CLOSED EXTENSIONS OF THE LAPLACE OPERATOR DETERMINED BY A GENERAL CLASS OF BOUNDARY CONDITIONS, FOR UNBOUNDED REGIONS

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1. Introduction. In [1] certain closed operators in $L^2(G)$ were defined in terms of the Laplacian operator, and a class of generalized boundary conditions. There G was a bounded domain in E^n with $C^{1.1}$ boundary and the boundary conditions were of the form $\partial u/\partial n - Lu = 0$, where L was an arbitrary bounded operator on $L^2(\partial G)$. The operator so defined was denoted by T_L and it was shown that T_L was a closed linear operator with compact resolvent. Its adjoint $(T_L)^*$ was shown to be T_{L*} and its spectrum was shown to lie inside a parabola of the form

$$y^2 = ax + b \text{ where } a, b > 0 .$$

It is the purpose of this paper to extend the results of [1] by removing the restriction that G be bounded. The previous history of the problem is adequately covered in [1]. It is worth noting here that the underlying idea of [1] derives from a theorem of Calkin [4] and relies on the compactness of the resolvent of an operator S on $L^2(G) \oplus L^2(\partial G)$. For certain of the results of [1] it is only necessary to assume the closedness of the range of S but many of the results explicitly used the compactness of the resolvent. If G is unbounded, the resolvent of S is no longer compact and the methods of [1] can no longer be used.

The approach taken here is to use localization methods to reduce the problem for unbounded regions to one for bounded regions. We observe that the Laplacian is an operator of local type in the sense of Hörmander [7]. However in applying these methods we must distinguish between two cases. If ∂G is bounded we can separate the behavior of functions near the boundary from their behavior near infinity by multiplying them by functions which vanish in a neighborhood of infinity and are identically one near the boundary. When ∂G is unbounded this can no longer be done and any such attempt at localization must affect the behavior of the function on portions of the boundary. It turns out to be necessary to require that any localization function, regarded as an operator on $L^2(\partial G)$, must commute with the boundary operator L. For this reason when ∂G is unbounded we restrict our attention to

multiplication operators on the boundary.

It is our hope for the future to extend these results to uniformly elliptic, second-order operators on $L^p(G)$ where $1 < p < \infty$.

In § 1 we list some geometric preliminaries. In § 2 we list results centering about the Lax-Milgram theorem [8] and preliminary localization results. The main results are contained in §§ 3 and 4. In § 3 we treat operators for unbounded regions with bounded boundaries and in § 4 we treat the situation where the boundary is unbounded.

1. Preliminaries. In what follows G will be a possibly unbounded open connected set in E^n. Its boundary will be denoted by ∂G. Points in G will be denoted by $x, y, \cdots$ and points in ∂G will be denoted by $\theta, \phi, \cdots$. We will denote by $dx, dy, \cdots$ and by $d\theta, d\phi, \cdots$ n-dimensional and $n - 1$ dimensional Lebesgue measures on G and ∂G, respectively. The exterior unit normal to ∂G at θ will be denoted by n_θ. As in [1] we will always assume that ∂G is an $n - 1$ dimensional $C^{1,1}$ manifold. A detailed treatment of such manifolds may be found in Lucas [9]. For convenience we will state the fundamental

1.1. THEOREM. *Let Γ be a $C^{1,1}$ manifold, then there exists a positive number, r_0, called the minimal radius of Γ such that:*

(i) *All segments of length r_0 centered at points of Γ and normal to Γ are mutually disjoint and exhaust a neighborhood of Γ, called an r_0 neighborhood of Γ.*

(ii) *At each point θ in Γ, spheres interior and exterior to Γ of radius r_0 may be drawn tangent to Γ at θ and containing no other points of Γ.*

(iii) *The normals to Γ satisfy a uniform Lipschitz condition with Lipschitz constant r_0^{-1}, i.e.,*

$$| n_\theta - n_\phi | \leqq r_0^{-1} | \theta - \phi |$$

for $\phi, \theta \in \partial G$.

(iv) *The set $\Gamma_\rho = \{\theta - \rho n_\theta : \theta \in \Gamma\}$ is a $C^{1,1}$ manifold for $\rho < r_0$ with minimal radius r_ρ satisfying $r_0 - | \rho | \leqq r_\rho \leqq r_0 + | \rho |$. Such surfaces will be called parallel surfaces to Γ and are denoted by Γ_ρ.*

(v) *The surface measures on Γ and Γ_ρ are mutually absolutely continuous with uniformly bounded Radon-Nikodym derivatives for $| \rho | < r_0/2$.*

We impose further restrictions on ∂G. We require that there exists a covering of every r neighborhood of ∂G, for $r < r_0$, by a possibly finite sequence of open sets $\{V_k\}$ such that

(a) For each k there exists a homeomorphism, T_k of V_k into E^n such that $T_k(V_k \cap G) = \Gamma \times \{0 < x_n < r\}$ where Γ is an $n - 1$ dimensional

sphere with center at the origin and diameter r.

(b) T_k, T_k^{-1} and their derivatives up to order j satisfy a uniform Lipschitz condition.

(c) There exists an integer N such that at most N of the sets $\{V_k\}$ have a nonempty intersection.

(d) $\bigcup_k T_k^{-1}\{1/2\Gamma x[0 < x_n < r/2]\}$ covers an $r/2$ neighborhood of ∂G.

1.2. DEFINITION. If ∂G satisfies conditions (a) $-$ (d) above we say that G is a *uniformly $C^{j,1}$ domain.*

Usually j can be 1, however later in § 4 we shall need j to be two. If the derivatives up to order j of T_k, T_k^{-1} are merely continuous and bounded, we speak of a *uniformly C^j domain.*

In addition to the function space $L^2(G)$ we refer to certain other function spaces which are by now quite familiar. The space $C_0^\infty(G)$ is the set of all infinitely differentiable functions with compact support in G. We shall always consider complex valued functions unless stated to the contrary. In the notation of Browder [2], $W^{j,p}(G)$ is the set of all functions in $L^p(G)$ whose distribution valued derivatives up to order j are functions in $L^p(G)$. Using the familiar notation

$$\alpha = \langle \alpha_1, \cdots, \alpha_n \rangle \text{ and } |\alpha| = \sum_{i=1}^n \alpha_i$$

where the α_i are nonnegative integers, we write

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

for the generic differential operator of order $|\alpha|$. For $u \in W^{j,p}(G)$ we write

$$\| u \|_{j,p} = \left(\sum_{|\alpha| \leq j} \| D^\alpha u \|_p^p \right)^{1/p}, \qquad 1 < p < \infty$$

where $\| \cdot \|_p$ is the L^p norm. Here we shall mainly be concerned with the L^2 norm which we shall write as $\| \cdot \|$. The following well known facts we state as a

1.3. LEMMA. *$W^{j,p}(G)$ is a Banach space in the norm $\| \cdot \|_{j,p}$, and a Hilbert space for $p = 2$. $C_0^\infty(E^n)$ is dense in $W^{j,p}(G)$ in the $\| \cdot \|_{j,p}$ norm.*

Let Γ be a subset of ∂G. For $0 < \rho < r_0$ and for $\phi \in \Gamma$ define the set G_ρ to be $\{x \in G : x = \phi - \rho n_\phi\}$. For fixed $\rho < r_0$ let $\Gamma_\rho = \{x \in G : x = \phi - \rho n_\phi, \phi \in \Gamma\}$. If u is a function in $W^{1,p}(G_\rho)$ we write $u_\rho(\phi) = u(\phi - \rho n_\phi)$, $\phi \in \Gamma$. Thus u_ρ is the restriction of u to Γ_ρ. We now make the following definition

1.4. DEFINITION. u_ρ has an L^p-*boundary value* on Γ if

(i) $u_\rho \in L^p(\Gamma_\rho)$ for $\rho < r_0$.

(ii) There exists a function $\tilde{u} \in L^p(\Gamma)$ such that

$$\lim_{\rho \to 0} \int_\Gamma |u_\rho(\phi) - \tilde{u}(\phi)|^p \, d\phi = 0 .$$

We now introduce the class of functions with which we shall be dealing throughout the remainder of this paper.

1.5. DEFINITION. $\mathscr{D}_1(G)$ is the set of all functions u in $W^{1,2}(G)$ such that

(i) $u \in W^{2,2}(G_1)$ for each open set G_1 whose closure is compact and lies in G,

(ii) $\varDelta u \in L^2(G)$,

(iii) u has an L^2 boundary value $\tilde{u}$ on ∂G, and $\partial u/\partial n$ has an L^2 boundary value on every compact subset Γ of ∂G.

It is well known [5] that every function in $W^{1,2}$ has an L^2 boundary value and an examination of harmonic functions on the unit circle shows that the inclusion $W^{2,2}(G) \subseteqq D_1(G) \subseteqq W^{1,2}(G)$ is proper.

Recall that the "minimal operator" associated with the Laplacian is the closure of the Laplacian restricted to $C_0^\infty(G)$ and the "maximal operator" is its adjoint. Its domain is the set of functions in $W^{1,2}(G)$ satisfying conditions (i) and (ii) of Definition 1.5.

2. **Lax-Milgram theorem.** We here present the Lax-Milgram theorem in a form suitable for our purposes. It is the generalization of the Friedrichs' extension theorem [6] to the nonsymmetric case.

Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Hilbert spaces with norms $\| \cdot \|$ and $\| \cdot \|_1$, respectively. We suppose

(i) $\mathscr{H}_1$ is densely contained in $\mathscr{H}$.

(ii) There exists a number $k > 0$ such that for all u in $\mathscr{H}_1$

$$k \| u \| \leqq \| u \|_1 .$$

Let B be a not necessarily Hermitian symmetric bilinear form whose domain is all of $\mathscr{H}_1$ and suppose

(a) $|B(u, v)| \leqq K \| u \|_1 \| v \|_1$ for $u, v \in \mathscr{H}_1$

(b) $Re\, B(u, u) \geqq C \| u \|_1^2$ for $u \in \mathscr{H}_1$.

Lax and Milgram dealt with real Hilbert spaces whereas here and throughout this paper we deal with complex Hilbert spaces. Condition (b) is the obvious modification of their condition, and the necessary changes to their proof resulting from (b) are of a trivial nature.

    2.1. THEOREM (*Lax-Milgram*). *For any bounded linear functional*
l on $\mathscr{H}_1$ *there exist two vectors* v, v^* *in* $\mathscr{H}_1$ *such that*

    (i) $l(u) = B(u, v) = B(v^*, u)$ *for* $u \in \mathscr{H}_1$.

    (ii) *If* $w \in \mathscr{H}_1$ *and* $l = (\cdot, w)_1$ *and* $v = P_1 w$, *then* P_1 *is a linear*
homeomorphism of $\mathscr{H}_1$ *onto* $\mathscr{H}_1$.

    (iii) *If* U *is a proper closed subspace of* $\mathscr{H}_1$, *then there exists a*
$v \in \mathscr{H}_1$ *such that for every* $u \in U$, $B(u, v) = 0$.

    2.2. THEOREM. *There exists a closed densely defined linear trans-*
formation T *of* $\mathscr{H}$ *into itself such that* T *and its adjoint* T^* *have*
domains contained in $\mathscr{H}_1$ *and have bounded everywhere defined inverses*
in $\mathscr{H}$. *Moreover*

$$B(u, v) = (u, Tv) \ for \ u \in \mathscr{H}_1, v \in \mathscr{D}(T)$$

and

$$B(u, v) \quad (T^*u, v) \ for \ u \in \mathscr{D}(T^*), v \in \mathscr{H}_1 \ .$$

    The fact the T is densely defined in $\mathscr{H}$ and closed as a map of $\mathscr{H}$
into $\mathscr{H}$ was not explicitly stated in Lax-Milgram. The density follows
from the fact that T^{-1} is everywhere defined in $\mathscr{H}$ and bounded. In
fact if z is orthogonal to $\mathscr{D}(T)$ then for $u \in \mathscr{D}(T)$, $(u, z) = 0$ and since
$(., z)$ is a bounded linear functional on $\mathscr{H}_1$ and $\mathscr{D}(T) \subseteqq \mathscr{H}_1$ it follows
that $(u, z) = B(u, T^{-1}z) = 0$ for $u \in \mathscr{D}(T) = \mathscr{R}(T^{-1})$. Thus $B(T^{-1}u, T^{-1}z) =$
0 and in particular $B(T^{-1}z)^1 = 0$ and thus $T^{-1}z = 0$. Hence $z = 0$. To
show that T is closed as a map of $\mathscr{H}$ into $\mathscr{H}$, let $\{v_n\}$ be a sequence
in $\mathscr{D}(T)$ and suppose $v_n \to v \in \mathscr{H}$ and $Tv_n \to w \in \mathscr{H}$. Now for each
n and all $u \in \mathscr{H}_1$ $B(u, v_n) = (u, Tv_n)$ and thus $B(v_n) = (v_n, Tv_n) \to (v, w)$.
But then v_n converges in $\mathscr{H}_1$ to say z and thus in $\mathscr{H}$. Hence $v = z$.
Since T is closed as a map of $\mathscr{H}_1$ into $\mathscr{H}$, $Tz = w = Tv$.

    The facts about T^* follow easily if one recalls that T^{-1} is every-
where defined and bounded.

    We shall now apply Theorem 2.2 to a study of the Laplacian oper-
ator, considered as an operator on $L^2(G)$. With the aid of Theorem 2.2
we shall prove the existence of a closed densely defined operator with
domain in $W^{1,2}(G)$ and satisfying certain boundary conditions in a gener-
alized sense. Since the domains lie in $W^{1,2}(G)$ we know that the functions
have L^2 boundary values. It will then be the task of the remainder of
this paper to show in what sense the boundary conditions are satisfied.
To begin we require the following

    2.3. LEMMA. *Let* G *be a uniformly* $C^{1,1}$ *manifold. Then for any*
$u \in W^{1,2}(G)$, u *has an* L^2 *boundary value* $\tilde{u}$ *and for each* $\varepsilon > 0$, *there*
exists a $C(\varepsilon) > 0$ *such that* $\| \tilde{u} \|_0^2 \leqq K[\varepsilon \| \nabla u \|^2 + C(\varepsilon) \| u \|^2]$.

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[1] We write $B(x)$ for $B(x_j, x)$.

Here ∇ represents the gradient of u and $\| \cdot \|_\partial$ is the norm on $L^2(\partial G)$.

This lemma appears widely in the literature, see for example Ehrling [5] and Nirenberg [10] where G is assumed to be bounded. When G is unbounded the proof can be suitably modified without too much difficulty with the aid of a partition of unity. For an outline of this proof see Poulsen [11].

Suppose u is a smooth function and that L is a bounded linear operator on $L^2(\partial G)$ and that $\tilde{u}_n - L\tilde{u} = 0$. Let

$$\int_G \nabla u \cdot \overline{\nabla v}\, dx = \int_G \sum_{i=1}^n \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_i}}\, dx$$

and apply Green's formula to obtain

$$(-\Delta u, u) = \| \nabla u \|^2 - (L\tilde{u}, \tilde{u})_\partial .$$

2.4. Definition. For $u \in W^{1,2}(G)$ let

$$B_G(u) = \| \nabla u \|^2 - (L\tilde{u}, \tilde{u})_\partial$$
$$B_G^\gamma(u) = B_G(u) + \gamma \| u \|^2$$

where γ is any complex number. Let L be any bounded operator on a Hilbert space $\mathscr{H}$. Form $L_1 = 1/2(L + L^*)$ and $L_2 = 1/2(L - L^*)$ so that $L_1^* = L_1$, $L_2^* = -L_2$ and $L = L_1 + L_2$. Moreover, $R_l(Lu, u) = (L_1 u, u)$ and $Im(Lu, u) = 1/i(L_2 u, u)$. L_1 and L_2 are respectively the symmetric and antisymmetric parts of L.

2.5. Lemma. *If the real part of γ is sufficiently large, then for all $u \in W^{1,2}(G)$*

$$R_l B_G^\gamma(u) \geqq \frac{1}{2} \| u \|_{1,2}^2 .$$

Proof. Let $R_l \gamma = \alpha$. Then

$$R_l B_G^\gamma(u) = \| \nabla u \|^2 + \alpha \| u \|^2 - (L_1 \tilde{u}, \tilde{u})_\partial$$
$$\geqq \| \nabla u \|^2 + \alpha \| u \|^2 - \| L_1 \| \, \| \tilde{u} \|_\partial^2 .$$

Now by Lemma 2.3

$$R_l B_G^\gamma(u) \geqq \| \nabla u \|^2 + \alpha \| u \|^2 - \| L_1 \| K[\varepsilon \| \nabla u \|^2 + C(\varepsilon) \| u \|^2]$$
$$= (1 - \| L_1 \| K\varepsilon) \| \nabla u \|^2 + (\alpha - \| L_1 \| KC(\varepsilon)) \| u \|^2 .$$

Choose ε so small that $1 - \| L_1 \| K\varepsilon > 1/2$ and α so large that

$$\alpha - \| L_1 \| KC(\varepsilon) > \tfrac{1}{2} .$$

Note that when L is the zero operator γ may be taken to be real,

for the form B_G is symmetric.

2.6. LEMMA. *There exists a positive real number C such that for every u and $v \in W^{1,2}(G)$*

(2.6.1)
$$| B_G^\gamma(u, v) | \leqq C \| u \|_1 \| v \|_1 .$$

Proof.

$$| B_G^\gamma(u, v) | \leqq \left| \int_G \nabla u \cdot \overline{\nabla v} dx \right| + | \gamma(u, v) | + | (Lu, v)_\partial |$$

$$\leqq \| \nabla u \| \| \nabla v \| + | \gamma | \| u \| \| v \| + \| L \| \| u \|_\partial \| v \|_\partial$$

$$\leqq \| \nabla u \| \| \nabla v \| + | \gamma | \| u \| \| v \| + \| L \| [K(\varepsilon \| \nabla u \|^2$$

$$+ C(\varepsilon) \| u \|^2)]^{1/2} [K(\varepsilon \| \nabla u \|^2 + C(\varepsilon) \| v \|^2)]^{1/2}$$

for any $\varepsilon > 0$. Choose $\varepsilon > 0$ and let $\delta = \max(\varepsilon, C(\varepsilon))$. Let $\eta = \max(| \gamma |, 1)$ and $C = 2\eta + \delta \| L \|$. Then a simple computation yields (2.6.1).

Let $\mathscr{H} = L^2(G)$ and $\mathscr{H}_1 = W^{1,2}(G)$ and apply the Lax-Milgram theorem to the form B_G^γ. The result is the following:

2.7. THEOREM. *Let L be any bounded linear operator on $L^2(\partial G)$. Then there exists a closed, densely defined, linear operator M_L in $L^2(G)$ with domain in $W^{1,2}(G)$ such that for every $v \in \mathscr{D}(M_L)$ and $u \in W^{1,2}(G)$, $B_G(u, v) = (u, M_L v)$. Its adjoint M_L^* enjoys the relation $B_G(u, v) = (M_L^* u, v)$ for $u \in \mathscr{D}(M_L^*)$ and $v \in W^{1,2}(G)$. Moreover, if γ is any complex number such that $R_l \gamma \leqq -(\delta + \| L_1 \| KC(\varepsilon))$ where $0 < \delta < 1$ and $C(\varepsilon)$ is the number of Lemma 2.3 with $\varepsilon < (1 - \delta)/\| L_1 \|$, then γ lies in the resolvent set of M_L.*

Proof. When γ is as above, Theorem 2.2 shows that $M_L + \gamma I$ is closed and densely defined in $L^2(G)$ with domain in $W^{1,2}(G)$. Moreover, zero is in its resolvent set.

We shall now give a characterization of $\mathscr{D}(M_L)$.

2.8. THEOREM. *Let $v \in W^{1,2}(G)$, then a necessary and sufficient condition that $v \in \mathscr{D}(M_L)(\mathscr{D}(M_L^*))$ is that $-\Delta v$ is a distribution in $L^2(G)$, and for $u \in W^{1,2}(G)$, $(u, -\Delta v) = B(u, v)((-\Delta v, u) = B(v, u))$.*

Proof. We give the proof for M_L, the proof for M_L^* being virtually identical.

Let $v \in \mathscr{D}(M_L)$. Then for every $u \in W^{1,2}(G)$, and hence for every u in $C_0^\infty(G)$, $(u, M_L v) = B(u, v)$. Recalling the definition of the derivative of a distribution, we have $(u, M_L v) = B(u, v) = \int_G \nabla u \cdot \overline{\nabla v} dx = (u, -\nabla v)$

for u in $C_0^\infty(G)$. Since $C_0^\infty(G)$ is dense in $L^2(G)$ it follows that $M_L v = -\Delta v$. Now suppose v is in $W^{1,2}(G)$ and $-\Delta v$ is a distribution in $L^2(G)$ such that for $u \in W^{1,2}(G)$, $(u, \Delta v) = B_G(u, v)$. Then for a suitable complex number γ the form B_G^γ defined by $(u, -\Delta v + \gamma v) = B_G^\gamma(u, v)$ is bounded in $W^{1,2}(G)$ and $R_l B_G^\gamma$ is bounded from below. It then follows from the proof of Theorem 2.2 that $v \in \mathscr{D}(M_L)$ and $M_L v = -\Delta v$.

We now present a series of localization lemmas which will be used in what follows. These results are by now quite common in the literature and we follow for the most part the presentation given in Poulsen [11].

2.9. LEMMA. *Let ω be in $C^1(G)$ and suppose K is an upper bound for ω and $|\nabla \omega|$. Then for every $u \in W^{1,2}(G)$, ωu is also in $W^{1,2}(G)$.*

Proof. $W^{1,2}(G)$ is the closure of $C_0^1(E^n)$ in the $W^{1,2}$ norm. If $\{u^j\}$ is a sequence in $C_0^1(E^n)$ such that $\| u^j - u \|_{1,2} \to 0$. Then the inequalities $\| \omega u^j - \omega u \| \leq K \| u^j - u \|$ and $\| \nabla(\omega u^j) - \nabla(\omega u) \| \leq K(\| u^j - u \| + \| \nabla u^j - \nabla u \|)$ show that $\| \omega u^j - \omega u \|_{1,2} \to 0$.

2.10. LEMMA. *Let $\omega \in C_0^{1,1}(E^n)$ be real valued and suppose that K is an upper bound for $|\omega|$, $|\nabla \omega|$, and $|-\Delta \omega|$ in G. If $u \in W^{1,2}(G)$ and $-\Delta u$ is a distribution in $L^2(G)$ then $\omega u \in W^{1,2}(G)$ and $-\Delta(\omega u)$ is a distribution in $L^2(G)$.*

Proof. Let $S(u, \omega) = (-\Delta \omega)u - 2\nabla \omega \cdot \nabla u - \omega \Delta u$ and observe that

$$\| S(u, \omega) \| \leq K(\| u \| + 2\| \nabla u \| + \| \Delta u \|) .$$

Thus $S(u, \omega)$ is clearly a distribution in $L^2(G)$ and by the definition of derivative of a distribution $S(\omega, u) = -\Delta(\omega u)$.

3. Regions with bounded boundaries. In the preceding section we have given a characterization of the operator M_L in terms of the Laplacian and the form $B_G(\cdot, \cdot)$. Hereafter the symbol M_L will always stand for this operator. Essentially these results concern themselves with regularity in the interior of G. It is the purpose of this section to study the boundary behavior of functions in $\mathscr{D}(M_L)$. Explicitly let $\mathscr{D}(T_L)$ be the set of $u \in \mathscr{D}_1(G)$ (cf. Definition 1.5) for which $\tilde{u}_n - L\tilde{u} = 0$ and let T_L be $-\Delta$ with domain $\mathscr{D}(T_L)$, then we shall show $T_L = M_L$.

3.1. DEFINITION. Let ϕ be a function in $C_0^\infty(E^n)$ satisfying

$$\phi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases}$$

and $0 \leqq \phi(x) \leqq 1$. Let $\omega^N(x) = \phi(x/N)$ and let K be an upper bound for $|\nabla \phi|$ and $|\Delta \phi|$.

3.2. LEMMA.

$$|\nabla \omega^N| \leqq K/N$$

$$|\Delta \omega^N| \leqq K/N^2 .$$

3.3. LEMMA. *Let $u \in W^{1,2}(G)$ and suppose $-\Delta u$ is a distribution in $L^2(G)$. Then as $N \to \infty$*

$$\| \omega^N u - u \|_{1,2} \to 0$$

$$\| -\Delta(\omega^N u) - (-\Delta u) \| \to 0 .$$

Proof. For each N, $\omega^N u \in W^{1,2}$ and $-\Delta(\omega^N u)$ is a distribution in $L^2(G)$ by Lemma 2.10. The results follow from an obvious computation, Lemma 3.2, and the dominated convergence theorem.

3.4. THEOREM (*Green's Formulas*). *Let u and v be in $\mathscr{D}_1(G)$. Then*

(3.4.1)
$$\int_G (-\Delta u)\bar{v}dx = \int_G \nabla u \cdot \overline{\nabla v} - \int_{G\partial} \tilde{u}_n \bar{\tilde{v}} d\theta$$

(3.4.2)
$$\int_G (-\Delta u)\bar{v}dx - \int_G u(-\overline{\Delta v})dx = \int_{\partial G}(\tilde{u}\bar{\tilde{v}}_n - \tilde{u}_n\bar{\tilde{v}})d\theta .$$

Proof. Let B_N be the sphere with center at the origin and radius N and let ω^N be the function of Lemma 3.2. Define v^N to be $\omega^N v$ so that v vanishes outside of B_{2N}. Then

$$\int_G (-\Delta u)\bar{v}^N dx = \int_{B_{3N} \cap G}(-\Delta u)\bar{\tilde{v}}^N dx = \int_{B_{3N} \cap G} \nabla u \cdot \overline{\nabla v^N}dx$$

$$- \int_{\partial B_{3N} \cup \partial G} \tilde{u}_n \overline{\tilde{v}}^N d\theta$$

$$= \int_G \nabla u \cdot \overline{\nabla v^N}dx - \int_{\partial G} \tilde{u}_n \bar{\tilde{v}}^N d\theta .$$

The last of the equalities follows from the fact that ω^N vanishes on and near ∂B_{3N}.[2] Let $N \to \infty$ and apply Lemma 3.3 to obtain Eq. (3.4.1). Formula (3.4.2) can be obtained in a similar manner.

3.5. THEOREM (*Localization*). *Let ω^N be as in 3.1. If $v \in \mathscr{D}(M_L)(\mathscr{D}(M_L^*))$, then $\omega^N v \in \mathscr{D}(M_L)(\mathscr{D}(M_L^*))$.*

Proof. We give the proof for M_L. By Theorem 2.8 and Lemma 2.10 it suffices to show that for $u \in W^{1,2}(G)$ or even for $u \in C_0^2(G)$

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[2] Observe that $\tilde{v}^N = \tilde{v}$ on ∂G.

(3.5.1) $(u, -\Delta(\omega^N v)) = B(u, \omega^n v)$.

Here of course N is assumed to be sufficiently large that $\partial G \subseteqq B_N$. We note that $C_0^2(E^n)$ is dense in $W^{1,2}(G)$ and, moreover, that if $\{u^j\}$ is a sequence in $C_0^2(E^n)$ such that $\|u^j - u\|_{1,2} \to 0$ and if $\tilde{u}^j$ is the corresponding boundary value, then $\|\tilde{u}^j - \tilde{u}\|_0 \to 0$. Hence it suffices to prove (3.5.1) for $u \in C_0^2(E^n)$. Now by Lemma 2.10, for all $u \in W^{1,2}(G)$

$$(u, -\Delta(\omega v)) = (u, \omega(-\Delta v)) + (u, v(-\Delta \omega)) - 2 \int_G u \overline{\nabla \omega \cdot \nabla v} \, dx$$

since ω is real valued. Now

$$(u, \omega(-\Delta v)) = \int_G u \omega(-\overline{\Delta v}) dx = B_G(u\omega, v) .$$

Moreover,

$$(u, v(-\Delta \omega)) = \int_G u \bar{v}(-\Delta \omega) dx = \int_\Omega u \bar{v}(-\Delta \omega) dx = \int_\Omega \nabla(u\bar{v}) \cdot \nabla \omega \, dx .$$

Here $\Omega = G \cap B_{3N}$ and we note that ω vanishes on and near ∂B_{3N}. Thus

$$(u, -\Delta(\omega v)) = B_G(u\omega, v) + \int_G \nabla(u\bar{v}) \cdot \nabla \omega \, dx - 2 \int_G u \nabla \omega \cdot \overline{\nabla v} \, dx .$$

Since $B_G(u\omega, v) = \int_G \nabla(u\omega) \cdot \overline{\nabla v} \, dx - \int_{\partial G} u\omega \overline{L v} \, dx$ and $\omega = 1$ on ∂G and

$$\int_G \nabla(u\omega) \cdot \overline{\nabla v} \, dx + \int_G \nabla(u\bar{v}) \cdot \nabla \omega \, dx - 2 \int_G u \nabla \omega \cdot \overline{\nabla v} \, dx$$

$$= \int_G \omega \nabla u \cdot \overline{\nabla v} \, dx + \int_G \nabla u \cdot \nabla \omega \bar{v} \, dx = \int_G \nabla u \cdot \nabla(\omega v) dx ,$$

we have finally

$$(u, -\Delta(\omega v)) = B(u, \omega v) \text{ for } u \in C_0^2(E^n) .$$

3.6. DEFINITION. Let $\mathscr{D}(T_L)(\mathscr{D}(T_{L*}))$ be the set of those $u \in \mathscr{D}_1(G)$ for which $\tilde{u}_n - L\tilde{u} = 0$ ($\tilde{u}_n - L^*\tilde{u} = 0$) and for u in $\mathscr{D}(T_L)(\mathscr{D}(T_{L*}))$, let $T_L u(T_{L*}u) = -\Delta u$.

When G was bounded it was shown in [1] that T_L was closed and densely defined and that $(T_L)^* = T_{L*}$. In the remainder of this section we shall identify M_L with T_L and M_L^* with T_{L*}, thus proving that for G unbounded, but with ∂G bounded, we again have $(T_L)^* = T_{L*}$.

It is immediately clear by Green's formula that $T_L \subseteqq M_L$ and $T_{L*} \subseteqq M_L^*$. The reverse inclusions will be obtained by localization. Consider a sphere B_{3N} with N so large that $\partial G \subseteqq B_N$, and let $\Omega = G \cap B_{3N}$, so that $\partial \Omega = \partial G \cup \partial B_{3N}$. We define an operator, L_0 on $L^2(\partial \Omega)$ by

$$L_0 \tilde{u} = \begin{cases} L\tilde{u} & \tilde{u} \in L^2(\partial G) \\ 0 & \tilde{u} \in L^2(\partial B_{3N}) \end{cases}$$

and the operator T_{L_0} analogously as in Definition 3.6 with L_0 replacing L and Ω replacing G.

In what follows N will be chosen so large that B_N contains ∂G in its interior. The corresponding ω^N will be denoted by ω. Moreover, we define M_{L_0}, $(M_{L_0}^*)$ analogously as $M_L(M_L^*)$ with $\Omega = G \cap B_{3N}$ replacing G and L_0 replacing L. We need the following

3.7. LEMMA. *Let G be a bounded domain with a $C^{1,1}$ boundary. Then $M_L = T_L$ and $M_L^* = T_{L*}$.*

Proof. By Green's formulas it is clear that $M_L \supseteq T_L$ and $M_L^* \supseteq T_{L*}$. But since all the operators in question are closed and $T_{L*} = (T_L)^* \supseteq M_L^* \supseteq T_L^*$ and as a consequence $T_{L*} = M_{L*}$, it also follows that $T_L = (T_{L*})^* = M_L^{**} = M_L$.

We return now to unbounded regions.

3.8. LEMMA. *Let $v \in \mathscr{D}(M_L)(\mathscr{D}(M_L^*))$, then ωv restricted to Ω is in $\mathscr{D}(M_{L_0})$ $(\mathscr{D}(M_{L_0}^*))$.*

Proof. By Theorem 3.5, $\omega v \in W^{1,2}(G)$ and $-\varDelta(\omega v)$ is a distribution in $L^2(G)$. The same is also true therefore if G is replaced by Ω. It suffices to show therefore that for $u \in C_0^\infty(E^n)$

$$(u, -\varDelta(\omega v))_\Omega = B_\Omega(u, \omega v) .$$

The proof of this is similar to that of the corresponding statement of Theorem 3.5 and will be omitted.

3.9. THEOREM. *$M_L = T_L$ and $M_L^* = T_{L*}$. Thus $(T_L)^* = T_{L*}$.*

Proof. By Lemma 3.8, $\omega v \in \mathscr{D}(M_{L_0})$ whenever $v \in \mathscr{D}(M_L)$. Thus by Lemma 3.7, $\omega v \in \mathscr{D}(T_L)$. It then follows that $v \in \mathscr{D}_1(G)$ and $\tilde{v}_n - L\tilde{v} = 0$, since $\omega = 1$ on $B_N \cap G$. Thus $v \in \mathscr{D}(T_L)$ and $M_L v = T_L v$ and the result follows. The proof that $M_L^* = T_{L*}$ is similar and will be omitted. It is clear from Theorem 2.7 that the spectrum of T_L lies in a half plane. We can sharpen this estimate considerably to obtain the results of [1].

3.10. THEOREM. *The spectrum of T_L lies inside the the parabolic region $\tau^2 = a\sigma + b$ with $a, b > 0$ where a complex number $\lambda = \sigma + i\tau$.*

Proof. We follow very closely the proof of Theorem 4.14 of [1]

first recalling the definition of the operators S, $\hat{L}$ and $P(\lambda)$. $\mathscr{D}(S) =$ $\{[u, \tilde{u}] : u \in \mathscr{D}_1(G)\}$ and $S[u, \tilde{u}] = [-\varDelta u, \tilde{u}_n]$ for $[u, \tilde{u}] \in \mathscr{D}(S)$. $\hat{L}$ and $P(\lambda)$ have all of $L^2(G) \oplus L^2(\partial G)$ as domains and $\hat{L}[f, g] = [0, Lg)$, $P(\lambda)[f, g] =$ $[\lambda f, 0]$. It was shown in [1] that when λ is such that

$$(3.10.1) \qquad |([S - P(\lambda)][u, \tilde{u}], [u, \tilde{u}])| > \| L \| \| [u, \tilde{u}] \|^2 ,$$

then zero is in the resolvent set of $S - \hat{L} - P(\lambda)$, and, moreover, for $[u, \tilde{u}] \in \mathscr{D}(S)$

$$(3.10.2) \qquad |([S - \hat{L} - P(\lambda)][u, \tilde{u}], [u, \tilde{u}])| \geqq K(\lambda) \| [u, \tilde{u}] \|^2 ,$$

where $K(\lambda)$ is a positive number.

But for $[u, \tilde{u}] \in \mathscr{D}(S)$

$$([S - \hat{L} - P(\lambda)][u, \tilde{u}], [u, \tilde{u}]) = \| \nabla u \|^2 - \lambda \| u \|^2 - (L\tilde{u}, \tilde{u})_\partial$$

and for $u \in \mathscr{D}(T_L)$

$$((T_L - \lambda)u, u) = \| \nabla u \|^2 - \lambda \| u \|^2 - (L\tilde{u}, \tilde{u})_\partial .$$

Thus if (3.10.2) holds for some $K(\lambda)$, then for the same $K(\lambda)$

$$|((T_L - \lambda I)u, u)| \geqq K(\lambda) \| u \|^2 , \qquad u \in \mathscr{D}(T_L) .$$

It only remains to determine conditions on λ such that (3.10.1) is valid. At this point we may proceed exactly as was done in [1] and we arrive at the same conclusions.

4. Regions with unbounded boundaries.

As was remarked in the introduction we shall restrict L to be multiplication by a bounded measurable function. We shall then use a result of Poulsen [11] to obtain most of the results of the preceding section.

4.1. DEFINITION. The operator M_L, when L is the zero operator, will be denoted by N. N is the operator corresponding to the Neumann boundary condition.

4.2 THEOREM. *Suppose* $\omega \in C_0^{1,1}(E^n)$ *is real valued and such that* $|\omega|$, $|\nabla \omega|$, *and* $|\varDelta \omega|$ *are all uniformly bounded on* G *by say* K. *Then if* $\omega \in \mathscr{D}(N)$ *and* $v \in \mathscr{D}(M_L)(\mathscr{D}(M_{L*}))$, *if follows that* ωv *also in* $\mathscr{D}(M_L)(\mathscr{D}(M_{L*}))$.

Proof. Let B_N be a sphere with center at the origin and radius N so large that it contains the support of ω in the interior. $\Omega = G \cap B_N$ and observe that as in the proof of Theorem 3.5, it suffices to show that for all $u \in C_0^2(E^n)$

(4.2.1) $$(u, \Delta(\omega v)) = B(u, \omega v)$$

The proof is almost identical to that of Theorem 3.5. We observe first that if the support of ω is contained in G we may simply apply Green's formulas. If the support of $\omega \cap \partial G$ is not empty we follow the arguments of Theorem 3.5. Note that since $\omega \in \mathscr{D}(N)$ and $u \in C_0^2(E^n)$ we find that $(u, v(-\Delta\omega)) = \int_G \nabla(u\bar{v}) \cdot \nabla\omega dx$.

Following the computations in the proof of Theorem 3.5 yields (4.2.1). The proof for M_L^* is almost identical.

4.3. LEMMA. (Poulsen [11]). *Let G be a uniformly C^3 domain and let θ be any point in ∂G. There exist neighborhoods A_1 and A_2 of θ with compact closures such that $\bar{A}_1 \subsetneqq A_2$. Moreover, there exists a real valued function $\omega \in C_0^2(E^n)$ such that ω is identically 1 in $G \cap A_1$ and vanishes in $G \cap A_2^c$. Moreover, $\omega \in \mathscr{D}(N)$ and $|\omega|$, $|\nabla\omega|$, and $|\Delta\omega|$ are uniformly bounded on G.*

4.4. DEFINITION. Let $\mathscr{D}(T_L)$ be the set of those functions in $\mathscr{D}_1(G)$ for which $\tilde{u}_n - L\tilde{u} = 0$ on every compact subset of ∂G. For $u \in \mathscr{D}(T_L)$ let $T_L u = -\Delta u$.

4.5. LEMMA. $T_L \subsetneqq M_L$ and $T_{L*} \subsetneqq M_{L*}$.

Proof. As usual we note that it suffices to show that for every $u \in C_0^2(E^n)$, $(u, -\Delta v) = B_G(u, v)$.

We give the proof for T_L, the proof for T_{L*} being similar. We note that G need only be a $C^{1,1}$ manifold for this proof. As usual let B_N be a sphere with center at the origin and with radius N sufficiently large that for a fixed $u \in C_0^2(E^n)$, support of u is contained in B_N. Let $\Omega = B_N \cap G$. Then

$$\int_G u(-\overline{\Delta v})dx = \int_\Omega u(-\overline{\Delta v})dx = \int_\Omega \nabla u \cdot \overline{\nabla v}dx - \int_{\partial\Omega} \tilde{u}\bar{\tilde{v}}_n d\theta$$

$$= \int_\Omega \nabla u \cdot \overline{\nabla v}dx - \int_{\partial\Omega \cap \partial G} \tilde{u}\bar{\tilde{v}}_n d\theta$$

$$= \int_G \nabla u \cdot \overline{\nabla v}dx - \int_{\partial G} \tilde{u}\bar{\tilde{v}}_n d\theta \ .$$

Since u was arbitrary in $C_0^{1,1}(E^n)$ the result follows.

Referring to Definition 1.2, let V_k be one of the covering neighborhoods so that $T_k(V_k \cap G) = \Gamma \times \{0 < x_n < r\}$. Now let $S_1, S_2,$ and S_3 be three C^3 manifolds such that

$$S_1 \cong \tfrac{1}{2}\Gamma \times \overline{\{0 < x_n < r/2\}}$$

$$S_2 \cong \bar{S}_1$$
$$S_3 \cong \bar{S}_2$$
$$\Gamma \times \{0 < x_n < r\} \cong \bar{S}_3 \;.$$

Let $A_{ik} = T_k^{-1}(S_i)$ for $i = 1, 2, 3$ and observe that $\bigcup_k A_{ik}$ contains an $r/2$ neighborhood of ∂G. Now let ω be the function of Lemma 4.3 with A_{1k} and A_{2k} playing the roles of A_1 and A_2.

Let $\partial A_{3k} \cap \partial G = \Gamma_1$ and $\Gamma_2 = \partial A_{3k} \sim \Gamma_1$.

We define an operator on ∂A_{3k} by

$$L_0 f(\theta) = \begin{cases} L f(\theta) & \text{for } \theta \in \Gamma_1 \\ 0 & \text{for } \theta \in \Gamma_2 \;. \end{cases}$$

Let $M_{L_0} \; T_{L_0}$ be the operator corresponding to L_0 and A_{3k} as M_L and T_L correspond to L and G. Then

4.6. LEMMA. *Let* $v \in \mathscr{D}(M_L)(\mathscr{D}(M_{L*}))$, *then* $\omega v \in \mathscr{D}(M_{L_0})(\mathscr{D}(M_{L_0^*}))$

Proof. The proof is almost identical to that of Theorem 4.2. By ωv we mean of course ωv restricted to A_{3k}.

Now A_{3k} is a bounded C^3 manifold and thus by Lemma 3.8, $M_{L_0} = T_{L_0}$ and $M_{L_0^*} = T_{L_0}^*$. Thus ωv is in $\mathscr{D}(T_{L_0})$. Hence on Γ_1, $(\omega v)_n - L(\omega v) = 0$. Thus in particular on $\partial G \cap \partial A_{ik}$, $\tilde{v}_n - L\tilde{v} = 0$, since here ω is identically one. Since $\cup A_{ik}$ covers an $r/2$ neighborhood of ∂G, it follows that $\tilde{v}_n - L\tilde{v} = 0$ on every compact subset of ∂G. Combining these facts with Lemma 4.5, we finally obtain

4.7. THEOREM. *If* G *is a uniformly* C^3 *manifold then* $T_L = M_L$ *and* $T_{L*} = M_{L*}$. *Thus* $(T_L)^* = T_{L*}$.

The fact that $T_{L*} = M_{L*}$ follows by using almost the same argument as the above.

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