, \beta) = \eta_s(\exp \alpha \operatorname{ad} z)(\exp \beta \operatorname{ad} w)$$

satisfies $\mathfrak{L}_1^{\zeta(\alpha,\beta)} \subseteqq \mathfrak{L}_1$, $\mathfrak{R}^{\zeta(\alpha,\beta)} \subseteqq \mathfrak{R}$. Since $\eta_s = 1$ in $\mathfrak{L}_1$ the restriction of $\zeta(1, 1)$ to $\mathfrak{L}_1$ is not unipotent. Also the restriction of $\zeta(0, 0) - 1 = \eta_s - 1$ to $\mathfrak{R}$ is nonsingular. It follows by a standard argument that α, β can be chosen so that the multiplicity of the characteristic root 1 for $\zeta(\alpha, \beta)$ is less than $\dim \mathfrak{L}_1$. Since $\zeta(\alpha, \beta)$ is in the same component as η this means that η is not regular contrary to assumption. Hence $\mathfrak{L}_1$ is abelian. Then every $ad_{\mathfrak{L}}l_1$, $l_1 \in \mathfrak{L}_1$ is semi-simple. On the other hand, $\log \eta_u$ is a nilpotent derivation in $\mathfrak{L}$. Since the derivations of $\mathfrak{L}$ are all inner, $\log \eta_u = adz$, $z \in \mathfrak{L}$. Since $[\eta, \log \eta_u] = 0$, $z \in \mathfrak{L}_1$. Hence adz is nilpotent and semi-simple. Then $adz = 0$ which implies that $\eta_u = 1$ and $\eta = \eta_s$ is semi-simple.

We wish to prove the converse of Theorem 2. For this we shall need the following

LEMMA. *Let η be an automorphism of a semi-simple Lie algebra $\mathfrak{L}$ such that the subalgebra $\mathfrak{L}_1$ is abelian. Let H be the subgroup of the automorphisms group G of $\mathfrak{L}$ of elements commuting with η, H_0 the algebraic component of 1 of H, G_0 the algebraic component of 1 of G. Then*

$$K = \{\tau\eta\zeta\tau^{-1} \mid \tau \in G_0, \zeta \in H_0\} \tag{2}$$

contains a nonvacuous open subset of ηG_0.

Proof. We note first that the proof of Theorem 2 shows that every $\zeta \in H_0$ is semi-simple; hence every $\eta' = \eta\zeta = \zeta\eta$ is semi-simple. The Lie algebra of G_0 is $ad\,\mathfrak{L}$ and the Lie algebra of H_0 is $ad_{\mathfrak{L}}\mathfrak{L}_1$ since $\mathfrak{L}_1$ is the

[1] For example, we can take z, w to satisfy $[zw] = h \neq 0$, $[zh] = 2z$, $[wh] = -2w$. See the proof of the lemma to Th. 5.

set of fixed points under η. If $\xi, \zeta \in H_0$ then the mapping $\xi \to \zeta^{-1}\xi\zeta$ is the identity; hence the induced mapping $x \to \zeta^{-1} \times \zeta$ is the identity in $ad_{\mathfrak{L}}\mathfrak{L}_1$. Since $\zeta^{-1}(adl_1)\zeta = adl_1^{\zeta}$ it follows that $l_1^{\zeta} = l_1$. This implies that the space of fixed points $\mathfrak{L}_1(\eta') \supseteqq \mathfrak{L}_1(\eta)$ for any $\eta' = \eta\zeta$. We note next that K is the orbit of ηH_0 under the group of mappings $\xi \to \tau\xi\tau^{-1}$, $\tau \in G_0$. This implies that K is épais in the sense of Chevalley: K is irreducible and K contains a nonvacuous open subset of its closure (see Chevalley [3] tome III, p. 193). The result we wish to prove will now follow by showing that K is dense in ηG_0. (This is all which will be needed for the proof of Theorem 3.) Let $\eta' = \eta\zeta$ be any element of ηH_0. Then the tangent space $\mathfrak{T}(K; \eta')$ of K at η' contains the tangent space $\mathfrak{T}(\eta H_0; \eta')$ of ηH_0 at η' as well as the image of η' under the Lie algebra of mappings $x \to [x, adl]$, $l \in L$ (Chevalley, loc. cit. p. 192). This is the set of mappings $[\eta', adl]$, $l \in \mathfrak{L}$. Now $\mathfrak{T}(H_0; 1)$ is the Lie algebra $ad_{\mathfrak{L}}\mathfrak{L}_1$ so $\mathfrak{T}(H_0, \eta') = \mathfrak{T}(\eta' H_0; \eta') = \eta' ad_{\mathfrak{L}}\mathfrak{L}_1$. We wish to show that $\mathfrak{T}(K; \eta') = \mathfrak{T}(\eta G_0; \eta') = \eta' \, ad\mathfrak{L}$. Since $\mathfrak{T}(K, \eta')$ contains $\eta' \; ad_{\mathfrak{L}}\mathfrak{L}_1$ and $[\eta', ad\mathfrak{L}] = \{[\eta', adl], l \in \mathfrak{L}\}$, it is enough to show that $ad \; \mathfrak{L} = ad_{\mathfrak{L}}\mathfrak{L}_1 + (\eta')^{-1}[\eta', ad \; \mathfrak{L}]$. Since $(\eta')^{-1}[\eta', adl] = adl - adl^{\eta'} = ad(l - l^{\eta'})$, it suffices to show that $\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}^{1-\eta'}$. But this is clear since η' is semi-simple. We have therefore proved that the tangent spaces to K and to ηG_0 at any point $\eta' = \eta\zeta$ coincide. Since K is the orbit of ηH_0 under the set of mappings $\xi \to \tau\xi\tau^{-1}$ it follows that the tangent space to K and to ηG_0 at any point of K coincide. This implies that K is dense in ηG_0.

THEOREM 3. *If $\mathfrak{L}_1$ is abelian then η is regular.*

Proof. Let K be the set defined by (2). Then the lemma implies that K contains a regular element $\tau\eta\zeta\tau^{-1}$, $\zeta \in H_0$, $\tau \in G_0$. Then $\eta\zeta$ is a regular element contained in the component $G_\iota = \eta G_0$ containing η. The foregoing proof shows also that $\mathfrak{L}_1(\eta\zeta) \supseteqq \mathfrak{L}_1(\eta)$. Since $\eta\zeta$ is regular we have $\mathfrak{L}_1(\eta\zeta) = \mathfrak{L}_1(\eta)$ so η is regular also.

Let η be a regular element and, as before, let $H_0(\eta)$ be the algebraic component of 1 in the subgroup $H(\eta)$ of G of elements commuting with η. Then $H_0(\eta)\eta = \eta H_0(\eta)$ is the component of η in $H(\eta)$. As we have seen in the preceding proof, the Lie algebra $H_0(\eta)$ is $ad_{\mathfrak{L}}\mathfrak{L}_1(\eta)$ where $\mathfrak{L}_1(\eta)$ is the set of fixed points under η. Moreover, $\mathfrak{L}_1(\eta)$ and $H_0(\eta)$ are abelian. The fact that $H_0(\eta)$ is abelian implies that $H_0(\eta) \subseteqq H_0(\eta^{\zeta})$ for any $\zeta \in H_0(\eta)$. If $\eta\zeta$ is regular also then we have $H_0(\eta\zeta) = H_0(\eta)$. In a similar manner, we see that the elements of $\mathfrak{L}_1(\eta)$ are fixed under $\eta\zeta$. If $\eta\zeta$ is regular also then $\mathfrak{L}_1(\eta) = \mathfrak{L}_1(\eta\zeta)$, $\zeta \in H_0(\eta)$. We note also that the argument used in the proof of Theorem 2 shows that every element of $H_0(\eta)$ and hence of $H_0(\eta)\eta$ is semi-simple.

We now use the full force of the lemma to prove the following

THEOREM 4. *Let η_1 and η_2 be two regular automorphisms contained in the same component of the automorphism group. Let $\mathfrak{L}_1(\eta_j)$ be the subalgebra of $\mathfrak{L}$ of fixed elements under η_j, $j = 1, 2$, and let $H_0(\eta_j)$ be the algebraic component of 1 in the subgroup $H(\eta_j)$ of automorphisms commuting with η_j. Then there exists a $\tau \in G_0$ such that $\mathfrak{L}_1(\eta_1)^\tau = \mathfrak{L}_1(\eta_2)$ and $\eta_2 H_0(\eta_2) = \tau^{-1}(\eta_1 H_0(\eta_1))\tau$.*

Proof. Let K_1 and K_2 be the sets (2) defined by η_1 and η_2 respectively. Then K_1 and K_2 contain open subsets of the component G_i containing η_1 and η_2. It follows that there exists a regular element in $K_1 \cap K_2$. Hence there exist $\tau_j \in G_0$, $\zeta_j \in H_0(\eta_j)$ such that

$$\eta = \tau_1(\eta_1\zeta_1)\tau_1^{-1} = \tau_2(\eta_2\zeta_2)\tau_2^{-1}$$

is regular. Then $\eta_1\zeta_1$ and $\eta_2\zeta_2$ are regular elements of G_i and $\eta_2\zeta_2 = \tau^{-1}(\eta_1\zeta_1)\tau$ where $\tau = \tau_1^{-1}\tau_2$. Then $\mathfrak{L}_1(\eta_2) = \mathfrak{L}_1(\eta_2\zeta_2) = \mathfrak{L}_1(\eta_1\zeta_1)^\tau = \mathfrak{L}_1(\eta_1)^\tau$. In a similar manner we see that $\tau^{-1}(\eta_1 H_0(\eta_1))\tau = \eta_2 H_0(\eta_2)$.

We have noted that if η is regular every element of $\eta H_0(\eta)$ is semi-simple. We wish to prove that conversely any semi-simple automorphism belongs to an $\eta H_0(\eta)$ where η is regular. The proof of this result in the complex case given by Gantmacher is based on the use of exponentials of elements adl_1, $l_1 \in \mathfrak{L}_1$. These are not available in the algebraic case. However, a suitable substitute for these has been found by Seligman and we shall use these.

Let $\mathfrak{K}$ be a semi-simple subalgebra of a Lie algebra $\mathfrak{L}$, $\mathfrak{H}$ a Cartan subalgebra of $\mathfrak{K}$. Let α_i, $i = 1, 2, \cdots, l$ be a simple system of roots for $\mathfrak{K}$ relative to $\mathfrak{H}$, $\mathfrak{K}_{\alpha_i}$ the corresponding one dimensional root spaces. Then there exists a canonical set of generators e_i, f_i, h_i, $i = 1, 2, \cdots, l$, for $\mathfrak{K}$ such that the h_i form a basis for $\mathfrak{H}$, $e_i \in \mathfrak{K}_{\alpha_i}$, $f_i \in \mathfrak{K}_{-\alpha_i}$ and the following relations hold:

$$(3) \qquad \begin{aligned} [h_i h_j] &= 0 \\ [e_i f_j] &= \delta_{ij} h_i \\ [e_i h_j] &= A_{ji} e_i \\ [f_i h_j] &= -A_{ji} f_i \end{aligned}$$

where (A_{ij}) is the Cartan matrix of the simple system α_i. We have $A_{ii} = 2$ and A_{ij} is a nonpositive integer if $i \neq j$. It is known that the mappings ade_i and adf_i are nilpotent (in $\mathfrak{L}$). Following Seligman we introduce the automorphism in $\mathfrak{L}$:

$$(4) \qquad \sigma_i(\xi) = (\exp ad\xi e_i)(\exp ad\xi^{-1} f_i)(\exp ad\xi e_i)$$

where ξ is any nonzero element of Φ and $i = 1, 2, \cdots, l$. Also we set

$$\omega_i(\xi) = \sigma_i(\xi)\sigma_i(-1) \tag{5}$$

and we let H be the group of automorphisms of $\mathfrak{L}$ generated by the $\omega_i(\xi)$, $\xi \neq 0$ in Φ, $i = 1, 2, \cdots, l$. Clearly the $\omega_i(\xi)$ map $\mathfrak{K}$ into itself. It has been shown by Seligman that the restriction $\tilde{H}$ of H to $\mathfrak{K}$ coincides with the group of automorphism in $\mathfrak{K}$ such that

$$e_i \to \mu_i e_i, f_i \to \mu_i^{-1} f_i, h_i \to h_i \tag{6}$$

where the μ_i are arbitrary nonzero elements.[2] Then $\tilde{H}$ is an irreducible abelian algebraic group of automorphisms in $\mathfrak{K}$ (an l-dimensional torus). We shall now prove the

LEMMA. *H is an abelian group.*

Proof. If we recall the form of the irreducible $\mathfrak{K}$-modules we see that $\mathfrak{L}$ is generated by the f and a set X of elements x such that $[xe_i] = 0$, $i = 1, 2, \cdots, l$ and $[xh] = \Lambda(h)x$, $\Lambda(h) \in \Phi$ ([5] p. 44). We fix i and write $F = ad\, f_i$, $E = ad\ e_i$. Then if $x_0 = x \in X$ and we define $x_j = x_0 F^j$, it is known that $x = x_0, x_1, \cdots, x_m$ are linearly independent and satisfy:

$$\begin{aligned} x_j F = x_{j+1}\ , \quad & j = 0, \cdots, m-1\ , \quad x_m F = 0 \\ x_0 E = 0\ , \quad x_j E = -j(m-j+1)x_j\ , \quad & j = 1, \cdots, m\ . \end{aligned} \tag{7}$$

Hence $x_0 \exp \xi E = x_0$,

$$x_0 \exp \xi^{-1} F = \sum_0^m \frac{\xi^{-j}}{j!} x_j$$

$$x_j(\exp \xi E) = \sum_{l=0}^{j} (-1)^l \prod_{k-1}^{l} (m-j+k)\binom{j}{l}\xi^l x_{j-l}\ .$$

Hence $x_0(\exp \xi E)(\exp \xi^{-1}F)(\exp \xi E) = \sum_{r=0}^m a_r \xi^{-r} x_r$ where

$$\begin{aligned} a_r &= \sum_{l=0}^{m-r} (-1)^l \binom{r+l}{l} \prod_{k=1}^{l} (m-r-l+k) \frac{1}{(r+l)!} \\ &= \frac{1}{r!} \sum_{l=1}^{m-r} (-1)^l \binom{m-r}{l}\ . \end{aligned}$$

It follows that $a_r = 0$ if $r \neq m$ and $a_m = 1/m!$ Hence

$$x_0(\exp \xi E)(\exp \xi^{-1}F)(\exp \xi E) = (1/m!)\xi^{-m}x_m.$$

A similar calculation shows that

$$x_m(\exp \xi E)(\exp \xi^{-1}F)(\exp \xi E) = (-1)^m m! \xi^m x_0\ .$$

Hence

[2] [8] p. 446. A simpler proof will be given in a forthcoming book on Lie algebras by the author.

$$x\omega_i(\xi) = \xi^{-m}x \tag{8}$$

where m is a nonnegative integer. This implies that $x\omega_i(\xi)\omega_{i'}(\xi') = x\omega_{i'}(\xi')\omega_i(\xi)$ if $x \in X$. In view of (6), $\omega_i(\xi)$ and $\omega_i(\xi')$ commute also in their action on the f_j. It follows that the group H generated by the $\omega_i(\xi)$ is abelian.

Now let ρ be a semi-simple automorphism of the semi-simple Lie algebra $\mathfrak{L}$ and let $\mathfrak{L}_1(\rho)$ be the set of ρ-fixed elements. We have $\mathfrak{L}_1(\rho) = \mathfrak{K} \oplus \mathfrak{C}$ where $\mathfrak{K}$ is the derived algebra and $\mathfrak{C}$ is the center. Then $\mathfrak{K}$ is semi-simple and we can apply the above considerations. Then let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{K}$, e_i, f_i, h_i canonical generators of the type indicated such that the h_i form a basis for $\mathfrak{H}$. Let H be the abelian group of automorphisms of $\mathfrak{L}$ generated by the $\omega_i(\xi)$, $\tilde{H}$ its restriction to $\mathfrak{K}$. Then if $\mu_1, \mu_2, \cdots, \mu_l$ are arbitrary nonzero elements of Φ, the automorphism $\zeta(\mu_1, \cdots, \mu_l)$ of $\mathfrak{K}$ such that $e_i \to \mu_i e_i, f_i \to \mu_i^{-1} f_i$ belongs to $\tilde{H}$. It is known that $\mathfrak{K}$ has a basis consisting of the h_i, certain products $[\cdots [e_{i_1} e_{i_2}] \cdots e_{i_r}]$ and certain products $[\cdots [f_{i_1} f_{i_2}] \cdots f_{i_r}]$. The first of these is a characteristic vector of $\zeta(\mu_1, \cdots, \mu_l)$ belonging to $\mu_{i_1}\mu_{i_2} \cdots \mu_{i_r}$ and the second belongs to the root $(\mu_{i_1}\mu_{i_2} \cdots \mu_{i_r})^{-1}$. It follows that the μ's can be chosen so that $\mathfrak{H}$ is the characteristic space of the root 1 of $\zeta(\mu_1, \cdots, \mu_l)$. It is clear that any $\zeta \in H$ commutes with ρ and so it respects the decomposition: $\mathfrak{L} = \mathfrak{L}_1(\rho) \oplus \mathfrak{L}^{1-\rho}$, $\mathfrak{L}^{1-\rho} = \{x - x^\rho\}$. It follows by the standard specialization argument that there exists an automorphism $\eta = \rho\zeta$, $\zeta \in H$, such that $\mathfrak{L}_1(\eta) = \mathfrak{H} \oplus \mathfrak{C}$. Since $\mathfrak{L}_1(\eta)$ is abelian we see that η is regular.

Let $H_0(\eta)$ be the algebraic component of 1 in the group of automorphisms of $\mathfrak{L}$ commuting with η. Since H is abelian, $\omega_i(\xi)$ commutes with η. It follows from (8) that for fixed $i, \xi \neq 0$ in Φ, the $\omega_i(\xi)$ form an irreducible algebraic group. Hence $\omega_i(\xi) \in H_0(\eta)$ and $H \subseteqq H_0(\eta)$. Then $\zeta^{-1} \in H_0(\eta)$ and $\rho = \eta\zeta^{-1}$.

We have therefore proved

THEOREM 5. *If ρ is a semi-simple automorphism of a semi-simple Lie algebra then ρ has the form $\eta\zeta$ where η is regular and $\zeta \in H_0(\eta)$ the component of* 1 *of the group of automorphisms commuting with η.*

4. Determination of the indices. Let $\mathfrak{L}$ be semisimple, $\mathfrak{H}$ a Cartan subalgebra, $e_i, f_i, h_i, i = 1, 2, \cdots, l$ canonical generators such that the h_i form a basis for $\mathfrak{H}$ and (3) hold. We define the *group of automorphisms of the Cartan matrix* (A_{ij}) to be the subgroup of the symmetric group S_l on $1, 2, \cdots, l$ of the permutations s such that $A_{s(i),s(j)} = A_{i,j}$, $i, j = 1, 2, \cdots, l$. If s is in this group there is a unique automorphism σ of $\mathfrak{L}$ such that $e_i^\sigma = e_{s(i)}, f_i^\sigma = f_{s(i)}$. The set of these automorphisms is a finite group F isomorphic to the group of automorphisms of the

Cartan matrix. It is known that $G = G_0F = FG_0$.[3]

Let $\sigma \in F$, s the corresponding automorphism of the Cartan matrix. If μ_i, $i = 1, 2, \cdots, l$, are arbitrary nonzero elements of Φ then there exists an automorphism ζ of $\mathfrak{L}$ such that

$$e_i^{\zeta} = \mu_i e_i\ , \qquad f_i^{\zeta} = \mu_i^{-1} f_i\ . \tag{9}$$

The argument used in proving Theorem 5 shows that $\zeta \in G_0$. We choose μ_i so that

$$\mu_{s(i)} = \mu_i \tag{10}$$

for every i, and for every positive root $\alpha = \Sigma \kappa_i \alpha_i$, κ_i nonnegative integral,

$$(\mu_1^{\kappa_1}\mu_2^{\kappa_2} \cdots \mu_l^{\kappa_l})^m \neq 1\ , \tag{11}$$

m the order of s. Clearly such a choice of the μ_i can be made. Also it is evident that ζ is semi-simple and $\mathfrak{L}_1(\zeta^m) = \mathfrak{H}$. We have $\sigma\zeta = \zeta\sigma$ and $\sigma^m = 1$ so that σ is semi-simple. Hence $\eta = \sigma\zeta$ is semi-simple and $\mathfrak{L}_1(\eta)$ is the set of η-fixed points. If $x \in \mathfrak{L}_1(\eta)$, $x^{\eta^m} = x^{\zeta^m} = x$ so $x \in \mathfrak{H}$. Since $x^{\zeta} = x$ for $x \in \mathfrak{H}$ we have $x^{\sigma} = x^{\zeta\sigma} = x^{\eta} = x$. Conversely, if $x \in \mathfrak{H}$ and $x^{\sigma} = x$ then $x \in \mathfrak{L}_1(\eta)$. Hence $\mathfrak{L}_1(\eta)$ is the subspace of $\mathfrak{H}$ of σ-fixed elements. We have $h_i^{\sigma} = h_{s(i)}$ for the basis $(h_1, h_2, \cdots, h_l)$ of $\mathfrak{H}$. Let

$$s = (i_1 \cdots i_{m_1})(j_1 \cdots j_{m_2}) \cdots (u_1 \cdots u_{m_p}) \tag{12}$$

be the decomposition of the permutation s into disjoint cycles of length $\geqq 1$ $(m_1 + m_2 + \cdots + m_p = l)$. Then it is clear that the elements

$$g_1 = h_{i_1} + \cdots + h_{i_{m_1}}, \cdots, g_p = h_{u_1} + \cdots + h_{u_{m_p}} \tag{13}$$

constitute a basis for the subspace of $\mathfrak{H}$ of σ-fixed points. Hence $\dim \mathfrak{L}_1(\eta) = p$, the number of cycles in the decomposition of s. Since $\zeta \in G_0$, η and σ are in the same component of the automorphism group. Since $\mathfrak{L}_1(\eta) \subseteqq \mathfrak{H}$, $\mathfrak{L}_1(\eta)$ is abelian and so η is a regular element in the same component as σ. We can therefore state

THEOREM 6. *Let $G_i = \sigma_i G_0$ be a component of the group of automorphisms where $\sigma_i \in F$ and corresponds to the automorphism s_i of the Cartan matrix. Then the index of G_i is the number of cycles in the decomposition of s_i into disjoint cycles of lengths $\geqq 1$.*

For G_0 we may take $\sigma_0 = 1$ and we obtain that the index of G_0 is l, the dimensionality of the Cartan subalgebra $\mathfrak{H}$. On the other hand. it is clear that if $\sigma_i \neq 1$ then the index of $\sigma_i G_0$ is positive but is less than l. Hence $\sigma_i G_0 \neq G_0$ and the decomposition $G = FG_0$ is semidirect.

[3] The arguments of [8] can be used to prove this. A detailed discussion will be given in Chapter IX of the author's forthcoming book.

COROLLARY. *The index of G_0 is l and the index of any $G_i \neq G_0$ is positive and less than l. The decomposition $G = FG_0$ is semi-direct.*

Our analysis shows also that any Cartan subalgebra is the space $\mathfrak{L}_1(\eta)$ for a regular automorphism belonging to G_0. Hence Theorem 4 proves again the conjugacy theorem for Cartain subalgebras by means of an element of G_0. Theorem 4 can be considered as a natural generlization of the classical conjugacy theorem.

Let $\eta = \sigma\zeta$ be the automorphism which we constructed for the proof of Theorem 6. If $\nu_1, \nu_2, \cdots, \nu_l$ are nonzero elements of Φ such that $\nu_{s(i)} = \nu_i$, $i = 1, 2, \cdots, l$, then it is clear that the automorphism $\zeta(\nu_1, \nu_2, \cdots, \nu_l)$ such that

$$e_i^{\zeta(\nu_1,\cdots,\nu_l)} = \nu_i e_i\,, \qquad f_i^{\zeta(\nu_1,\cdots,\nu_l)} = \nu_i^{-1} f_i \tag{14}$$

commutes with η. The set of these automorphisms is an irreducible algebraic group of p dimensions where p is the index of ηG_0. It follows that this group coincides with $H_0(\eta)$ the component of 1 in the group of automorphisms commuting with η. Theorem 4 therefore implies that if ρ is any semi-simple automorphism then ρ has the form $\tau(\sigma_i\xi)\tau^{-1}$ where $\tau \in G_0$, $\sigma_i \in F$ and ξ is of the form $\zeta(\nu_1, \cdots, \nu_l)$ as in (14). This is Gantmacher's "canonical form" for the semi-simple automorphism ρ. It is clear from the definition of σ_i that $\mathfrak{H}^{\sigma_i} \subseteqq \mathfrak{H}$. In fact, if we choose the basis for $\mathfrak{H}$ as before, then $h_j^{\sigma_i} = h_{s_i(j)}$ where s_i is the permutation of $1, 2, \cdots, l$ associated with σ_i. It is clear that the restriction of σ_i to $\mathfrak{H}$ is periodic and the subspace of σ_i-fixed points of $\mathfrak{H}$ has l_i-dimensions where l_i is the index of the component $G_0^{\sigma_i}$. Since $h^\xi = h$ for every $h \in \mathfrak{H}$ these results hold also for $\sigma_i\xi$. Since $\rho = \tau(\sigma_i\xi)\tau^{-1}$ we have the following

THEOREM 7. *If ρ is a semi-simple automorphism of a semi-simple Lie algebra, then there exists a Cartan subalgebra $\mathfrak{H}$ such that $\mathfrak{H}^\rho \subseteqq \mathfrak{H}$, the restriction of ρ to $\mathfrak{H}$ is periodic and the subspace of ρ-fixed points of $\mathfrak{H}$ has dimensionality equal to the index of the component ρG_0.*

We look next at the indices of the components $G_i \neq G_0$ and for the sake of simplicity we confine our attention to the simple algebras. Outer automorphisms (that is, automorphisms not in G_0) exist in the following cases: A_l, $l > 1$, D_l, $l \geqq 4$ and E_6 and only in these cases, The group of automorphisms of the Cartan matrix can be identified with the group of automorphisms of the associated Dynkin diagram. For A_l the automorphism $\neq 1$ of the Dynkin diagram (suitably labelled) is $i \to l + 1 - i$. If l is even the cycle decomposition is $(1l)(2, l-1) \cdots (l/2, l/2+1)$ and for odd l it is $(1l)(2, l-1) \cdots ((l-1)/2, (l-1)/2+2)((l-1)/2+1)$. In both cases, if $G_1 \neq G_0$, then index $G_1 = [(l+1)/2]$. For $D_l, l > 4, G = G_0 \cup G_1, G_1 \neq G_0$ and the permutation associated with G_1 is $i \to i$ if $i \leqq l-2, l \to l-1, l-1 \to l$. The cycle decomposition is $(1) \cdots (l-2)$ $(l-1, l)$. Hence index $G_1 = l-1$. For D_4 the group of automorphisms of the Dynkin diagram is the symmetric group on 1, 3, 4 if $\alpha_1, \alpha_3, \alpha_4$ are the end points of the diagram. If the permutation associated with G_i is of order two then index $G_i = 3$. If the permutation is of order three then index $G_i = 2$. For a suitable ordering of the vertices the automorphism $\neq 1$ of the Dynkin diagram of E_6 is (15) (24) (3) (6). Hence index $G_1 = 4$.

THEOREM 8. *For $A_l, l > 1, G = G_0 \cup G_1$ and index $G_1 = [(l+1)/2]$. For $D_l, l > 4, G = G_0 \cup G_1$ and index $G_1 = l-1$. For D_4, G/G_0 is the symmetric group S_3 and index $G_i = 2$ if the coset of G_i is of order 3 and index $G_i = 3$ if the coset of G_i is of order 2. For $E_6, G = G_0 \cup G_1$ and index $G_1 = 4$.*

5. Application to fixed points. In the applications to fixed points we can for the most part relax the assumption on the base field Φ and suppose only that Φ is of characteristic 0. If η is an automorphism in $\mathfrak{L}$ over Φ and P is an extension field of Φ then η has a unique extension to an automorphism η of $\mathfrak{L}_P$ and the space of η-fixed points of $\mathfrak{L}_P$ has the form $\mathfrak{F}(\eta)_P$ where $\mathfrak{F}(\eta)$ is the space of η-fixed points of $\mathfrak{L}$. This remark reduces most considerations of fixed points to the algebraically closed case.

The following result is due to Borel and Mostow ([1] p 398) for semi-simple automorphisms.

THEOREM 9. *If $\mathfrak{L}$ is a nonsolvable Lie algebra over a field of characteristic 0 then any automorphism η of $\mathfrak{L}$ has a fixed point.*

Proof. It suffices to assume the base field is algebraically closed. Let $\mathfrak{R}$ be the radical of $\mathfrak{L}$. Then η induces an automorphism $\bar{\eta}$ in the semi-simple Lie algebra $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{R}$. By the Corollary to Th. 6, $\dim \bar{\mathfrak{L}}_1(\bar{\eta}) \geqq 1$. This means that 1 is a characteristic root of $\bar{\eta}$. Hence 1 is a characteristic root of η and there exists a nonzero fixed point.

It is convenient at this point to introduce another type of index $m(G_i)$ of a component G_i of the automorphism group in the algebraically closed case. We set $m(G_i)$ = minimum of $\dim \mathfrak{F}(\eta)$ for $\eta \in G_i$ where $\mathfrak{F}(\eta)$ is the space of η-fixed points. If (α) is a matrix of η then $\dim \mathfrak{F}(\eta) = n - \text{rank}\,((\alpha) - 1), n = \dim \mathfrak{L}$. Hence $m(G_i) = k_i$ means that for every $\eta \in G_i$ every minor of order $n - k_i + 1$ of $(\alpha) - 1$ vanishes but there exists an $\eta \in G_i$ such that $(\alpha) - 1$ has a nonvanishing minor of order $n - k_i$. If is clear from this that the elements η of G_i such that

$\dim \mathfrak{F}(\eta) = k_i \equiv m(G_i)$ form an open set in G_i. Hence this set contains a regular element η. If $\mathfrak{L}$ is semisimple it follows that such an η is semi-simple. Then $\mathfrak{F}(\eta) = \mathfrak{L}_1(\eta)$ and consequently $m(G_i) = n(G_i)$. We can state this result in the following way.

THEOREM 10. *Let $\mathfrak{L}$ be a semi-simple Lie algebra over an algebraically closed field of characteristic* 0 *and let l_i be the index of the component G_i of the group of automorphisms of $\mathfrak{L}$. Let $\eta \in G_i$ and let $\mathfrak{F}(\eta)$ be the space of fixed points under η. Then* $\dim \mathfrak{F}(\eta) \geqq l_i$ *and there exists $\eta \in G_i$ such that* $\dim \mathfrak{F}(\eta) = l_i$.

This result can also be applied to the case of an arbitrary base field of charactetistic 0. A given automorphism η has its extension to an automorphism η of $\mathfrak{L}_\Omega$, Ω the algebraic closure of Φ. Also $\mathfrak{L}_\Omega$ is semi-simple if $\mathfrak{L}_\Omega$ is semi-simple. The result just proved gives a lower bound for $\dim \mathfrak{F}(\eta)$ once the component of η in the group of automorphisms of $\mathfrak{L}_\Omega$ is known. Even without this information we can say that $\dim \mathfrak{F}(\eta) \geqq m$ where m is the minimum of the indices of the components G_i of the group of automorphisms of $\mathfrak{L}_\Omega$.

Again let $\mathfrak{L}$ be semi-simple over any field of characteristic 0 and let η be semi-simple in the sense that its minimum polymonial is a product of distinct prime factors. Since the base field is perfect this property is preserved under field extension. It follows from this that $\mathfrak{L}_1(\eta) = \mathfrak{F}(\eta)$ is reductive. This implies that any Cartan subalgebra $\mathfrak{H}_1$ of $\mathfrak{L}_1(\eta)$ is abelian and reductive in $\mathfrak{L}$. Moreover, any reductive abelian subalgebra of $\mathfrak{L}_1(\eta)$ can be imbedded in a Cartan subalgebra and any two Cartan subalgebras of $\mathfrak{L}_1(\eta)$ have the same dimensionality. It follows by a field extension argument and Th. 7 that the dimensionality of $\mathfrak{H}_1$ is not less than the index of the component of η in the group of automorphisms of $\mathfrak{L}_\Omega$, Ω the algebraic closure of the base field.

The result just indicated holds also for arbitrary $\mathfrak{L}$ and semi-simple η by virtue of a result of Mostow's that there exists a Levi-decomposition $\mathfrak{L} = \mathfrak{K} + \mathfrak{R}$ where $\mathfrak{R}$ is the radical and $\mathfrak{K}$ is a semi-simple subalgebra invariant under η ([7]). It is known that if $a \in \mathfrak{K}$ and $ad_{\mathfrak{K}}\, a$ is semi-simple then $ad_{\mathfrak{L}} a$ is semi-simple. We can therefore state the following extension of a theorem of de Siebenthal-Borel-Mostow ([1] p. 498).

THEOREM 11. *Let η be a semi-simple automorphism of a Lie algebra $\mathfrak{L}$ over a field of characteristic* 0. *Let $\mathfrak{R}$ be the radical, $\bar{\mathfrak{L}} = \mathfrak{L}/\mathfrak{R}$ and let m be the minimal index of the components of the group of automorphism of $\bar{\mathfrak{L}}_\Omega$, Ω the algebraic closure of the base field. Then $m \geqq 1$ and there exists an abelian reductive subalgebra of m dimensions whose elements are fixed under η.*

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