INSTABILITY AND ASYMPTOTICITY IN TOPOLOGICAL DYNAMICS

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It is the purpose of this paper to sharpen some results obtained by the author [1] and to extend to more general groups results obtained by Schwartzman [4, 10,36] and Bryant [3]. The original source from which many of these notions spring is Utz [8]. The principal result of the paper is this: If X is a compact Hausdorff space, T a generative group, (X, T, π) an unstable (i.e. expansive) transformation group, and P a replete semigroup in T, then there exists a replete semigroup Q in P and a pair of distinct points of X which are Q-asymptotic. In one sense this is the best result that can be hoped for, since under the above hypotheses it is not necessarily the case that there exist a pair of distinct points of X which are P-asymptotic. A study of the symbolic transformation group over the lattice points of the plane [1, § 3] shows that there does not exist a pair of distinct points of the space N which are asymptotic relative to the replete semigroup $P = \{(x, y) | x, y \text{ integers}, x > y\}$ or to any of its translates, pp, $p \in p$.

The pertinent definitions for what follows are contained in either [4] or [1]. In §1 we prove some general results about instability and asymptoticity; in §2 we prove the principal result mentioned above. Throughout the paper we assume that X is a Hausdorff space, and more often than not, compact. We feel free to use the fact that X is a uniform space, if it is compact; and we assume tacitly that the (Hausdorff) topology of X is the one induced by the uniformity.

1. Theorems on instability and asymptoticity.

1.01 THEOREM [*Inheritance of instability*]. *Let X be a compact space, (X, T, π) a transformation group. Let S be a left syndetic subgroup of T, then (X, T, π) is unstable if and only if (X, S, π) is unstable.*

Proof. Let (X, T, π) be unstable. There is a compact set $K \subset T$ such that $T = SK$. Let δ be the index of instability of (X, T, π), then there is an index β of X such that $(x, y) \in \delta'$ (the complement of δ), $k^{-1} \in K^{-1}$ implies $(xk^{-1}, yk^{-1}) \in \beta'$ for all $k \in K$ [4, 1.21]. Now let $x, y \in X$, then there exists $t \in T$ such that $(xt, yt) \in \delta'$; further $t = sk$ for some $s \in S$ and $k \in K$, whence $(xsk, ysk) \in \delta'$ and $(xs, ys) = (xskk^{-1}, yskk^{-1}) \in \beta'$.

Received August 19, 1960, and in revised form March 22, 1961. The author is a National Science Foundation Science Faculty Fellow.

Thus (X, S, π) is unstable with instability index β.

Conversely let (X, S, π) be unstable with instability index α. Let $x, y \in X$, then there exists $s \in S \subset T$ such that $(xs, ys) \in \alpha'$, i.e. (X, T, π) is unstable with instability index α.

1.02 LEMMA. *Let S be a closed syndetic subgroup of T, an abelian topological group, and let P be a replete semigroup in T. Then there exists a compact subset K of T and an element $r \in P$ such that*

(1) $T = SK = SK^{-1}$,

(2) $K \subset P$,

(3) $(P \cap S) K \supset rP$.

Proof. Since S is syndetic, there exists a compact set $K_0 \subset T$ such that $T = SK_0$. Since P is replete, so also is P^{-1} and there exists $t \in T$ such that $tK_0 \subset P^{-1}$. Further there exists $r \in T$ such that $r\{tK_0 \cup e\} \subset P$. Let $K = r\{tK_0 \cup e\}$, then it is clear that $K \subset P$, $r \in P$, and $T = SK = S^{-1}K^{-1} = SK^{-1}$.

Let $p \in P$, then since $T = PP^{-1}$ [4, 6.04] there exist $q_1, q_2 \in P$ such that $p = q_1q_2^{-1}$. Further since S is syndetic $q_1q_2t^{-1}K_0^{-1} \cap S \neq \phi$, and we may select $k^{-1} \in K_0^{-1}$ such that $q_1q_2t^{-1}k^{-1} \in S$. We remark that since $tk \in P^{-1}$, $t^{-1}k^{-1} \in P$, thus $q_1q_2t^{-1}k^{-1} \in P$. Now $rp = rq_1q_2^{-1} = rtk(q_1q_2t^{-1}k^{-1})$ and $rtk \in K$ and $q_1q_2t^{-1}k^{-1} \in P \cap S$. Thus $rP \subset (P \cap S)K$.

1.03. THEOREM [*Inheritance of Asymptoticity*]. *Let X be a compact space, T an abelian topological group, S a closed syndetic subgroup of T and (X, T, π) a transformation group, then*

I. *If A is a closed invariant set in $X, x\uparrow_P A$ if and only if $x\uparrow_{P\cap S} A$.*

II. *$x\uparrow_P y$ if and only if $x\uparrow_{P\cap S} y$.*

Proof. I. Suppose $x\uparrow_P A$. Let U be a neighborhood of A, then there is a $q \in P$ such that $xqP \subset U$. Select K according to Lemma 1.02, then there is an $s \in S$ and a $k \in K \subset P$ such that $q = sk^{-1}$, i.e. $s = qk \in P$, whence $sP = qkP \subset qP$, and $xsP \subset xqP \subset U$. Thus there is an $s \in P \cap S$ such that $xs(P \cap S) \subset xsP \subset U$ and $x\uparrow_{P\cap S} A$.

Conversely, let $x\uparrow_{P\cap S} A$. Let U be a neighborhood of A and let K and r be selected according to Lemma 1.02. Then since A is invariant $AK = A$, and since A is compact, there is a neighborhood V of A such that $VK \subset U$. Let $q \in P \cap S$ such that $xq(P \cap S) \subset V$, let $t = qr \in PP \subset P$, then $xtP = xqrP \subset xq(P \cap S)K \subset VK \subset U$. Thus $x\uparrow_P A$.

II. Let $x\uparrow_P y$, Let α be an index of X, then there exists $q \in P$ such that $(xqp, yqp) \in \alpha$ for all $p \in P$. Select K according to Lemma 1.02, then $q = sk^{-1}$, where $s \in S$ and $k \in K$, whence $s = qk \in P \cap S$. Since

$kP \subset P$, it follows that $(xqkp, yqkp) \in \alpha$ for all $p \in P$. Thus there is an $s \in P \cap S$ such that $(xsp, ysp) \in \alpha$ for all $p \in P \cap S \subset P$, and $x \uparrow_{P \cap S} y$.

Conversely, let $x \uparrow_{P \cap S} y$. Let α be an index of X and select K and r according to Lemma 1.02. Let β be an index of X such that $x \in X$ and $k \in K$ implies $x\beta k \subset xk\alpha$. Let $q \in P \cap S$ such that $yqs \in xqs\beta$ for all $s \in P \cap S$. Let $t = qr \in PP \subset P$ and let $p \in P$. Then since $(P \cap S)K \supset rP$, $rp = sk$, where $s \in P \cap S$ and $k \in K$. Then $ytp = yqrp = yqsk \in xqs\beta k \subset xqsk\alpha = xqrp\alpha = xtp\alpha$ and $(xtp, ytp) \in \alpha$ for each $p \in P$, whence $x \uparrow_P y$.

1.04 LEMMA. *Let X be compact and let (X, T, π) be unstable, then the set of fixed points of (X, T, π) is finite.*

Proof. Suppose the set of fixed points is infinite. Let y be a limit point of the set of fixed points, then clearly y is also a fixed point. Let δ be the index of instability of (X, T, π), then there exists a fixed point $x \neq y$ such that $(x, y) \in \delta$; and since x and y are both fixed $(xt, yt) = (x, y) \in \delta$ for all $t \in T$. This contradicts the instability of (X, T, π).

1.05. LEMMA. *Let X be compact, and let (X, T, π) be unstable, then the set of points with period P is finite.*

Proof. Let x be a point with period P, i.e. $xP = x$ and P is syndetic. We remark that P is a subgroup of T. Consider (X, P, π), then by 1.01 (X, P, π) is unstable and by 1.04 the set of fixed points of (X, P, π) is finite. However, the set of fixed points of (X, P, π) contains the set of points of period P of (X, T, π).

1.06. COROLLARY. *Let X be compact, let T have at most countably many syndetic subgroups, and let (X, T, π) be unstable, then the set of periodic points is at most countable.*

1.07. THEOREM. *Let X be compact and dense-in-itself, and let (X, T, π) be unstable. Let T have at most countably many syndetic subgroups, then*
(1) *T is not pointwise periodic and*
(2) *T is not almost periodic.*

Proof. (1) Follows from Lemma 1.05. (2) Follows from [4, 4.35 and 4.38].

1.08. REMARK. Let (X, T, π) be a transformation group, P a replete semigroup in T, and let $\mathscr{P} = \{(x, y) \mid x \uparrow_P y\}$, then
(1) $\mathscr{P}$ is a relation in $X \times X$,
(2) $\mathscr{P}$ is symmetric and nonreflexive,

(3) if T is abelian $\mathscr{P}$ is transitive.

2. The relation of instability and asymptoticity.

2.01. REMARK. Throughout what follows we make frequent use of the fact that if T is abelian and P is a replete semigroup in T, then $T = PP^{-1}$ [4, 6.04]. We feel free to use this fact without further explicit mention.

2.02. LEMMA. *Let P be a replete semigroup in T, abelian. Let X be uniform, (X, T, π) be a transformation group and let $x \neq y$, then the following statements are pairwise equivalent:*

(1) $x \uparrow_P y$

(2) *For all* $p \in P$, $x \uparrow_{pP} y$

(3) *There is a* $p \in P$ *such that* $x \uparrow_{pP} y$

(4) *There is a* $t \in T$ *such that* $xt \uparrow_P yt$

(5) *For all* $t \in T$, $xt \uparrow_P yt$.

Proof. The following implications are clear: (1)→(2)→(3)→(1)→(4) and (5)→(1). We prove (4)→(5). Let $s \in T$, then $t = pq^{-1}$ and $s = ru^{-1}$ where $p, q, r, u \in P$. Let α be an index of X, then there exists $v \in P$ such that $(xtvw, ytvw) \in \alpha$ for all $w \in P$. Then $(xtvwrq, ytvwrq) \in \alpha$ for all $w \in P$, since $wrq \in P$. But $tvwrq = pq^{-1}\, vwrquu^{-1} = (ru^{-1})\, (pvu)w = s(pvu)w$, Let $pvu = v_0 \in P$, then $(xsv_0w, ysv_0w) \in \alpha$ for all $w \in P$, whence $xs \uparrow_P ys$.

2.03. LEMMA. *Let X be compact, let T be a generative separable group, and let (X, T, π) be unstable. Let γ be the instability index of X, and let $N(\Delta)$ be a neighborhood of Δ, the diagonal of $X \times X$, such that the closure of $N(\Delta) \subset \gamma$. Then if there exists $p \in P$, a replete semigroup in T, such that $(xpq, ypq) \in N(\Delta)$ for all $q \in P$, then $x \uparrow_P y$, provided that $x \neq y$.*

Proof. Use 2.02 and [1, 2.16]

2.04. REMARK. If T is generative, $T = C \times R^m \times I^n$, $m \geqq 0$, $n \geqq 0$, where C is a compact abelian group, R is the reals, and I the integers [9, pg. 110]. Let e be the identity of C, then $S = e \times R^m \times I^n$ is a syndetic invariant subgroup of T. Since asymptoticity is inherited from S to T and vice versa [cf. 1.03] we may assume that T has the structure $R^m \times I^n$; however, in order to shorten proofs we shall assume that $T = R^n$, $n > 0$. Minor modifications will adapt the proofs to the cases $T = R^m \times I^n$ and $T = I^n$.

2.05. *Standing Notation.* $T = R^n$, $n > 0$. We use additive vector notation and assume an inner product $a \cdot b$ and a norm $|a|$ have been defined in T such that $|a| = (a \cdot a)^{1/2}$ and that the norm is consistent with the topology of T. 0 denotes the null vector in T, and Greek letters, $\alpha, \beta, \cdots$ denote scalars.

2.06. DEFINITION. Let p^* be a unit vector in T, let $0 < \alpha < 1$, then $C_0 = \{p \mid (p \cdot p^*/|p|) \geqq 1 - \alpha\}$ is called a *solid cone with vertex at the origin and axis* p^*. Let $C = C_0 + p$, where $p \in C_0$, then C is called a *solid cone.*

2.07. REMARK. A solid cone is a convex set and is a replete semigroup. Furthermore any replete semigroup P in $T = R^n$ contains a solid cone. The notion of solid cone is intimately related to the notion of a wedge [5, 3.4]; in fact in view of [5, 3.5 and 3.6] every solid cone contains a wedge and conversely.

2.08. LEMMA. *Let A be a closed invariant set in X, a compact space, let $\{V_n \mid n = 1, 2, \cdots\}$ be a sequence of open sets and let U be an open set such that $A \subset V_n \subset V_{n-1} \subset U$ for $n > 1$ and such that $\bigcap_{n=1}^{\infty} V_n = A$. Let $N_i = \{t \mid t \in T, v_i t \notin U$ for some $v_i \in V_i\}$ be nonvacuous and let $n_i = \inf_{t \in N_i} |t|$, then $\lim_{i \to \infty} n_i = \infty$ and in fact monotonically.*

Proof. Let $S(n_i) = \{t \mid |t| < n_i\}$, then $t \in S(n_i)$ implies $v_i t \in U$ for all $v_i \in V_i$. Now $S(n_i) \supset S(n_{i-1})$, since $V_i \subset V_{i-1}$, hence $t \in S(n_{i-1})$ implies $v_i t \in U$ for each $v_i \in V_i$, since $V_i \subset V_{i-1}$. Thus $t \notin N_i$ and $\inf_{t \in N_i} |t| \geqq n_{i-1}$ and the sequence $\{n_i\}$ is monotonically increasing.

Suppose $n_i \to \infty$ is false, then there exists and $n > 0$ such that $n_i < n$ for all i. Let $S(n) = \{t \mid |t| < n\}$, then $AS(n) = A$, thus there is a neighborhood W of A such that $WS(n) \subset U$, and there is an i such that $V_i \subset W$, whence $V_i S(n) \subset WS(n) \subset U$. Thus $t \in S(n)$ implies $v_i t \in U$ for all $v_i \in V_i$, whence $t \notin N_i$ and this implies $n_i = \inf_{t \in N_i} |t| \geqq n$. This is a contradiction, therefore $n_i \to \infty$.

2.09. REMARK. Since we are considering asymptoticity, and since this property is invariant under translation [cf. 2.02] of asymptotic points, and since we shall be discussing solid cones in relation to asymptoticity, we may assume that all solid cones have their vertices at the origin.

2.10. LEMMA. *Let P be a solid cone in R^n with vertex at the origin and axis P^*, let Q be a solid cone with vertex at the origin and axis p^*, lying except for 0, in the interior of P. Let $\{-q_i\} \subset Q^{-1}$ be a sequence of points such that $|-q_i| \to \infty$ and $-q_i/|q_i| \to -q^*$ as $i \to \infty$, where $-q^*$ is some unit vector. Let $R_i = \{q_i - q \mid q \in Q, |q| \leqq |q_i|\}$.*

Then there exists a solid cone $R \subset P$, such that if $S(m) = \{s \mid |s| \leqq m\}$, there exists a subsequence of $\{q_i\}$ which we again call $\{q_i\}$ and an integer $N = N(m)$ such that $i > N$ implies $R_i \supset R \cap S(m)$.

Proof. Let $P = \{p \mid (p \cdot p^*/|p|) \geqq 1 - \alpha\}$, where $0 < \alpha < 1$, and let $0 < \beta < \alpha$ and $Q = \{q \mid (q \cdot p^*/|q|) \geqq 1 - \beta\}$. We assume without loss that $|q_i| = i$ hence that $\lim_{i \to \infty} q_i/|q_i| = \lim_{i \to \infty} q_i/i = q^*$. We wish to show that there exists a solid cone $R \subset P$ and an integer $N = N(m)$ such that $S(m) \cap R \subset R_i$ for $i > N$. There are two cases.

Case I. $q^* \cdot p^* > 1 - \beta$.

Let T be a solid cone with vertex 0, with axis q^* and lying except for 0, interior to P. Let R be a solid cone with vertex 0, with axis q^*, and lying except for 0, interior to T. Let $t \in T$, and consider $q_i - t$. Then for i sufficiently large $|q_i - t| > 0$ and

$$\lim_{i \to \infty} \frac{(q_i - t) \cdot p^*}{|q_i - t|} = \lim_{i \to \infty} \frac{|q_i|}{|q_i - t|} \left(\frac{q_i}{|q_i|} - \frac{t}{|q_i|} \right) \cdot p^* = q^* \cdot p^* > 1 - \beta \,.$$

Thus for i sufficiently large $(q_i - t) \in Q$.

We show that for i sufficiently large $|q_i - t| < |q_i|$.

This will follow if $t \cdot t < 2q_i \cdot t$, or if $t \cdot t < 2|q_i|(q_i \cdot t/|q_i|)$, but since $\lim_{i \to \infty} |q_i| = \infty$ and $\lim_{i \to \infty} q_i \cdot t/|q_i| = q^* \cdot t > 0$, we have the desired result. Thus $t \in R_i$ for i sufficiently large.

The set R_i is a convex set. Given any compact subset C of R there exists a finite set $F \subset T$ whose convex hull contains C. This finite set will be contained in R_i for i sufficiently large and thus C will be contained in R_i for i sufficiently large. Thus there exists $N = N(m)$ such that $n > N$ implies $S(m) \cap R \subset R_i$.

Case II. $q^* \cdot p^* = 1 - \beta$.

Consider $t^* = p^* + \delta(q^* - p^*)$. We can choose $\delta > 1$ so that t^* lies interior to P and let δ be so chosen. Then

$$\begin{aligned}(p^* \cdot p^*)&(q^* \cdot t^*) - p^* \cdot t^* \\ &= (p^* \cdot q^*)(q^* \cdot p^* + \delta - \delta(q^* \cdot p^*)) - 1 - \delta(p^* \cdot q^*) + \delta \\ &= (\delta - 1)(1 - (p^* \cdot q^*)^2) > 0 \,.\end{aligned}$$

Thus there exists a cone T, with axis $t^*/|t^*|$, with vertex at 0, which, except for 0, lies interior to P, such that if $t \in T$ then

$$(p^* \cdot q^*)(q^* \cdot t) > (p^* \cdot t) \,. \tag{1}$$

Let $t \in T$, and consider $q_i - t$. We show that for i sufficiently large

$$(q_i - t) \cdot p^* > p^* \cdot q^* |q_i - t| \, . \tag{2}$$

This will follow if

$$(q_i \cdot p^*)^2 - 2(q_i \cdot p^*)(t \cdot p^*) + (t \cdot p^*)^2 \geqq (p^* \cdot q^*)^2 [q_i \cdot q_i - 2q_i \cdot t + t \cdot t] \, . \tag{3}$$

This will follow if both of the following are true:

$$(q_i \cdot p^*)^2 \geqq (p^* \cdot q^*)^2 (q_i \cdot q_i) \, , \tag{4}$$

$$(t \cdot p^*)^2 - 2(q_i \cdot p^*)(t \cdot p^*) \geqq (p^* \cdot q^*)^2 [t \cdot t - 2q_i \cdot t] \, . \tag{5}$$

Now (4) is equivalent to

$$\frac{q_i}{|q_i|} \cdot p^* \geqq p^* \cdot q^* \tag{4'}$$

but since $q_i \in Q$

$$\frac{q_i}{|q_i|} \cdot p^* \geqq 1 - \beta = p^* \cdot q^* \, .$$

Thus (4) is valid.

Now (5) is equivalent to

$$\frac{1}{|q_i|}(t \cdot p^*)^2 - 2\left(\frac{q_i}{|q_i|} \cdot p^*\right)(t \cdot p^*) \geqq \frac{1}{|q_i|}(p^* \cdot q^*)^2 (t \cdot t) - 2(p^* \cdot q^*)^2 \left(\frac{q_i}{|q_i|} \cdot t\right) . \tag{5'}$$

As i become infinite, the left side of (5′) approaches

$$-2(q^* \cdot p^*)(t \cdot p^*)$$

and the right side of (5′) approaches

$$-2(p^* \cdot q^*)^2 (q^* \cdot t) \, .$$

Thus, in view of (1) it follows that (5′) is valid for i sufficiently large; and thus (2) is valid for sufficiently large i.

We show that for i sufficiently large $|q_i - t| < |q_i|$. This follows if $t \cdot t < 2q_i \cdot t$, or if

$$t \cdot t < 2|q_i| \frac{q_i \cdot t}{|q_i|} \, .$$

Now $\lim_{i \to \infty} |q_i| = \infty$, and $\lim_{i \to \infty} q_i \cdot t/|q_i| = q^* \cdot t$, but by (1)

$$q^* \cdot t > \frac{p^* \cdot t}{p^* \cdot q^*}$$

and further

$$\frac{p^* \cdot t}{p^* \cdot q^*} > \frac{1-\alpha}{1-\beta} > 0 .$$

Thus $t \in R_i$ for i sufficiently large. Now choose a cone R interior to T and argue by convexity as in Case I. This completes the proof.

The author wishes to thank Professor G. A. Hedlund for the suggestion for the preceding lemma and its proof.

At this point we drop the additive notation in T.

2.11. THEOREM. *Let X be a compact Hausdorff space, $T = R^n$, (X, T, π) a transformation group, and let P be a replete semigroup in T. Let $A \subset X$ be a closed invariant set, U open in X, and $A \subset U$. Then there exists a replete semigroup $R \subset P$ such that either*

1. *There exists E closed in X, $A \subset E \subset \bar{U}$, $E \not\subset U$, and $ER \subset E$, or*
2. *There exists V open in X, $A \subset V \subset U$ and $VR^{-1} \subset U$.*

Proof. By 2.07 we may assume P is a solid cone with vertex 0. Assume 2. fails, then since X is compact Hausdorff, we may select a sequence of open sets $\{V_i\}$ such that $A \subset V_i \subset V_{i-1}$ for all i and $\bigcap_{i=1}^{\infty} V_i = A$. Let Q be a solid cone with vertex 0 lying, except for 0, interior to P. Define $Q_i^{-1} = \{q^{-1} \mid q \in Q, v_i q^{-1} \notin U$ for some $v_i \in V_i\}$. By assumption $Q_i^{-1} \neq \phi$, thus let $d_i = \inf_{q^{-1} \in Q_i^{-1}} |q^{-1}|$. Let n_i be such that $V_i S(n_i) \subset U$, where $S(n_i) = \{t \mid |t| < n_i\}$, and such that there is a $t \in \text{Closure}(S(n_i))$ and a $v_i \in V_i$ such that $v_i t \notin U$. Now by Lemma 2.08, $n_i \to \infty$, and since $n_i \leqq d_i$, $d_i \to \infty$. Select $q_i^{-1} \in Q_i^{-1} \cap \text{Closure}(S(d_i))$ and $v_i \in V_i$ such that $v_i q_i^{-1} \notin U$, and by choosing appropriate subsequences we may assume $v_i q_i^{-1} \to x \in X$ and $q_i^{-1}/|q_i^{-1}| \to q^{*-1}$. Let R be the solid cone determined by Lemma 2.10, so that $R \subset P$ and $R_i = \{q_i q^{-1} \mid q^{-1} \in S(d_i) \cap Q^{-1}\} \supset R \cap S(m)$ for prescribed $m > 0$ and i sufficiently large. Let $E = A \cup \text{Closure}(xR) \cup x$, then clearly E is closed and further $A \subset E$.

Now since $A \subset U$, $A \subset \bar{U}$. Also $x \in \bar{U}$, for suppose not, then there is an open set W such that $x \in W$ and $W \cap \bar{U} = \phi$, and there exists $v_i \in V_i$ and q_i such that $v_i q_i^{-1} \in W$ and a spherical neighborhood S of q_i^{-1} such that $v_i S \subset W$. Select $q^{-1} \in S \cap Q^{-1}$ such that $|q^{-1}| < |q_i^{-1}|$, then $v_i q^{-1} \notin U$, and this contradicts the choice of q_i^{-1}.

Now let $r \in R$ with $|r| = m$, then there exists an integer i such that $j \geqq i$ implies $r \in R \cap S(m+1) \subset R_j$ by Lemma 2.10. Now $S(|q_i^{-1}|) \cap Q^{-1} \subset S(|q_j^{-1}|) \cap Q^{-1}$ for $j \geqq i$. Let $j > i$, then $r = q_j q_{r_j}^{-1}$ for some $q_{r_j}^{-1} \in S(|q_j^{-1}|) \cap Q^{-1}$. Then $v_j q_{r_j}^{-1} = v_j q_j^{-1} q_j q_{r_j}^{-1} = (v_j q_j^{-1}) r \to xr$ and since $v_j q_{r_j}^{-1} \in U$ for each $j > i$, $xr \in \bar{U}$. Thus $xR \subset \bar{U}$, whence also $\text{Closure}(xR) \subset \bar{U}$. Thus all told we have $E \subset \bar{U}$.

$E \not\subset U$, for x as a limit point of $\{v_i q_i^{-1}\}$ all of which are in the complement of U is also in the complement of U.

$ER \subset E$, for let $r \in R$, then if $y \in A$, $yr \in A \subset E$, if $y \in$ Closure (xR), $yr \in$ (Closure $(xR))r =$ Closure $(xrR) \subset$ Closure $(xR) \subset E$, and if $y = x$, $yr = xr \in$ Closure $(xR) \subset E$.

Thus R is the desired replete semigroup, and E the desired closed set which satisfy 1. This completes the proof.

The preceding theorem has a rather interesting history. The above is a modification of a theorem due to Montgomery [7], which occurs also in [4, 10.29], though in quite a different form. In this connection see also Kerékjártó [6].

2.12. THEOREM. *Let X be a compact infinite metric space, T a generative group, (X, T, π) an unstable transformation group. Let P be a replete semigroup in T, then there is a replete semigroup $Q \subset P$ and there are distinct points, x, y, such that $x \uparrow_Q y$.*

Proof. By 2.04 we may assume $T = R^n$. Consider the transformation group $(X \times X, T, \sigma)$, where $\sigma[(x, y), t] = (xt, yt)$. The diagonal Δ of $X \times X$ is a closed invariant set of $X \times X$. Let $U_n = \{(x, y) \mid \rho(x, y) < 1/n\}$ where ρ is the metric of X, then U_n is an open set such that $U_n \supset \Delta$. If conclusion 2. of Theorem 2.11 holds for all $n \geqq N$ and if α is an index of X, then there exists n such that $n \geqq N$ and $U_n \subset \alpha$. By Theorem 2.11, there is an open set V, $\Delta \subset V \subset U_n$ and a replete semigroup $R \subset P$ such that $VR^{-1} \subset U_n \subset \alpha$. Let β be an index of X such that $\beta \subset V$, then $\beta R^{-1} \subset \alpha$. This, however, implies that T is equicontinuous [2], and this contradicts the instability of (X, T, π).

Thus conclusion 1. of Theorem 2.11 holds for arbitrarily large values of n, i.e. for each $n > 0$, there exists an integer $m > n$ for which there exists a closed set E_m and a replete semigroup $R_m \subset P$ such that $\Delta \subset E_m \subset \bar{U}_m$, $E_m \not\subset U_m$, and $E_m R_m \subset E_m$. Let δ be the index of instability of X; select n so large that $n > 1/\delta$, and $m > n$ for which conclusion 1. of Theorem 2.11 holds. Select $(x, y) \in E_m$ such that $(x, y) \notin U_m$, whence $x \neq y$, and a replete semigroup $Q = R_m \subset P$, then $(x, y)Q = (x, y)R_m \subset E_m R_m \subset E_m \subset \bar{U}_m \subset \delta$, whence $(xq, yq) \in \delta$ for all $q \in Q$. By Lemma 2.02 this implies $x \uparrow_Q y$. This completes the proof.

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