SOME FURTHER EXTENSIONS OF A THEOREM OF MARCINKIEWICZ

INGE FUTTRUP CHRISTENSEN

1. Introduction. It is often of interest to decide whether a given function can be a characteristic function. Necessary and sufficient conditions are known which a complex-valued function of a real variable must satisfy in order to be a characteristic function (see e. g. [4], Chapter IV), but these general conditions are not easily applied. Therefore, various conditions have been derived which are restricted to certain classes of functions but which are applied more readily. One of the most important of these results was obtained by J. Marcinkiewicz and gives a necessary condition for an entire function of finite order to be a characteristic function [7]. As a special case, Marcinkiewicz considered entire functions of the form $f(t) = \exp[P_m(t)]$, where $P_m(t)$ is a polynomial of degree m, and obtained that if $m > 2$, then $f(t)$ can not be a characteristic function. This result, which is referred to as the Theorem of Marcinkiewicz, has been extended by E. Lukacs ([4], p. 146) to functions of the form $f_n(t) = k_n e_n[P_m(t)]$, where k_n is a constant determined by the condition that $f_n(0) = 1$, and where $e_1(z) = \exp(z)$, $e_2(z) = \exp[e_1(z)], \cdots$, $e_k(z) = \exp[e_{k-1}(z)]$. In the present paper, the Theorem of Marcinkiewicz is further extended to certain functions of the form

$$f_n(t) = k_n g(t) e_n[P_m(t)] \, , \tag{1.1}$$

where $g(t)$ is some specified characteristic function. In § 3 we shall consider certain entire functions of the form (1.1), while in § 4 we shall turn our investigation to certain analytic functions of the form (1.1) with $n = 1$ which are regular in a half-plane having the origin as an interior point.

2. Some auxiliary results. We now consider briefly the class of analytic characteristic functions and state some results which will be needed in our investigation.

A characteristic function $f(t)$ is said to be an analytic characteristic function if there exists a function $A(z)$ of the complex variable $z = t + iy$ (t and y real) which is regular in the circle $|z| < R$ ($R > 0$) and a constant $\Delta > 0$ such that $A(t) = f(t)$ for $|t| < \Delta$. D. A. Raikov has shown [8] that if a characteristic function $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$ is regular in a neighborhood of the origin, then it is also regular in the interior

Received May 4, 1961. This work was supported by the National Science Foundation through grant NSF-G-4220.

of a horizontal strip of the z-plane and can be represented in this strip by the Fourier integral $f(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$, where $z = t + iy$; the horizontal strip is either the whole plane, in which case we have an entire characteristic function, or it has one or two horizontal boundary lines.

One of the basic tools in our investigation will be the following theorem ([4], p. 134) which follows from Raikov's result quoted in the previous paragraph.

THEOREM A. *Let $f(z)$ be an analytic characteristic function. Then for any horizontal line located in the interior of the strip of regularity of $f(z)$, the modulus $|f(z)|$ attains its absolute maximum on the imaginary axis.*

The following theorem of Dugué [2] will also be useful.

THEOREM B. *Let $f(z)$ be an analytic characteristic function which is regular in the strip $-\alpha < Im(z) < \beta$. Then for any real number η in the interval $(-\alpha, \beta)$, the function $h(z) = f(z + i\eta)/f(i\eta)$ is also ananalytic characteristic function and is regular in the strip $-\alpha - \eta < Im(z) < \beta - \eta$.*

We shall also use the following result [5].

THEOREM C. *Let $f(z)$ be an analytic characteristic function which is regular in the strip $-\alpha < Im(z) < \beta$. Then*

$$|f(t + \tau + iy) - f(t + iy)|^2 \leqq 2f(2iy)[1 - Re\, f(\tau)] \tag{2.1}$$

for any real number τ, provided that $-\alpha/2 < y < \beta/2$.

In order to state the next theorem due to E. Lukacs, it will be necessary to introduce some notation. Let $z = t + iy$ and consider the functions $f_v(z) = k_v e_v[P_m(z)]$ $(v = 1, 2, \cdots, n)$ where $P_m(t) = \sum_{j=0}^{m} c_j t^j$, $c_j = \alpha_j + i\beta_j$, $c_m \neq 0$, α_j and β_j real, and where $k_v = [e_v(c_0)]^{-1}$. Introduce the functions $\phi_v(z) = k_{v-1}^{-1}[f_{v-1}(z) - 1]$ $(v = 2, 3, \cdots, n)$ and write $\alpha_v(t, y)$ for the real part and $\beta_v(t, y)$ for the imaginary part of $\phi_v(z)$. We define

$$A_v(t, y) = \alpha_v(t, y) - \alpha_v(0, y) \qquad (v = 1, 2, \cdots, n) \tag{2.2}$$

and are now prepared to state the theorem. (For a proof, see [6]).

LEMMA A. *Let $m \geqq 3$. Consider the following two cases:*
(i) *either $m > 3$, or $m = 3$ and $\beta_3 = 0$;*
(ii) *$m = 3$ and $\beta_3 \neq 0$.*
Then there exists a $\xi_m \geqq 0$ and a value $Y = Y(m)$ such that $A_n(y\sqrt{\xi_m}, y) \to \infty$

provided that, in case (i), $y \geqq Y$, *while in case* (ii) *it is required that* $(-\text{sign}\, \beta_3)y \geqq Y$.

We shall also need the following.

COROLLARY TO LEMMA A. *Under the conditions of Lemma A, there exist a* $\xi_m \geqq 0$ *and an* $A_m > 0$ *such that* $A_1(y\sqrt{\xi_m}, y) = A_m \mid y \mid^m [1 + o(1)]$, *where the estimate holds in case* (i) *as* $y \to \infty$ *but in case* (ii) *as* $(-\text{sign}\, \beta_3)y \to \infty$.

Furthermore, we shall need a result which we state as

LEMMA B. *If* $g(z) = \sum_{s=1}^{S} p_s e^{i\eta_s z}$, $(z = t + iy, p_s \geqq 0, \sum_{s=1}^{S} p_s = 1, \eta_s$ *real*, $S < \infty)$, *then there exists a constant* B $(0 < B < \infty)$ *such that*

$$\frac{g(2iy)}{g^2(iy)} < B\,. \tag{2.3}$$

Proof. The existence of such a finite positive constant B is assured from the fact that $g(2iy)/g^2(iy)$ is continuous on $-\infty < y < \infty$ and converges to finite positive limits as $y \to \pm\infty$.

3. Discussion of certain entire functions.

We now give the results obtained in the investigation of certain entire functions of the form (1.1) where $g(t)$ belongs to some specified discrete distributions. The proofs follow.

THEOREM 3.1. *Consider, for any integer* $n \geqq 1$, the *function* $f_n(t) = k_n \exp\,[g_1(t) - 1]e_n[P_m(t)]$, *where* $g_1(t)$ *is an entire characteristic function belonging to a lattice distribution with the origin as a lattice point. If* $m > 2$, *then* $f_n(t)$ *can not be a characteristic function.*

As a special case of this theorem we let $g_1(t) = \lambda_1(e^{it} - 1) + \lambda_2(e^{-it} - 1) + 1$ and obtain a slight generalization of a result due to E. Lukacs ([6], p. 489); if $\lambda_1 = \lambda_2 = 0$ and $n = 1$, we have the Theorem of Marcinkiewicz.

THEOREM 3.2. *Consider, for any integer* $n \geqq 1$, *the function* $f_n(t) = k_n g(t) e_n[P_m(t)]$, *where* $g(t)$ *is the characteristic function belonging to a discrete distribution having a finite number of discontinuity points. If* $m > 2$, *then* $f_n(t)$ *can not be a characteristic function.*

We give an indirect proof for each of these theorems and suppose that $f_n(t)$ is a characteristic function. Then, by definition, $f_n(t)$ must be an entire characteristic function, and we consider $f_n(t)$ also for complex values of the argument $z = t + iy$. By means of Lemma A, it is then possible to arrive at a contradiction of the fundamental maximum modulus property of analytic characteristic functions which was stated in Theorem A. Let us consider each theorem in turn.

In proving Theorem 3.1, where $g_1(t) = \sum_{j=-\infty}^{\infty} p_j e^{ij\wedge t}$, ($\wedge$ positive and real, $p_j \geqq 0$, $\sum_j p_j = 1$), we find that

$$(3.1) \quad R(t, y) = \left|\frac{f_n(t + iy)}{f_n(iy)}\right| = \exp\{\sum_j [p_j e^{-j\wedge y}(\cos j\wedge t - 1)] + A_n(t, y)\} ,$$

where $A_n(t, y)$ was defined by (2.2). We select an integer k according to the following rule, where Y and ξ_m are the quantities determined by Lemma A.

(a) If $m > 3$, or if $m = 3$ and $\beta_3 = 0$, then chooose $k > (\wedge Y\sqrt{\xi_m})/(2\pi)$.

(b) If $m = 3$ and $\beta_3 \neq 0$, then choose $(-\text{sign}\,\beta_3)k > (\wedge Y\sqrt{\xi_m})/(2\pi)$.

Let now $y^* = (2\pi k)/(\wedge\sqrt{\xi_m})$ and $t^* = y^*\sqrt{\xi_m} = (2\pi k)/\wedge$. Then from (3.1), $R(t^*, y^*) = \exp[A_n(t^*, y^*)]$. The conditions of Lemma A being satisfied, it follows that $R(t^*, y^*) > 1$.

We turn now to the proof of Theorem 3.2 and will investigate the function

$$(3.2) \qquad R(t, y) = \left|\frac{g(t + iy)}{g(iy)}\right| \exp[A_n(t, y)] ,$$

where $g(t) = \sum_{s=1}^{S} p_s e^{i\eta_s t}$ (η_s real, $p_s \geqq 0$, $\sum_{s=1}^{S} p_s = 1$, $S < \infty$). First we prove the following.

LEMMA 3.1. *Let* $g(z) = \sum_{s=1}^{S} p_s e^{i\eta_s z}$, $(z = t + iy)$, *and consider* $R_1(t, y) = |g(t + iy)/g(iy)|$. *Given any* $\varepsilon > 0$, *there exists a real number* $M = M(\varepsilon)$ *such that*

$$(3.3) \qquad 1 - \varepsilon < R_1(M, y) < 1 + \varepsilon.$$

Let any $\varepsilon > 0$ be given. We note that $g(t)$ is an almost periodic function of the real variable t. Then for every $\varepsilon' > 0$, there exists a real number $L = L(\varepsilon')$ such that every interval of length L contains at least one translation number $N = N(\varepsilon')$, that is, a number N satisfying the inequality $|g(t + N) - g(t)| \leqq \varepsilon'$, $-\infty < t < \infty$ ([1], p. 31). It is easy to see that $\overline{g(t)}$ is also an almost periodic function. Since the sum of two almost periodic functions is itself almost periodic, we have that $Re\, g(t)$ is an almost periodic function. Consider now Lemma B and take $\varepsilon'' = \varepsilon^2/(2B)$, where B refers to (2.3). It is possible to find a translation number $M = M(\varepsilon'') = M(\varepsilon)$ for $Re\, g(t)$ so that M satisfies one of the following conditions, where Y and ξ_m are the quantities determined by Lemma A.

(a) If $m > 3$, or if $m = 3$ and $\beta_3 = 0$, then $M > Y\sqrt{\xi_m}$.
(b) If $m = 3$ and $\beta_3 \neq 0$, then $(-\operatorname{sign} \beta_3)M > Y\sqrt{\xi_m}$.
We know that $|Re\, g(t + M) - Re\, g(t)| \leqq \varepsilon''$ for all t. Set $t = 0$, and we have

$$|Re\, g(M) - 1| \leqq \varepsilon'' . \tag{3.4}$$

We now use Theorem C and consider (2.1) for $\tau = M$. Then we see by (3.4) that $|g(t + M + iy) - g(t + iy)| \leqq \sqrt{2g(2iy)\varepsilon''}$. From this inequality and from (2.3) we obtain

$$|R_1(t + M, y) - R_1(t, y)| = \left|\frac{g(t + M + iy) - g(t + iy)}{g(iy)}\right|$$
$$\leqq \sqrt{\frac{2g(2iy)}{g^2(iy)}\varepsilon''} < \sqrt{2B\varepsilon''} = \varepsilon .$$

Setting $t = 0$, (3.3) follows immediately so that Lemma 3.1 is proven.

We turn now to function (3.2). Set $y^* = M/\sqrt{\xi_m}$, where M is selected according to the rule in the proof of Lemma 3.1; it is then clear that y^* satisfies either condition (i) or (ii) of Lemma A. Set $t^* = y^*\sqrt{\xi_m} = M$, and by Lemma A and Lemma 3.1 we have that $R(t^*, y^*) \to \infty$ as $y^* \to \infty$ or as $(-\operatorname{sign} \beta_3)y^* \to \infty$, referring to (i) and (ii), respectively, of Lemma A. This completes the proof of Theorem 3.2.

4. Discussion of certain analytic functions regular in a half-plane. In this section we shall consider the class of infinitely divisible characteristic functions. A characteristic function $g(t)$ is said to be infinitely divisible if, for every positive integer n, it is the nth power of some characteristic function $g_n(t)$ (which depends, of course, on n). Then $g_n(t)$ is uniquely determined by $g(t)$ according to the formula $g_n(t) = [g(t)]^{1/n}$ provided that we select for the nth root the principal branch (see e. g. [3], Chapter III).

Infinitely divisible characteristic functions admit canonical representations. We shall especially be interested in the well known result ([4], p. 189) that if an analytic characteristic function $g(z)$ is infinitely divisible and belongs to a distribution having a finite second moment, then $g(z)$ has the following unique representation in the interior of its strip of regularity:

$$\log g(z) = icz + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx)\frac{dK(x)}{x^2} , \tag{4.1}$$

where c is a real constant and $K(x)$ is a nondecreasing and bounded function such that $K(-\infty) = 0$ and $\int_{-\infty}^{\infty} dK(x) = K(+\infty) < \infty$; at $x = 0$, the integrand is defined by continuity to be $-z^2/2$. This result is an extension of the Kolmogorov Canonical Form ([3], Ch. III) for representing an infinitely divisible characteristic function $g(t)$.

The Theorem of Marcinkiewicz was extended to the following result.

THEOREM 4.1. *Consider the function* $f(z) = g(z)\exp[P_m(z)]$, *where* $g(z)$ *is an infinitely divisible characteristic function regular in a half-plane* $Im(z) > -\alpha (\alpha > 0)$. *If* $m > 3$, *or if* $m = 3$ *and* $\beta_3 \leqq 0$, *then* $f(z)$ *can not be a characteristic function.*

For the proof, we proceed as in the previous section. Let us then assume that $f(z)$ is a characteristic function. By (4.1) we find that

$$R_1(t, y) = \left|\frac{g(t+iy)}{g(iy)}\right| = \exp\left[\int_{-\infty}^{\infty} D(t, y, x)dK(x)\right], \tag{4.2}$$

where $D(t, y, x) = e^{-yx}((\cos tx - 1)/x^2)$; $D(t, y, 0)$ is defined by continuity to be $-t^2/2$. Using definition (2.2), we have

$$R_2(t, y) = \left|\frac{\exp[P_m(t+iy)]}{\exp[P_m(iy)]}\right| = \exp[A_1(t, y)] . \tag{4.3}$$

We now investigate

$$R(t, y) = R_1(t, y)R_2(t, y) . \tag{4.4}$$

Let an ε be given such that $0 < \varepsilon < 1$. By Lemma A and (4.3) we know that there exist a $\xi_m \geqq 0$ and a $Y = Y(m, \varepsilon)$ such that

$$R_2(y\sqrt{\xi_m}, y) > \frac{1}{1-\varepsilon} \qquad \text{for} \quad y \geqq Y . \tag{4.5}$$

We consider (4.2) for such t and y and have

$$R_1(y\sqrt{\xi_m}, y) = \exp\left[\int_{-\infty}^{\infty} D_1(y, x)dK(x)\right], \tag{4.6}$$

where $D_1(y, x) = D(y\sqrt{\xi_m}, y, x) = e^{-yx}((\cos yx\sqrt{\xi_m} - 1)/x^2)$. The proof of Theorem 4.3 will be completed if we can show that there exists a value $y^* > Y$ such that $R_1(y^*\sqrt{\xi_m}, y^*) > 1 - \varepsilon$. To do this, we first prove the following.

LEMMA 4.1. *Given any* $\varepsilon > 0$, *there exists an* $A = A(\varepsilon) > 0$ *such that*

$$\left|\int_A^{\infty} D_1(y, x)dK(x)\right| \leqq \frac{\varepsilon}{2}$$

and

$$\left|\int_{-\infty}^{-A} D_1(y, x)dK(x)\right| \leqq \frac{\varepsilon}{2}$$

for all y.

The proof follows easily if we note that, for $a > 0$,

$$\left|\int_A^{A+a} D_1(y, x)dK(x)\right| \leqq \int_A^{A+a} \frac{2}{x^2}dK(x) \leqq \frac{2}{A^2}\int_A^{A+a} dK(x) \leqq \frac{2C}{A^2}\ ,$$

where

$$C = \int_{-\infty}^{\infty} dK(x) < \infty\ .$$

Letting $a \to \infty$, $\left|\int_A^{\infty} D_1(y, x)dK(x)\right| \leqq 2C/A^2$. From this it is clear that, given any $\varepsilon > 0$, there exists an $A = A(\varepsilon)$ such that $2C/A^2 \leqq \varepsilon/2$. This proves the first result in the lemma, and the second follows in a similar manner.

For the fixed $\varepsilon\,(0 < \varepsilon < 1)$, we can then write

$$\int_{-\infty}^{\infty} D_1(y, x)dK(x) = \int_{-A}^{A} D_1(y, x)dK(x) + \varepsilon\theta\ , \tag{4.7}$$

where $|\theta| \leqq 1$, and where $A = A(\varepsilon)$ is determined by Lemma 4.1. By the first mean value theorem we obtain from (4.7) that

$$\int_{-\infty}^{\infty} D_1(y, x)dK(x) = MD_1(y, x^*) + \varepsilon\theta\ , \tag{4.8}$$

where $|x^*| < A$ and $M = M(A) = \int_{-A}^{A} dK(x) \leqq K(\infty) < \infty$.

We must distinguish between two cases. Suppose first that $x^* \neq 0$. We know from (4.6) that

$$D_1(y, x^*) = e^{-yx^*}\frac{\cos yx^*\sqrt{\xi_m} - 1}{(x^*)^2}\ . \tag{4.9}$$

Select an integer k such that $k > (Yx^*\sqrt{\xi_m})/2\pi$, where Y was introduced in (4.5), and choose $y^* = 2k\pi/(x^*\sqrt{\xi_m})$ so that $y^* > Y$. Then, by (4.9), $D_1(y^*, x^*) = 0$. It follows from (4.8) and (4.6) that $R_1(y^*\sqrt{\xi_m}, y^*) = \exp(\varepsilon\theta)$, where $|\varepsilon\theta| < 1$, and hence, by Maclaurin's series, that $R_1(y^*\sqrt{\xi_m}, y^*) > 1 + \varepsilon\theta \geqq 1 - \varepsilon$. This result together with (4.5) and (4.4) gives a contradiction of Theorem A, thus completing the proof when $x^* \neq 0$.

We now consider the case when $x^* = 0$. Since $D_1(y, 0) = -y^2\xi_m/2$, we have from (4.6), (4.8), (4.3) and (4.4) that

$$R(y\sqrt{\xi_m}, y) = \exp[-y^2\xi_m M/2 + \varepsilon\theta + A_1(y\sqrt{\xi_m}, y)]\ .$$

By the Corollary to Lemma A it follows immediately that $R(y\sqrt{\xi_m}, y) \to \infty$ as $y \to \infty$. This completes the proof of Theorem 4.1.

It is to be noted that Lemma A cannot be applied for the case $m = 3$ and $\beta_3 > 0$. However, for this case, the following result was obtained for two particular infinitely divisible characteristic functions, namely for those belonging to a Gamma distribution and to a negative-binomial distribution.

THEOREM 4.2. *Consider the function* $f(z) = g(z) \exp(c_1 z + c_2 z^2 + c_3 z^3)$, *where* $c_j = \alpha_j + i\beta_j$, *in the following two cases*:

Case I: $g(z) = (1 - (iz/\alpha))^{-\lambda}$, $(\alpha > 0, \lambda > 0)$.

Case II: $g(z) = (p/(1 - qe^{iz}))^r$, $(p > 0, q > 0, p + q = 1, r > 0)$.

If $f(z)$ is a characteristic function, and if $\beta_3 > 0$, then necessarily, in Case I, $\beta_3 \leqq |\alpha_2|/(3\alpha)$, and in Case II, $\beta_3 \leqq |\alpha_2/(3 \ln q)|$.

In the proof, we first note that if $f(t)$ is a characteristic function, then $P_3(t)$ takes on a special form. For, from some fundamental properties of characteristic functions, we must have $\alpha_1 = \alpha_3 = \beta_2 = 0$; it is no restriction to take $\beta_1 = 0$; and finally, since $f(t)$ is bounded, $\alpha_2 < 0$. Thus, letting $\gamma_2 = -\alpha_2$, we have $P_3(z) = -\gamma_2 z^2 + i\beta_3 z^3$, $(\gamma_2 > 0, \beta_3 > 0)$.

We now choose a real number η in the half-plane of regularity of $g(z)$ and construct the function $h(z) = f(z + i\eta)/f(i\eta)$. By Theorem B, $h(z)$ is an analytic characteristic function regular in a certain half-plane. But $h(-z)$ is also a characteristic function, and so, since the product of two characteristic functions is a characteristic function, $h_0(z) = h(z)h(-z)$ is also a characteristic function.

Case I. Choosing any real $\eta > -\alpha$, we obtain

$$h_0(z) = \left[1 + \frac{z^2}{(\alpha + \eta)^2}\right]^{-\lambda} \exp\left[-2(\gamma_2 + 3\beta_3\eta)z^2\right]$$

which is an analytic characteristic function regular in the strip $|y| < \alpha + \eta$. Setting $\eta = -\alpha\theta$, where $0 < \theta < 1$, we have

$$h_0(z) = [1 + z^2/\alpha^2(1 - \theta)^2]^{-\lambda} \exp\left[-2(\gamma_2 - 3\beta_3\alpha\theta)z^2\right] .$$

Consider $h_0(t)$ for real t and suppose $A = \gamma_2 - 3\beta_3\alpha\theta < 0$. Then $\exp(-2At^2) \to \infty$ as $t \to \infty$. Since $[1 + t^2/\alpha^2(1 - \theta)^2]^{\lambda} = o[\exp(-2At^2)]$ as $t \to \infty$, we have that $h_0(t) \to \infty$ as $t \to \infty$. This contradicts the fundamental property of boundedness for characteristic functions. We conclude therefore that $\gamma_2 - 3\beta_3\alpha\theta \geqq 0$f or any θ. This completes the proof of Case I.

It is interesting to note that the bound obtained for β_3 in no way depends on the parameter λ of the Gamma distribution.

Case II. Choosing any real number $\eta > \ln q$ and letting $\eta = \theta \ln q$ $(0 < q < 1)$, we obtain

$$h_0(z) = \left[\frac{(1-q^{1-\theta})^2}{1-q^{1-\theta}(e^{iz}+e^{-iz})+q^{2(1-\theta)}}\right]^r \exp\left[-2(\gamma_2+3\beta_3\theta\ln q)z^2\right],$$

which is an analytic characteristic function regular in the strip $(1-\theta)\ln q < y < (\theta-1)\ln q$. Again, consider $h_0(t)$ for real t. We observe that

$$\frac{(1-q^{1-\theta})^2}{1-q^{1-\theta}(e^{it}+e^{-it})+q^{2(1-\theta)}} \geqq \left(\frac{1-q^{1-\theta}}{1+q^{1-\theta}}\right)^2 > 0 .$$

Therefore, if $\gamma_2 + 3\beta_3\theta\ln q < 0$, then $h_0(t) \to \infty$ as $t \to \infty$ which is impossible. This completes the proof.

It may be pointed out that it would be interesting to know if the necessary conditions given in Theorem 4.2 are also sufficient conditions. It seems, however, to be not easy to arrive at a decision.

5. Acknowledgments. The results of this paper were submitted to the Catholic University of America in partial fulfillment of the requirements for the degree of Doctor of Philosophy. I would like to express my gratitude and indebtedness to Professor Eugene Lukacs for his invaluable help and suggestions.

Bibliography

1. H. Bohr, *Almost Periodic Functions*, New York: Chelsea Publ. Co., 1951.
2. D Dugué,, *Sur la théorème de Lévy-Cramér*, Publ. de l'Institut de Statistique de l'Université de Paris, Vol. VI (1957), 213-225.
3. B. V. Gnedenko, and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Cambridge: Addison-Wesley Publ. Co., Inc., 1954.
4. E. Lukacs, *Characteristic Functions*, London: Charles Griffin & Co., 1960.
5. ———, To be published in the Proceedings of the Thirty-Second Session of the International Statistical Institute held in Tokyo, May 30-June 10, 1960.
6. ———, *Some extensions of a theorem of Marcinkiewicz*, Pacific J. Math., **8** (1958), 487-501.
7. J. Marcinkiewicz, *Sur une propriété de la loi de Gauss*, Math. Zeitschrift, **44** (1938), 612-618.
8. D. A. Raikov, *On the decomposition of Gauss and Poisson laws*. Izvestija Akad. Nauk SSSR., ser. mat., **2** (1938), 91-124.

THE CATHOLIC UNIVERSITY OF AMERICA