A NOTE ON COOK'S WAVE-MATRIX THEOREM

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1. Introduction. Consider the linear operator H_0 defined by

$$[H_0u](\vec{x}) = -\nabla^2 u(\vec{x}) + V(\vec{x})u(\vec{x}) \tag{1.1}$$

over all $\vec{x} \in R_n$, n-dimensional Euclidean space, for each $u \in \mathscr{D}_0$. Here ∇^2 is the Laplacian and we take $\mathscr{D}_0$ as the set of all complex valued functions u over R_n which everywhere possess continuous partials of all orders $\leqq 2$ and which together with these partials are in absolute value $\leqq Q(|\vec{x}|)\exp(-2^{-1}|\vec{x}|^2)$ over R_n for some polynomial Q depending on u. Here V is a fixed, real valued, measurable function over R_n subject to additional assumptions below which will assure that H_0 takes $\mathscr{D}_0$ into $X = L_2(R_n)$ as a symmetric operator in the Hilbert space X.

Assuming that $V \in L_2(R_n)$ for $n = 3$, Cook [2] has shown that the unique existent (see Theorem I following) self-adjoint extension H of H_0 has the unitary operator

$$W(t) = e^{itH}e^{-it\tilde{H}} , \tag{1.2}$$

where $\tilde{H}$ is the similar extension of $\tilde{H}_0$ and $\tilde{H}_0$ differs from H_0 only by replacing $V(\vec{x})$ by zero in (1.1), to have existent isometric operators $W_\pm$ on X which are the strong limits of $W(t)$ as $t \to \pm\infty$. Moreover, $W_\pm\tilde{H} = HW_\pm$, the range spaces $Y_\pm = W_\pm X$ reduce H, and each H eigenvector is orthogonal to $Y_\pm$. In Theorem II bellow we give a significant sharpening of these results by weakening the restrictions upon V at ∞. Thus, with arbitrary $\rho > 0$, any function of the form $C|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geqq b$ will qualify under our assumptions (the Coulomb case $C|\vec{x}|^{-1}$ thus being borderline), while only such of form $C|\vec{x}|^{-3/2-\rho}$ there will do so under Cook's assumptions. In Theorem III we also generalize to dimension $n \geqq 3$. Cook's results are used by Ikebe [4] in showing $S = W_+^* W_-$, the "S-matrix", to be unitary with $Y_+ = Y_-$ and in showing the expected connection of the positive part of the spectrum of H with scattering theory under considerably more stringent conditions upon V. Our $n = 3$ existence result II for $W_\pm$ also includes that of Jauch & Zinnes ([5], p. 566), who assume $V(\vec{x}) = C|\vec{x}|^{-\beta}$ with $1 < \beta < 3/2$, and that of Hack [3], who replaces ${}_+\|V\|_\gamma < +\infty$ for some $\gamma \in [2,3)$ by the above noted stronger assumption that $|V(\vec{x})| \leqq M|\vec{x}|^{-1-\rho}$ over $|\vec{x}| \geqq b$ for some $\rho > 0$.*

2. Statements. As notation for our theorems, denote $D_b^+ = \{\vec{x} \in R_n \,|\, |\vec{x}| \geqq b\}$

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* *Note added in proof.* See also Kuroda, Nuovo Cim., **12**, (1959), p. 431–454 particularly Theorem 4.1), p. 444.

and $D_b^- = \{\vec{x} \in R_n \mid |\vec{x}| \leqq b\}$, $|\vec{x}| = [\sum_{j=1}^n x_j^2]^{1/2}$. Also for real $r \geqq 1$ and measurable u over D, let $f_r(u, D) = \left[\int_D |u|^r d\mu_n\right]^{1/r}$ with μ_n n-dimensional Lebesgue measure, and define $\|u\|_r = f_r(u, R_n)$ and $_+\|u\|_r = f_r(u, D_b^+)$ and $_-\|u\|_r = f_r(u, D_b^-)$ for specified real $b > 0$. Likewise $f_\infty(u, D) = (\operatorname{ess\,sup}_{\vec{x} \in D} |u(\vec{x})|)$ for measurable u over D defines $\|u\|_\infty$ and $_\pm\|u\|_\infty$ similarly. If r is suppressed, this denotes $\gamma = 2$, so that $\|u\|$ and $_\pm\|u\|$ are the $L_2(R_n)$ and $L_2(D_b^\pm)$ Hilbert space norms.

We also define on $X = L_2(R_n)$ the unitary Fourier-Plancherel transform operators U and $\tilde{U}$, having $\tilde{U} = U^* = U^{-1}$, by

$$(2.1) \qquad [\tilde{U}w](\vec{y}) = \lim_{T \to +\infty} (2\pi)^{-n/2} \int_{D_T^-} w(\vec{x}) e^{-i(\vec{x} \cdot \vec{y})} d\mu_n(\vec{x}) \, ,$$

$$(2.2) \qquad [Uw](\vec{x}) = \lim_{T \to +\infty} (2\pi)^{-n/2} \int_{D_T^-} w(\vec{y}) e^{i(\vec{x} \cdot \vec{y})} d\mu_n(\vec{y}) \, ,$$

for all $w \in X$, the limits being X norm limits. Here $(\vec{x} \cdot \vec{y}) = \sum_{j=1}^n x_j y_j$ is the R_n inner product. We also will need to consider the set G of all functions u of the form

$$(2.3) \qquad u = Uw \, , \; w(\vec{y}) = \exp(-a^2 |\vec{y} - \vec{z}|^2)$$

for some $\vec{z} \in R_n$ and real $a > 0$ depending upon u. With this notation our theorems are as follows.

THEOREM I. *Let real $b > 0$ and let η and γ be extended real satisfying $2 \leqq \eta$, $n/2 < \eta$, $\eta \leqq +\infty$ and $2 \leqq \gamma$, $n/2 < \gamma$, $\gamma \leqq +\infty$ for integer $n \geqq 1$, the dimension of R_n. Let real valued, measurable V over R_n satisfy both*

(i) $_-\|V\|_\eta < +\infty$,

(ii) $_+\|V\|_\gamma < +\infty$.

Then H_0 in (1.1) takes $\mathscr{D}_0$ into $X = L_2(R_n)$ as a symmetric operator, and H_0 possesses a unique self-adjoint extension operator H in X.

The special case of I where $\gamma = +\infty$ is our previous Theorem (T.1) of [1], except for the enlargement of the initial domain there to $\mathscr{D}_0$ here; the modification needed to take care of general γ is very slight. As there define $[Aw](\vec{y}) = |\vec{y}|^2 w(\vec{y})$ over $\vec{y} \in R_n$, the domain $\mathscr{D}_A$ of A being all $w \in X$ for which $|\vec{y}|^2 w(\vec{y})$ is also finitely square integrable. Then A is easily seen selfadjoint in X, and hence so is $\tilde{H} = UA\tilde{U}$ with domain $\mathscr{D} = U\mathscr{D}_A$; moreover, $\tilde{H}_0 \subseteqq \tilde{H}$ is now a consequence of standard Fourier transform theorems (or a simple use of Green's formula). With $\mathscr{D} = U\mathscr{D}_A$, and defining $[Vu](\vec{x}) = V(\vec{x})u(\vec{x})$, we have the following lemma.

LEMMA 2.4. *Let V satisfy the hypotheses of Theorem* I. *Then* *the function Vu is in X for all $u \in \mathscr{D}$. Moreover, for each real $\alpha > 0$ there exists real $\beta_\alpha > 0$ such that*

$$(2.5) \qquad ||Vu|| \leqq \alpha||\tilde{H}u|| + \beta_\alpha||u||$$

over $u \in \mathscr{D}$.

Since $\tilde{H}_0 \subseteqq \tilde{H}$ has $\mathscr{D}_0 \subseteqq \mathscr{D}$, from this lemma it follows that H_0 takes $\mathscr{D}_0$ into X, and Green's formula with the $\mathscr{D}_0$ exponential bound at ∞ shows that H_0 is symmetric. Also $Hu = \tilde{H}u + Vu$ for $u \in \mathscr{D}$ defines H from $\mathscr{D}$ into X, and $H_0 \subseteqq H$ follows from $\tilde{H}_0 \subseteqq \tilde{H}$. Also our Lemma 2.4 (replacing Lemma T.2 in [1]) shows H self-adjoint in X without any further change ([1], p.957). Likewise the previous approximation argument ([1], p.958) with Lemma 2.4 shows that H is the closure of $H_1 \subseteqq H_0 \subseteqq H$ and hence of H_0, and likewise $\tilde{H}$ is the closure of $\tilde{H}_1 \subseteqq \tilde{H}_0 \subseteqq \tilde{H}$ and hence of $\tilde{H}_0$. Thus H is the unique selfadjoint extension of H_0 and $\tilde{H}$ likewise of $\tilde{H}_0$, where H_1 and $\tilde{H}_1$ are the restrictions of H_0 and $\tilde{H}_0$ respectively to $\mathscr{D}_1 \subseteqq \mathscr{D}_0$, with $\mathscr{D}_1$ the Hermite functions. Thus Theorem I will be proved as soon as we prove Lemma 2.4 in the next section.

For our main Theorems II and III, we also need the following extension of Cook's [2] Lemma 2.

LEMMA 2.6. *If $u \in G$* (i.e. *of form* 2.3), *then with $0 < K_n < +\infty$ for real $r \geqq 1$ and real t*

$$(2.7) \quad |[e^{it\tilde{H}}u](\vec{x})| = [4(a^4 + t^2)]^{-n/4}\exp(-a^2[4(a^4 + t^2)]^{-1}|\vec{x} + 2t|\vec{z}^2) ,$$

$$(2.8) \qquad ||e^{it\tilde{H}}u||_r = [4(a^4 + t^2)]^{-(n/2)(1/2-1/r)}(a^2r)^{-n/2r}(K_n)^{1/r} ,$$

$$(2.9) \qquad ||e^{it\tilde{H}}u||_\infty = [4(a^4 + t^2)]^{-n/4}.$$

Moreover, for real valued, measurable V satisfying both (i) *and* (ii) *of Theorem* I *with extended real η and γ, there results for such u both*

$$(2.10) \qquad \int_{-\infty}^{\infty} ||Ve^{it\tilde{H}}u||dt < +\infty ,$$

$$(2.11) \qquad 0 = \lim_{|t|\to\infty} ||Ve^{it\tilde{H}}u|| ,$$

if $2 \leqq \eta$ and $2 \leqq \gamma < n$.

Since $2 \leqq \gamma < n$ in the last part of the lemma, this only applies when dimension $n \geqq 3$. From the crucial (2.10) and (2.11) (Corollary 2 and 1 of Cook's Lemma 2), the other arguments of Cook's paper [2] apply without other change and yeild all the conculsions of our following Theorems II and III, except for the unstated by Cook orthogonality of each H eigenvector in X to $Y_\pm$, which is an easy consequence of $W_\pm\tilde{H} = HW_\pm$ and hence $\tilde{H} = W_\pm^* HW_\pm$ and the reduction of H by $Y_\pm$.

Thus as soon as both Lemmas (2.4) and (2.6) are shown in the next section, all our Theorems I, II, and III will be proved.

THEOREM II. *Let* $n = 3$ *and for some real* $b > 0$ *let real valued, measurable* V *satisfy both* (i) *and* (ii) *of Theorem* I *with* $\eta = 2$ *and some real* γ *satisfying* $2 \leqq \gamma < 3$. *Then there exist isometric operators* W_+ *and* W_- *on* $X = L_2(R_3)$ *such that the unitary operator* $W(t)$ *in* (1.2) *has* $\lim_{t\to+\infty} \| W_+u - W(t)u\| = 0 = \lim_{t\to-\infty} \| W_-u - W(t)u\|$ *for every* $u \in X$. *Moreover,* $W_\pm \tilde{H} = HW_\pm$; $P_\pm = W_\pm W_\pm^*$ *are orthogonal projections whose range spaces* $Y_\pm = P_\pm X$ *reduce* H; *and every* $u \in \mathscr{D} = \mathscr{D}_H$ *satisfying* $Hu = \lambda u$ *for some scalar* λ *is orthogonal to* $Y_\pm$.

This is our new version of Cook's theorem, the special case here $\gamma = 2$ being exactly Cook's statement. Since in most applications the potential V will be bounded at ∞, and since

$$L_\infty(D_b^+) \cap L_2(D_b^+) \subset L_\infty(D_b^+) \cap L_\gamma(D_b^+)$$

properly for $\gamma > 2$ is easily seen, our version is essentially sharper than Cook's. As pointed out in the introduction it "almost" includes the Coulomb potential, which Cook's does not. (Actually, (2.10) fails for $V(\vec{x}) = C|\vec{x}|^{-1}$, $C \neq 0$.) We also remark that there would be no gain in allowing $2 \leqq \eta < 3$ in II instead of specifying $\eta = 2$, since ${}_-\| V\|_2 \leqq {}_-\| V\|_\eta [\mu_n(D_b^-)]^{1/2-1/\eta}$ follows from the Schwarz-Hölder inequality.

THEOREM III. *Let integer* $n \geqq 4$ *and for some real* $b > 0$ *let real valued, measurable* V *satisfy both* (i) *and* (ii) *of Theorem* I *with some real* η *and* γ *satisfying* $n/2 < \eta$ *and* $n/2 < \gamma < n$. *Then the Theorem* II *conclusions follow.*

As above, the assumptions in III are least restrictive with η as small as possible; and, for $V \in L_\infty(D_b^+)$ also holding, are then least restrictive with γ as large as possible.

3. **Proof of lemmas.** We start by proving Lemma 2.4, considering first the case $1 \leqq n \leqq 3$. For given $\alpha' > 0$, we see by taking $\omega > 0$ sufficiently small in equation (7) of [1] and by $\sqrt{a^2 + b^2} \leqq |a| + |b|$ that

$$\| u\|_\infty \leqq \alpha' \| \tilde{H}u\| + \beta'_{\alpha'} \| u\| \tag{3.1}$$

over all $u \in \mathscr{D}$ for some real $\beta'_{\alpha'} \geqq 1$. Now define real $r \geqq 2$ if $\gamma > 2$ in Theorem I (the Lemma (2.4) hypotheses) by requiring $2/\gamma + 2/r = 1$. Then (3.1) with $\beta'_{\alpha'} \geqq 1$ yields for $u \in \mathscr{D}$

$$\begin{aligned} \| u\|_r &\leqq [\| u\|_\infty^{r-2} \| u\|^2]^{1/r} = \| u\|^{2/r} (\alpha' \| \tilde{H}u\| + \beta'_{\alpha'} \| u\|)^{1-2/r} \\ &\leqq \alpha' \| \tilde{H}u\| + \beta'_{\alpha'} \| u\| . \end{aligned} \tag{3.2}$$

Thus (3.2), (ii) of I, and the Schwarz-Hölder inequality for the associated powers $r/2$ and $\gamma/2$ yield

$$(3.3)\qquad {}_{+}\|Vu\|^2 \leqq {}_{+}\|V\|_\gamma^2\|u\|_r^2 \leqq {}_{+}\|V\|_\gamma^2(\alpha'\|\tilde{H}u\| + \beta'_{\alpha'}\|u\|)^2 .$$

Also ${}_{-}\|V\|_2 \leqq [\mu_n(D_b^-)]^{1/2-1/\eta}{}_{-}\|V\|_\eta < +\infty$, using (i) of I and the Schwarz-Holder inequality with $\eta \geqq 2$, gives from (3.1)

$$(3.4)\qquad {}_{-}\|Vu\|^2 \leqq {}_{-}\|V\|_2^2\|u\|_\infty^2 \leqq {}_{-}\|V\|_2^2(\alpha'\|\tilde{H}u\| + \beta'_{\alpha'}\|u\|)^2$$

over $u \in \mathscr{D}$. (3.3) and (3.4) and $\|Vu\|^2 = {}_{+}\|Vu\|^2 + {}_{-}\|Vu\|^2$ and $\sqrt{a^2 + b^2} \leqq |a| + |b|$ yield (2.5), with $\alpha = M\alpha'$ freely chosen > 0 by choice of α', and $Vu \in X$ as desired if $\gamma > 2$. If $\gamma = 2$, then ${}_{-}\|V\|_2 < +\infty$ above with (ii) of I yields $\|V\|_2 < +\infty$; hence (3.1) yields (3.4) with the-script dropped, proving (2.5) and $Vu \in X$. Thus Lemma 2.4 has been shown if $1 \leqq n \leqq 3$.

Now consider the remaining case $n \geqq 4$ of Lemma 2.4. Here $2 \leqq n/2 < s \leqq +\infty$ for $s = \eta$ and $s = \gamma$, and hence real $\tau \geqq 2$ and $\mu \geqq 2$ are defined by the requirements $2/\gamma + 2/\tau = 1$ and $2/\eta + 2/\mu = 1$ respectively. Moreover, using $(n + \rho)2^{-1} = \gamma$ or η respectively, we see in [1] at the top of p. 956 that $r' = 4\gamma(2\gamma - 4)^{-1} = 2(1 - 2/\gamma)^{-1} = \tau$ or $r' = 4\eta(2\eta - 4)^{-1} = 2(1 - 2/\eta)^{-1} = \mu$ respectively, and equation (8) there becomes

$$(3.5)\qquad \|u\|_\tau \leqq \alpha'\|\tilde{H}u\| + \beta'_{\alpha'}\|u\| ,$$

$$(3.6)\qquad \|u\|_\mu \leqq \alpha'\|\tilde{H}u\| + \beta''_{\alpha'}\|u\|$$

respectively over $u \in \mathscr{D}$, with real $\beta'_{\alpha'} > 0$ and $\beta''_{\alpha'} > 0$ existing for each real $\alpha' > 0$. From (3.5) and (3.6) respectively, from (ii) and (i) respectively of I, and from the Schwarz-Hölder inequality we obtain respectively

$$(3.7)\qquad {}_{+}\|Vu\|^2 \leqq {}_{+}\|V\|_\gamma^2\|u\|_\tau^2 \leqq {}_{+}\|V\|_\gamma^2(\alpha'\|\tilde{H}u\| + \beta'_{\alpha'}\|u\|)^2 ,$$

$$(3.8)\qquad {}_{-}\|Vu\|^2 \leqq {}_{-}\|V\|_\eta^2\|u\|_\mu^2 \leqq {}_{-}\|V\|_\eta^2(\alpha'\|\tilde{H}u\| + \beta''_{\alpha'}\|u\|)^2$$

over $u \in \mathscr{D}$. Thus (3.7) and (3.8) and $\|Vu\| \leqq \sqrt{{}_{+}\|Vu\|^2 + {}_{-}\|Vu\|^2} \leqq {}_{+}\|Vu\| + {}_{-}\|Vu\|$ yields (2.5), with $\alpha = M\alpha' > 0$ freely chosen, and $Vu \in X$ as desired when $n \geqq 4$, completing the proof of Lemma 2.4.

Finally we must prove Lemma 2.6. Here from the proof of I (independently of any condition on V), we have $\tilde{H} = UA\tilde{U}$ to be the unique self-adjoint extension of $\tilde{H}_0$. Hence $e^{it\tilde{H}} = Ue^{itA}\tilde{U}$ and for u of form (2.3) we compute directly, since the L_1 Fourier transform and the L_2 Fourier-Plancherel transform are well known to coincide almost everywhere for functions in $L_1(R_n) \cap L_2(R_n)$,

$$
(3.9)\quad \begin{aligned}
[e^{it\tilde{H}}u](\vec{x}) &= (2\pi)^{-n/2}\int_{R_n}\exp(-a^2|\vec{y}-\vec{z}|^2+it|\vec{y}|^2+i(\vec{y}\cdot\vec{x}))d\mu_n(\vec{y}) \\
&= \prod_{j=1}^{n}\left\{(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(-a^2(y-z_j)^2+ity^2+iyx_j)dy\right\} \\
&= \exp\left(-a^2|\vec{z}|^2+4^{-1}(a^2-it)^{-1}\sum_{j=1}^{n}(2a^2z_j+ix_j)^2\right) \\
&\quad \prod_{j=1}^{n}\left\{(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-(a^2-it)y^2}dy\right\} \\
&= [2(a^2-it)]^{-n/2}\exp\left(-a^2|\vec{z}|^2+4^{-1}(a^2-it)^{-1}\sum_{j=1}^{n}(2a^2z_j+ix_j)^2\right).
\end{aligned}
$$

From (3.9) we readily obtain (2.7), from which (2.9) is obvious and (2.8) follows by the direct computation

$$
(3.10)\quad \begin{aligned}
\|e^{it\tilde{H}}u\|_r &= [4(a^4+t^2)]^{-n/4}\left[\int_{R_n}\exp(-a^2r4^{-1}(a^4+t^2)^{-1}|\vec{y}|^2)d\mu_n(\vec{y})\right]^{1/r} \\
&= [4(a^4+t^2)]^{-n/4}[a^{-2}r^{-1}4(a^4+t^2)]^{n/2r}(K_n)^{1/r}
\end{aligned}
$$

with $K_n = \int_{Rn} e^{-|\vec{y}|^2}d\mu_n(\vec{y})$ positive and finite.

Finally to prove last statement of Lemma 2.6 with conclusions (2.10) and (2.11), we here are given V to satisfy (i) and (ii) of I with $2 \leqq \gamma < n$ and $2 \leqq \eta$. Thus $_-\|V\|_2 \leqq {}_-\|V\|_\eta[\mu_n(D_b^-)]^{1/2-1/\eta} < +\infty$, as noted just before III, and by (2.9) for our $u \in G$

$$
(3.11)\quad {}_-\|Ve^{it\tilde{H}}u\| \leqq {}_-\|V\|_2[4(a^4+t^2)]^{-n/4}\,.
$$

Since $n > 2$ here, the right side of (3.11) is in $L_1(-\infty,\ \infty)$ over t. If $\gamma = 2$, then $_+\|V\|_2 < +\infty$ and (3.11) with the $-$ script replaced by $+$ shows $_+\|Ve^{itH}u\| \in L_1(-\infty,\ \infty)$ over t. If $\gamma > 2$, then the requirement $2/\gamma + 2/r = 1$ defines real $r \geqq 2$, and the Schwarz-Hölder inequality for this r yields from (2.8) and (ii) of I for our $u \in G$

$$
(3.12)\quad {}_+\|Ve^{it\tilde{H}}u\| \leqq {}_+\|V\|_\gamma M'(a^4+t^2)^{-(n/2)(1/2-1/r)} = M(a^4+t^2)^{-n/2\gamma}\,,
$$

which is in $L_1(-\infty, \infty)$ by $\gamma < n$. Hence (3.11) and (3.12) and $\|w\| \leqq {}_+\|w\| + {}_-\|w\|$ prove (2.10) and (2.11), and the proof of Lemma 2.6 is complete.

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