OSCILLATION AND NONOSCILLATION CRITERIA FOR $y''(x) + f(y(x))p(x) = 0$

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1. In recent papers Moore and Nehari [6] and Atkinson [1] studied the differential equation

$$y''(x) + y(x)^{2n+1}p(x) = 0 \tag{1.1}$$

where n is a positive integer and $p(x)$ is positive and continuous for $0 < x < \infty$. In the latter paper a necessary and sufficient condition for the existence of a nonoscillatory solution of (1.1) is established while in the former paper results concerning the nonoscillatory and asymptotic characteristics of the solutions of (1.1) are obtained by use of variational (and other) techniques. As pointed out in [6] an equation of type (1.1) may possess solutions, y_1 and y_2 say, such that y_1 has an infinite number of zeros for $0 < x < \infty$ while y_2 has only a finite number of zeros for $0 < x < \infty$. The object here is to give sufficient conditions for a differential equation of the type

$$y''(x) + f(y(x))p(x) = 0 \tag{1.2}$$

to have all of those solutions existing for $0 < x < \infty$ oscillatory (possessing an infinite number of zeros in $a < x < \infty$, for all $a > 0$) or to have some of those solutions existing for $0 < x < \infty$ nonoscillatory (possessing a finite number of zeros in $a < x < \infty$, for some $a > 0$) where $f(y)$ as a function of y is of class C' for $-\infty < y < \infty$ and $p(x)$ is continuous and nonnegative for $0 < x < \infty$, but this last requirement will be relaxed in certain theorems. This definition of oscillation is used since it seems natural, for example, to classify the solution $y(x) = \exp(g(x)) - 1$ of the equation $y''(x) + (y(x) + 1)p(x) = 0$, where $p(x) = -(g'(x))^2 + g''(x)$, $g(x) = (x-1)^5 \sin(1/(x-1))$, $x \neq 1$, $g(1) = g'(1) = g''(1) = 0$, as nonoscillatory rather than oscillatory, even though it has an infinite number of zeros in $0 < x < \infty$. For a further discussion of what should be called a "nonoscillatory" solution see [6]. For $f(y) \equiv y$ the results reduce to known criteria for this type of problem. Cf. [2,4,5,7 & 8].

2. In this section we prove the following oscillation theorem for equation (1.2).

THEOREM 1. *If*

1. $y(x)$ *is a solution of* (1.2) *existing for* $0 < x < \infty$.

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2. *$f(y)/y \geqq \gamma > 0$ for $-\infty < y < \infty$, γ a constant,*
3. *for some*
   $a > 0$, $k(x) \equiv \int_a^x [\gamma g(t)p(t) - \frac{1}{4}(g'(t))^2/g(t)]dt + \frac{1}{2}(g'(x)) \to \infty$
   monotonely as $x \to \infty$, where
4. *$p(x) \geqq 0$ for $a \leqq x < \infty$ and*
5. *$g(x) > 0$, $g(x) \in C'$ for $a \leqq x < \infty$ with $\int_a^x (1/g(t))dt \to \infty$ as $x \to \infty$, then $y(x)$ is an oscillatory solution of (1.2).*

The proof is by contradiction. Suppose the solution $y(x)$ of hypothesis 1 possesses a last zero at $x = x_0$ where we may clearly assume without loss of generality that $0 < x_0 < a$. Changing to a new dependent variable $w = w(x)$ in (1.2), where $w(x)/g(x) = -(y'(x))/y(x)$ and $x \geqq a$, we get

$$(2.1) \qquad w'(x) = (1/g(x))(w(x) + \tfrac{1}{2}g'(x))^2 + \\ + [g(x)\{f(y(x))/y(x)\}p(x) - \tfrac{1}{4}(g'(x))^2/g(x)].$$

Eliminating the term in square brackets of the second member of (2.1) by setting $w(x) = z(x) + h_1(x)$ where

$$h_1(x) \equiv \int_a^x [g(t)\{f(y(t))/y(t)\}p(t) - \tfrac{1}{4}(g'(t))^2/g(t)]\, dt$$

we get

$$(2.2) \qquad z'(x) = (1/g(x))(z(x) + h(x))^2, \ h(x) = h_1(x) + \tfrac{1}{2}g'(x), \ x \geqq a .$$

By use of hypotheses 2 and 4 we have $h(x) \geqq k(x)$, $x \geqq a$. Noting $z(x)$ is an increasing function and using hypothesis 3 we conclude that there exists a point $a_1 > a$ such that

$$z(x) + h(x) \geqq z(x) + k(x) \geqq z(x) + k(a_1) \geqq z(a_1) + k(a_1) = 1/a_2 > 0, \ x \geqq a_1 .$$

Thus for $x \geqq a_1$ we must have

$$z'(x) = (1/g(x))(z(x) + h(x))^2 \geqq (1/g(x))(z(x) + k(a_1))^2 .$$

Integrating $(z'(x))/(z(x)) + k(a_1))^2 \geqq 1/g(x)$ from a_1 to $x > a_1$ we get

$$-1/(z(x) + k(a_1)) \geqq -a_2 + \int_{a_1}^x (1/g(t))\,dt$$

and, therefore,

$$(2.3) \qquad z(x) \geqq -k(a_1) + 1/[a_2 - \int_{a_1}^x (1/g(t))dt] \qquad x \geqq a_1 .$$

By use of hypothesis 5 we see there exists a point $a_3 > a_1$ such that

$\int_{a1}^{a3}(1/g(t))dt = a_2$. But we know $y(x)$ is continuous and not zero in $[a_1, a_3]$, and an inspection of the transformations used to obtain (2.2) from (1.2) shows that $z(x)$ must also be continuous in the interval $[a_1, a_3]$. Hence $|z(x)|$ must be bounded in $[a_1, a_3]$. But (2.3) shows $z(x)$ assumes arbitrarily large positive values in $[a_1, a_3)$, a contradiction. This proves the theorem.

3. In this section we prove another oscillation theorem for equation (1.2).

THEOREM 2. *If*

1. $y(x)$ *is a solution of* (1.2) *existing for* $0 < x < \infty$,
2. $f'(y) \geqq \gamma_1 > 0$ *for* $-\infty < y < \infty$, γ_1 *a constant,*
3. $f(y) = 0$ *if and only if* $y = 0$,
4. *for some* $a > 0$ $k(x, \gamma) \equiv \int_a^x [g(t)p(t) - \frac{1}{4}(g'(t))^2/\gamma g(t)]\,dt$ *is such that* $k(x, \gamma_1) - \frac{1}{2}(|g'(x)|)/\gamma_1 \to \infty$ *monotonely as* $x \to \infty$, *where*
5. $g(x)$ *is as in hypothesis* 5, *Theorem* 1, *then* $y(x)$ *is an oscillatory solution of* (1.2). *It is to be noted that there is no restriction on the sign of* $p(x)$.

The proof is similar to the proof of Theorem 1. As before we assume the existence of a last zero of the solution $y(x)$ of hypothesis 1 at x_0, $0 < x_0 < a$ and show this leads to a contradiction. By use of hypothesis 3 we see a new dependent variable $w = w(x)$ is well defined for equation (1.2) by $w(x)/g(x) = -y'(x)/f(y(x))$, $x \geqq a$. Equation (1.2) becomes

$$w'(x) = [f'(y(x))/g(x)]\,[w(x) + \tfrac{1}{2}(g'(x))/f'(y(x))]^2 + g(x)p(x) - \tfrac{1}{4}(g'(x)^2/g(x)f'(y(x)) , \qquad x \geqq a . \tag{3.1}$$

By setting $w(x) = z(x) + h_1(x)$ where

$$h_1(x) \equiv \int_a^x [g(t)p(t) - \tfrac{1}{4}(g'(t))^2/g(t)f'(y(t))]dt$$

we get

$$z'(x) = [f'(y(x))/g(x)]\,[z(x) + H(x)]^2 , \quad H(x) = h_1(x) + \tfrac{1}{2}(g'(x))/f'(y(x)) , \qquad x \geqq a . \tag{3.2}$$

By combining hypotheses 2 and 4 and noting $z(x)$ is an increasing function we see there exists a point $a_1 > a$ such that for $x \geqq a_1$

$$\begin{aligned} z(x) + H(x) &\geqq z(x) + k(x, \gamma_1) - \tfrac{1}{2}(|g'(x)|)/\gamma_1 \\ &\geqq z(a_1) + k(a_1, \gamma_1) - \tfrac{1}{2}(|g'(a_1)|)/\gamma_1 = 1/a_2 > 0 ; \end{aligned}$$

therefore we have

$$z'(x) \geqq (\gamma_1/g(x))[z(x) + k(a_1, \gamma_1) - \tfrac{1}{2}(|g'(a_1)|)/\gamma_1]^2$$

and, integrating from a_1 to $x > a_1$ we get

$$z(x) \geqq -\left\{(1/a_2) - z(a_1) + 1\Big/\left[a_2 - \gamma_1\int_{a_1}^{x}(1/g(t))dt\right]\right\} \quad \text{for } x \geqq a_1 . \tag{3.3}$$

By use of hypothesis 5 we have the existence of a point $a_3 > a_1$ such that $a_2 = \gamma_1\int_{a_1}^{a_3}(1/g(t))dt$. Hence we reach a contradiction as before by noting that $z(x)$ is continuous in a closed interval, by its construction, but unbounded by (3.3), an impossibility. This proves the theorem.

4. In this section we prove an oscillation theorem that applies to equation (1.1). Clearly, neither of the two theorems proved so far may be used to give results concerning the oscillatory behavior of solutions of equation (1.1).

THEOREM 3. *If*

1. $y(x)$ *is a solution of* (1.2) *existing for* $0 < x < \infty$,
2. $f(y)$ *is odd and* $f(y) > 0$ *if* $y > 0$,
3. $f(y)/y \geqq \gamma_1 > 0$ *if* $y \geqq \gamma_2 > 0$, γ_1 *and* γ_2 *constants,*
4. *for some* $a > 0$ $k(x) \equiv \int_a^x[\gamma g(t)p(t) - \tfrac{1}{4}(g'(t))^2/g(t)]dt + \tfrac{1}{2}(g'(x)) \to \infty$ *monotonely as* $x \to \infty$, *for any* $\gamma > 0$, *where*
5. $p(x) \geqq 0$ *for* $a \leqq x < \infty$, *and*
6. $g(x)$ *is as in hypothesis* 5, *Theorem* 1, *then* $y(x)$ *is an oscillatory solution of* (1.2).

The proof is almost a duplicate of the proof of Theorem 1. If $y(x)$ is assumed to have a last zero at x_0 where $0 < x_0 < a$ then there is no loss in generality in assuming that $y(x)$ eventually positive, since $f(y)$ is odd. If $y(a) = \gamma_2 > 0$ then since $y''(x) = -f(y(x))p(x) \leqq 0$ for $x \geqq a$, $y(x)$ must be an increasing function, for if $y'(x) < 0$ for some $x \geqq a$ then, using $y''(x) \leqq 0$ we have $y(x) = 0$ for some $x > a$, a contradiction. Hence $y(x) \geqq \gamma_2$ for $x \geqq a$. From this point the remainder of the proof parallels the proof of Theorem 1 exactly and details are accordingly omitted.

5. In this section we prove a final oscillation theorem for equations of type (1.2).

THEOREM 4. *If*

1. $y(x)$ *is a solution of* (1.2) *existing for* $0 < x < \infty$,

2. *$f(y) = 0$ if and only if $y = 0$,*
3. *$\gamma_2 \geqq f'(y) \geqq \gamma_1 > 0$ for $-\infty < y < \infty$, γ_1 and γ_2 constants,*
4. *for some $a > 0$ $k(x, \gamma)$ is as in hypothesis 4, Theorem 2, with*

$$\alpha = \limsup_{x \to \infty} [k(x, \gamma_1) - \tfrac{1}{2}(|g'(x)|)/\gamma_1] >$$
$$\beta = \liminf_{x \to \infty} [k(x, \gamma_2) + \tfrac{1}{2}(|g'(x)|)/\gamma_1] ,$$
$$\liminf_{x \to \infty} k(x, \gamma_1) > -\infty, \text{ where}$$

5. *$g(x) > 0$, $g(x) \in C^2$ for $a \leqq x < \infty$, $\int_a^x (1/g(t))dt \to \infty$ as $x \to \infty$, there exists a constant L such that $g(x) + |g'(x)| + |g''(x)| < L$ for $a \leqq x < \infty$, and*
6. *either the integrand of $k(x, \gamma_1)$ is bounded universally from below, or the integrand of $k(x, \gamma_2)$ is bounded universally from above, for $a \leqq x < \infty$, then $y(x)$ is an oscillatory solution of* (1.2).

The proof is again by contradiction. Suppose, as before, $y(x)$ has a last zero at x_0, $0 < x_0 < a$. Then we repeat the argument initially used in the proof of Theorem 2 and transform (1.2) into, for $x \geqq a$,

$$z'(x) = [f'(y(x))/g(x)]\,[z(x) + H(x)]^2 \tag{5.1}$$

where

$$H(x) \equiv \int_a^x [g(t)p(t) - \tfrac{1}{4}(g'(t))^2/g(t)f'(y(t))]dt + \tfrac{1}{2}(g'(x))/f'(y(x)) .$$

It is readily verified by use of hypotheses 3, 4, 5 and 6 that

$$H(x) \geqq k(x, \gamma_1) - \tfrac{1}{2}(|g'(x)|)/\gamma_1,\ k(x, \gamma_2) + \tfrac{1}{2}(|g'(x)|)/\gamma_1 \geqq H(x),\ x \geqq a, \tag{5.2}$$

and

$$\liminf_{x \to \infty} H(x) > -\infty .$$

We shall next show that $\lim_{x\to\infty} z(x) = \infty$. Suppose this is not so. Since $z(x)$ increases, $\lim_{x\to\infty} z(x)$ would exist and have value N, say. Then $z(x) = N - \varepsilon(x)$, where $\varepsilon(x) \to 0^+$ as $x \to \infty$. We shall only treat the case in which the integrand of $k(x, \gamma_1)$ is bounded universally from below, by $-M^2$, say, since the details for the case of the integrand bounded from above are completely analogous. There are several possibilities, of which only two need be considered. Details are provided below for the situations in which $N + \alpha = \delta_1 > 0$, or $N + \beta = \delta_2 < 0$, for if $N + \alpha \leqq 0$, then $N + \beta < 0$ or if $N + \beta \geqq 0$, then $N + \alpha > 0$.

In this paragraph we treat the case in which $N + \alpha = \delta_1 > 0$, and the integrand, of $k(x, \gamma_1)$ is bounded from below by $-M^2$. Suppose for the moment that $\alpha < \infty$. By definition of α we know there exists a sequence of points, $\{b_i\}$ say, where $b_1 > a$, $b_{i+1} > b_i + 1$, $i = 1, 2, \cdots$, and $k(b_i, \gamma_1) - \frac{1}{2}(|g'(b_i)|)/\gamma_1 > \alpha - \delta_1/4$. Moreover we have, for all i sufficiently large, $z(b_i) > N - \delta_1/4$, since $z(x) = N - \varepsilon(x)$, where $\varepsilon(x) \to 0^+$ as $x \to \infty$. Now consider the interval $I_i = [b_i, b_i + \xi]$ where $0 < \xi < 1$, for the moment, but to be chosen specifically below. For any $x \in I_i$ we have $k(x, \gamma_1) = k(b_i, \gamma_1) + \int_{b_i}^{x} [g(t)p(t) - \frac{1}{4}(g'(t))^2/\gamma_1 g(t)]\,dt \geqq k(b_i, \gamma_1) + (\xi)(-M^2)$; and $\frac{1}{2}(|g'(x)|)/\gamma_1 \leqq \frac{1}{2}(|g'(b_i)|)/\gamma_1 + (L/2)(\xi/\gamma_1)$ by an application of the mean value theorem. Combining our results we have that for any i sufficiently large, if $x \in I_i$, then $z(x) + H(x) \geqq z(x) + k(x, \gamma_1) - \frac{1}{2}(|g'(x)|)/\gamma_1 \geqq z(b_i) + k(b_i, \gamma_1) - \xi M^2 - \frac{1}{2}(|g'(b_i)|)/\gamma_1 - (L/2)(\xi/\gamma_1) \geqq N - \delta_1/4 + \alpha - \delta_1/4 - \xi M^2 - (L/2)(\xi/\gamma_1) = \delta_1/2 - \xi M^2 - (L/2)(\xi/\gamma_1)$. Now choose ξ so small that $\xi M^2 + (L/2)(\xi/\gamma_1) < \delta_1/4$. Hence if $x \in I_i$, (any i sufficiently large) $z'(x) > (\gamma_1/L)(\delta_1^2/16)$ and we conclude as $x \to \infty$ $z'(x)$ is bounded away from zero on a set whose measure $\to \infty$. Hence $z(x)$ eventually exceeds N, a contradiction. Thus $\lim z(x)_{x\to\infty} = \infty$. If $\alpha = \infty$ we need merely replace δ_1 by any positive number greater than $N + \beta$ and repeat the argument.

In this paragraph we treat the case in which $N + \beta = \delta_2 < 0$, and the integrand of $k(x, \gamma_1)$ is bounded from below by $-M^2$. We note first that since $k(x, \gamma_1) \leqq k(x, \gamma_2)$ that $\beta > -\infty$ by use of hypythesis 4, and by use of the definition of $k(x, \gamma)$ that the integrand of $k(x, \gamma_2)$ is bounded below by $-M^2$ also. As before, by definition of β we know there exists a sequence of points, $\{b_i\}$ say, where $b_1 > a$, $b_{i+1} > b_i + 1$, $i = 1, 2, \cdots$, such that $k(b_i, \gamma_2) + \frac{1}{2}(|g'(b_i)|)/\gamma_1 < \beta - \delta_2/2$. Now consider the interval $I_i = [b_i - \xi, b_i]$ where $0 < \xi < 1$, for the moment, but to be chosen specifically below. For any $x \in I_i$ we have $k(x, \gamma_2) = k(b_i, \gamma_2) - \int_{x}^{b_i} [g(t)p(t) - \frac{1}{4}(g'(t))^2/g(t)\gamma_2]\,dt$. But $\int_{x}^{b_i} [g(t)p(t) - \frac{1}{4}(g'(t))^2/g(t)\gamma_2]\,dt \geqq (b_i - x)(-M^2) \geqq -\xi M^2$. Hence $k(x, \gamma_2) \leqq k(b_i, \gamma_2) + \xi M^2$. Combining our results we have that for any i sufficiently large, if $x \in I_i$, then $z(x) + H(x) \leqq z(x) + k(x, \gamma_2) + \frac{1}{2}(|g'(x)|)/\gamma_1 \leqq N + k(b_i, \gamma_2) + \xi M^2 + \frac{1}{2}(|g'(b_i)|)/\gamma_1 + (L/2)(\xi/\gamma_1) \leqq N + \beta - \delta_2/2 + \xi M^2 + (L/2)(\xi/\gamma_1) = \delta_2/2 + \xi M^2 + (L/2)(\xi/\gamma_1)$. Now choose ξ so small that $\xi M^2 + (L/2)(\xi/\gamma_1) < -\delta_2/4$. Hence if $x \in I_i$, (any i sufficiently large) $z'(x) \geqq (\gamma_1/L)(\delta_2^2/16)$. As before we have, eventually, $z(x) > N$, a contradiction. Hence we must have $\lim_{x\to\infty} z(x) = \infty$.

Now the proof may be concluded just as in the proof of Theorem 2. For we have, since $\lim_{x\to\infty} z(x) = \infty$, and $\liminf_{x\to\infty} H(x) = \zeta > -\infty$, that there exists a point $a_1 > a$ such that, for $x \geqq a_1$, $z(x) + H(x) \geqq z(x) + \zeta - 1 \geqq z(a_1) + \zeta - 1 = 1/a_2 > 0$. Hence for $x \geqq a_1$, $z'(x) \geqq [\gamma_1/g(x)]\,[z(x) + \zeta - 1]^2$ and we have, after an integration,

$$z(x) \geqq -[(1/a_1) - z(a_1)] + 1\Big/\Big[a_2 - \gamma_1 \int_{a_1}^{x} (1/g(t))\,dt\Big] \text{ for } x \geqq a_1 .$$

As before a contradiction is easily reached from this last inequality. The details are omitted. This concludes the proof of Theorem 4.

6. The same type of reasoning may be used to show the existence of nonoscillatory solutions of equations of the form (1.2). The following two lemmas will be found useful.

LEMMA 1. *If*

1. $y(x)$ *is a solution of equation* (1.2) *existing for* $0 < x < \infty$,
2. $y(x_0) \neq 0$, $y(x_1) = 0$, *with* x_1 *the first root of the equation* $y(x) = 0$ *to the right of* x_0,
3. $f(y) = 0$ *if and only if* $y = 0$,
4. $u_1(x) = |y'(x)/y(x)|$, $u_2(x) = |y'(x)/f(y(x))|$ *then* $\limsup_{x \to x_1^-} u_1(x) = \limsup_{x \to x_1^-} u_2(x) = \infty$.

It is clear that if we prove the result for $u_2(x)$ we shall have proved it for $u_1(x)$, by choosing $f(y) \equiv y$. The proof is by contradiction. Suppose there exists a constant $k > 0$ such that $u_2(x) > k$ for all x in $[x_0, x_1)$. For such x we have $|y'(x)/f(y(x))| < k$ so $|[y'(x)f'(y(x))]/f(y(x))| \leqq k|f'(y(x))| < k_1$ by the continuity of the function $f'(y(x))$ in $[x_0, x_1]$. Thus we have $\left|\int_{x_0}^{x} [y'(t)f'(y(t))/f(y(t))]dt\right| = |Ln|f(y(t))|_{x0}^{x}| \leqq k_1(x - x_0)$ which is a contradiction, since as $x \to x_1^-$ we have $|Ln|f(y(t))|_{x_0}^{x}| \to \infty$ by use of hypothesis 2 and hypothesis 3, while the right hand side of the last inequality is bounded. This proves the lemma.

LEMMA 2. *If*

1. $z(x)$ *is a solution of the differential equation* $z'(x) = F(x, z(x))$ *for* $0 < a \leqq x \leqq b$, *where* $f(x, z)$ *is continuous in the* $x - z$ *plane.*
2. *there exists a function* $\varphi(x)$ *of class* C' *for* $a \leqq x \leqq b$ *such that* $\varphi'(x) > F(x, \varphi(x))$ *for* $a \leqq x \leqq b$ *with* $\varphi(a) \geqq z(a)$, *then* $\varphi(x) > z(x)$, *for* $a < x \leqq b$.

For a proof of this result see [3].

We have the following nonoscillation theorem as typical of the results that can be obtained.

THEOREM 5. *If*

1. $y(x)$ *is a solution of equation* (1.2) *existing for* $0 < x < \infty$, *with* $y(a) \neq 0$, $\underline{a}$ *some positive number,*
2. $g(x) > 0$, $g(x) \in C'$ *for* $a \leqq x < \infty$,
3. *for every* $b > a$, $h(x) \equiv \int_a^x [g(t)p(t)f(y(t))/y(t) - \frac{1}{4}(g'(t))^2/g(t)]dt + \frac{1}{2}g'(x)$, *where* $a \leqq x \leqq b$, *can be appraised so that one can dem-*

onstrate the existence of functions $\varphi(x)$ *and* $\psi(x)$, *of class* C' *and* C *for* $a \leqq x \leqq b$, *respectively, with the properties that, for* $a \leqq x \leqq b$, $-\varphi(x) < h(x) < -\varphi(x) + \psi(x)$, $\varphi'(x) \geqq (1/g(x)(\psi(x))^2$, *and* $\varphi(a) = c$, c *an arbitrarily large positive number, then* $y(x)$ *is a nonoscillatory solution.*

The proof is by contradiction. Suppose $y(x)$ is an oscillatory solution. Then there exists a first zero of the equation $y(x) = 0$ after the point $\underline{a}$ mentioned in hypothesis 1, at $x = b$, say. Therefore for x in $a \leqq x < b$ we can transform equation (1.2) into equation (2.2) as in the proof of Theorem 1. By use of Lemma 1 we see (with the notation of Theorem 1) that $|u(x)| = |-y'(x)/y(x)|$ must assume arbitrarily large values in $[a, b)$. The same must be true of $|z(x)|$ by an inspection of the transformations used to obtain (2.2) for (1.2) and our knowledge, by use of hypothesis 3, that $h(x)$ is bounded in $[a, b]$. But from hypothesis 3 we have the existence of a function $\varphi(x)$ such that $\varphi'(x) \geqq 1/g(x)[\psi(x)]^2 > (\varphi(x) + h(x))^2$ with $\varphi(a) = c \geqq z(a)$ since c was arbitrarily large and positive. Hence by Lemma 2 $z(x)$, the solution of (2.2), is bounded for $a \leqq x \leqq b$, a contradiction. This proves the theorem.

7. Precisely the same type of reasoning may be used to prove the following nonoscillation theorem.

THEOREM 6. *If*

1. $y(x)$ *is a solution of equation* (1.2) *existing for* $0 < x < \infty$, *with* $y(a) \neq 0$, $\underline{a}$ *some positive number,*
2. $g(x) > 0$, $g(x) \in C^1$ *for* $a \leqq x < \infty$,
3. $\gamma_1 \geqq f'(y)$ *for* $-\infty < y < \infty$, *and* $y = 0$ *is an isolated zero of the equation* $f(y) = 0$,
4. *for every* $b > a$, $H(x) \equiv \int_a^x [g(t)p(t) - \frac{1}{4}(g'(t))^2/g(t)f'(y(t))]\,dt + \frac{1}{2}(g'(x))/f'(y(x))$, *where* $a \leqq x \leqq b$, *can be appraised so that one can demonstrate the existence of functions* $\varphi(x)$ *and* $\psi(x)$, *of class* C' *and* C *for* $a \leqq x \leqq b$, *respectively, with the properties that, for* $a \leqq x \leqq b$, $-\varphi(x) < H(x) < -\varphi(x) + \psi(x)$, $\varphi'(x) \geqq (\gamma_1/g(x))(\psi(x))^2$, *and* $\varphi(a) = c$, c *an arbitrarily large positive number, then* $y(x)$ *is a nonoscillatory solution.*

The proof is very similar to the proof of Theorem 5. Suppose $y(x)$ is an oscillatory solution. Then there exists a first zero of the equation $y(x) = 0$ after the point $\underline{a}$ mentioned in hypothesis 1, at $x = b$, say. Assume that $\underline{a}$ is so close to b that $f(y(x)) \neq 0$ for $a \leqq x < b$ and then transform equation (1.2) into equation (3.2) as in the proof of Theorem 2. It is easy to modify the proof of Lemma 1 so that the conclusion of that lemma holds under the hypotheses on $f(y)$ as given in hypothesis 3 above. As in the proof of Theorem 5 we can conclude that $|z(x)|$ must assume arbitrarily large values in $[a, b)$. But by hypothesis 4 we have the existence of a function $\varphi(x)$ such that $\varphi'(x) \geqq (\gamma_1/g(x))(\psi(x))^2 > [f'(y(x))/g(x)][\varphi(x) + H(x)]^2$ with $\varphi(a) = c \geqq z(a)$. Hence, by Lemma 2, $z(x)$ is bounded in $[a, b]$, a contradiction. This proves the theorem.

References

1. F. V. Atkinson, *On second-order non-linear oscillations*, Pacific J. Math., **5** (1955), 643-647.
2. R. Bellman, *Stability theory of differential equations*, McGraw-Hill, 1953, New York.
3. N. G. de Bruijn, *Asymptotic methods in Analysis*, North-Holland, 1958, Amsterdam.
4. E. Hille, *Non-Oscillation theorems*, Trans. Amer. Math. Soc., **64** (1948), 234-252.
5. W. Leighton, *The detection of the oscillation of solutions of a second-order linear differential equation*, Duke Math. Journal, **17**, No. 1, (1950), 57-62.
6. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of non-linear differential equations*, Trans. Amer. Math. Soc., **93**, No. 1, (1959), 30-52.
7. C. Olech, Z. Opial, and T. Wazewski, *Sur le Probleme d'oscillation des integrales de l'equation* $y''+g(t)y=0$, Bulletin de l'Academie Polonaise des Sciences, Classe Troisieme, **5** (1957), 621-626.
8. Z. Opial, *Sur une Critere d'oscillation des Integrales de l'equation Differentielle* $(Q(t)x')'+f(t)x=0$, Annales Polonici Mathematici, **6** (1959-60), 99-104.

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