THE GENERALIZED WHITEHEAD PRODUCT

MARTIN ARKOWITZ

Introduction. In this paper we investigate an operation which is a generalization of the Whitehead product for homotopy groups. Let $\pi(R, S)$ denote the collection of homotopy classes of base point preserving maps of R into S, let ΣR denote the reduced suspension of R, and let $R \mathbin{\#} S$ be the identification space $R \times S/R \vee S$ (see § 1 for complete definitions). Then this generalized Whitehead product (written GWP) assigns to each $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ an element $[\alpha, \beta] \in \pi(\Sigma(A \mathbin{\#} B), X)$, where A and B are polyhedra and X is a topological space. In the case when A and B are spheres $[\alpha, \beta]$ is essentially the Whitehead product. In this paper we generalize known results on spheres and Whitehead products to polyhedra and GWPs.

The paper is divided into six parts. After the preliminaries of § 1 we present two definitions of the GWP in § 2. The first definition, which was given by Hilton in [8 ; pp. 130-131], is closely related to a commutator of group elements. The second definition is essentially a generalization of the ordinary Whitehead product. It first appeared, stated in the language of carrier theory, in a paper by D. E. Cohen [3]. We prove in Theorem 2.4 that these two definitions coincide. This generalizes a result of Samelson [11 ; p. 750].

In § 3 we establish some properties of the GWP such as anti-commutativity and bi-additivity. With the exception of Proposition 3.1 the results of this section have been obtained by Cohen [3]. However, the proofs that we give are based on the first definition and facts about commutators. Moreover, we believe that our proofs are quite elementary.

In the next section we show that $\Sigma A \times \Sigma B$ has the same homotopy type as the space obtained by attaching a cone by means of the GWP map. We then deduce a few simple consequences of this. In § 5 we consider the different ways that the GWP may be trivial. We study the following situations : (i) $[\alpha, \beta] = 0$ (ii) the GWP map is null-homotopic (iii) X is a space in which all GWPs vanish. With regard to (iii) we see that such spaces are not necessarily H-spaces.

The final section is devoted to a product which is dual (in the sense of Eckmann and Hilton) to the GWP. Two definitions of the dual product are given and they are shown to be equivalent. We also indicate some properties of the dual product.

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A recent application of GWPs has been given by Barcus [1].

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1. Preliminaries. In this section we recall some facts about homotopy theory. The general reference is [8] (see also [4]). Throughout this paper we consider topological spaces with base points. The base point is always denoted by $*$ but will often not be explicitly mentioned. By a polyhedron we shall mean the underlying space of a locally-finite, connected CW-complex [13] with some vertex as base point. All maps and homotopies are to preserve base points. If R and S are spaces (with base points) then $\pi(R, S)$ denotes the collection of homotopy classes of maps of R into S. If $f, g : R \to S$ are maps then $f \simeq g$ signifies that f and g are homotopic. The homotopy class of $f : R \to S$ is written $\{f\} \in \pi(R, S)$. We denote by $0 : R \to S$ the constant map which is defined by $0(R) = * \in S$.

If I denotes the closed interval [0,1], then *cone on a space* R, TR, is the space obtained from $R \times I$ by "factoring out" the equivalence relation $(r, 1) \sim (r', 1)$ for all $r, r' \in R$. The base point of TR is $(*, 0)$. (Note that a point in an identification (quotient) space is designated by its pre-image under the identification map.) Clearly R is embedded in TR by the map $r \to (r, 0)$. We also consider CR, the *reduced cone* on R. This is the space obtained from $R \times I$ by pinching $R \times 1 \cup * \times I$ to $*$; briefly, $CR = R \times I/R \times 1 \cup * \times I$. A third important identification space is ΣR, the *reduced suspension* of R. This space is defined by $\Sigma R = R \times I/R \times 0 \cup R \times 1 \cup * \times I$ or, equivalently, by $\Sigma R = CR/R$. Notice that T, C, and Σ can be applied to maps. For example, if $f : R \to S$ is any map, then $\Sigma f : \Sigma R \to S$ is defined by $\Sigma f(r, t) = (f(r), t)$. Thus we have a transformation $\Sigma_* : \pi(R, S) \to \pi(\Sigma R, \Sigma S)$. We also define ΩS, the *loop space* of S, as the collection of maps $l : I \to S$ with the compact-open topology such that $l(0) = * = l(1)$. Of course we may iterate suspensions and loop spaces and define, for any integer $n > 1$, $\Sigma^n R = \Sigma(\Sigma^{n-1} R)$ and $\Omega^n S = \Omega(\Omega^{n-1} S)$. We also set $\Sigma^0 R = R$ and $\Omega^0 S = S$.

In [8; p. 4] it is proved that $\pi(\Sigma R, S)$ has group structure. The product (or sum) of two maps $f, g : \Sigma R \to S$ is a map $f \cdot g : \Sigma R \to S$ defined by

$$
\begin{aligned}
(f \cdot g)(r, t) &= f(r, 2t) && \text{for } 0 \leqq t \leqq \frac{1}{2} \\
&= g(r, 2t-1) && \text{for } \frac{1}{2} \leqq t \leqq 1 ,
\end{aligned}
$$

where $r \in R$ and $t \in I$. The inverse of a map $f : \Sigma R \to S$ is a map f^{-1}

(or $-f$) $: \Sigma R \to S$ defined by $f^{-1}(r, t) = f(r, 1-t)$. This product and inverse provide $\pi(\Sigma R, S)$ with group structure.

PROPOSITION 1.1. *For all spaces R and S, the group $\pi(\Sigma R, S)$ is abelian if*

(i) *R is a suspension*

or

(ii) *S is an H-space.*

See [8; pp. 5–6] or [7; p. 377].

It is also not difficult to show that $\pi(R, \Omega S)$ is a group, for all R and S (see [8; p. 2]). Furthermore, there is a transformation which assigns to a map $f: \Sigma R \to S$, a map $\kappa(f): R \to \Omega S$ defined by $\kappa(f)(r)(t) = f(r, t)$.

PROPOSITION 1.2. *The transformation $\kappa_*: \pi(\Sigma R, S) \to \pi(R, \Omega S)$ defined by $\kappa_*\{f\} = \{\kappa(f)\}$ is a natural isomorphism.*

Next if $h: S \to S'$ then for every space R there is an induced transformation $h_*: \pi(R, S) \to \pi(R, S')$ defined by $h_*\{f\} = \{hf\}$. If R is a suspension then h_* is a homomorphism. Dually, if $k: R' \to R$ then for every space S there is an induced transformation $k^*: \pi(R, S) \to \pi(R', S)$ defined by $k^*\{g\} = \{gk\}$. If S is a loop space then k^* is a homomorphism. Observe also that for $k: R \to R'$ and any space S, $(\Sigma k)^*: \pi(\Sigma R', S) \to \pi(\Sigma R, S)$ is a homomorphism.

For any two spaces R and S we define $R \vee S$ to be the subset $R \times * \cup * \times S$ of $R \times S$. We then define $R \# S$ as the quotient space $R \times S/R \vee S$. We also define the *join* $R * S$ of R and S to be the space obtained from $R \times S \times I$ by factoring out the relations $(r, s, 0) \sim (r, s', 0)$ for all $s, s' \in S$ and $(r, s, 1) \sim (r', s, 1)$ for all $r, r' \in R$. The base point of $R * S$ is $(*, *, 1/2)$.

Let R be a subpolyhedron of S (i.e., the complex of R is a subcomplex of the complex of S) and let F be the space S/R. Let $i: R \to S$ be the inclusion map and let $q: S \to F$ be the projection.

THEOREM 1.3. *For any space X there is an exact sequence,*

$\cdots \to \pi(\Sigma^{n+1}R, X) \to \pi(\Sigma^n F, X) \xrightarrow{(\Sigma^n q)^*} \pi(\Sigma^n S, X) \xrightarrow{(\Sigma^n i)^*} \pi(\Sigma^n R, X)$, *where* $n \geqq 0$.[1]

A proof appears in [8; § 4] (see also [4]).

2. The definitions of the GWP and their equivalence. We now turn to the two definitions of the generalized Whitehead product (GWP). We are given $\alpha \in \pi(\Sigma A, X)$ represented by $f: \Sigma A \to X$ and $\beta \in \pi(\Sigma B, X)$

[1] When $n = 0$ exactness is in the sense of sets with distinguished elements and their transformations.

represented by $g : \Sigma B \to X$ where A and B are polyhedra and X is any space. Letting p_A and p_B be the projections of $A \times B$ onto A and B respectively, we define $f' = f \circ \Sigma p_A : \Sigma(A \times B) \to X$ and $g' = g \circ \Sigma p_B : \Sigma(A \times B) \to X$ and then we define the commutator

$$k' = (f'^{-1} \cdot g'^{-1}) \cdot (f' \cdot g') : \Sigma(A \times B) \to X ,$$

where the products and inverses come from the suspension structure of $\Sigma(A \times B)$ (see § 1). Clearly $k' \mid \Sigma(A \vee B) \simeq 0$ since $k' \mid \Sigma(A \times *) \simeq 0$ and $k' \mid \Sigma(* \times B) \simeq 0$. By the homotopy extension property for the polyhedral pair $\Sigma(A \times B)$, $\Sigma(A \vee B)$ [**6**; p. 97] there is a map $k : \Sigma(A \times B) \to X$ such that $k \simeq k'$ and $k \mid \Sigma(A \vee B) = 0$. Thus k induces $\tilde{k} : \Sigma(A \divideontimes B) = \Sigma(A \times B)/\Sigma(A \vee B) \to X$ with the property $k = \tilde{k}\Sigma q$, where $q : A \times B \to A \divideontimes B$ is the projection. We show that the homotopy class of $\tilde{k}$ does not depend on the choice of the map k.

LEMMA 2.1. *Given maps $r, s : \Sigma(A \times B) \to X$ with $r \mid \Sigma(A \vee B) = 0$ and $s \mid \Sigma(A \vee B) = 0$, where A and B are polyhedra; r and s induce $\tilde{r}, \tilde{s} : \Sigma(A \divideontimes B) \to X$ with $r = \tilde{r}\Sigma q$ and $s = \tilde{s}\Sigma q$. If $r \simeq s$ then $\tilde{r} \simeq \tilde{s}$.*

Proof. If $j : A \vee B \to A \times B$ is the injection then it is easily seen that there is a map $p : \Sigma(A \times B) \to \Sigma(A \vee B)$ such that $p\, \Sigma j \simeq 1$. (We may set $p = (\Sigma p_1) \cdot (\Sigma p_2)$, where $p_1, p_2 : A \times B \to A \vee B$ are defined by $p_1(a, b) = (a, *)$ and $p_2(a, b) = (*, b)$.) Thus $\Sigma p\, \Sigma^2 j = \Sigma(p\, \Sigma j) \simeq \Sigma 1 = 1$ and so $(\Sigma^2 j)^*(\Sigma p)^* = 1$. This shows that, for any space X, $(\Sigma^2 j)^*$: $\pi(\Sigma^2(A \times B), X) \to \pi(\Sigma^2(A \vee B), X)$ is onto. By applying Theorem 1.3 to the inclusion $j : A \vee B \to A \times B$ we obtain an exact sequence

$$\cdots \longrightarrow \pi(\Sigma^2(A \times B), X) \xrightarrow{(\Sigma^2 j)^*} \pi(\Sigma^2(A \vee B), X) \longrightarrow$$
$$\pi(\Sigma(A \divideontimes B), X) \xrightarrow{(\Sigma q)^*} \pi(\Sigma(A \times B), X) \longrightarrow \cdots .$$

We infer from the fact that $(\Sigma^2 j)^*$ is onto, that $(\Sigma q)^*$ is one-to-one. Since $(\Sigma q)^*\{\tilde{r}\} = (\Sigma q)^*\{\tilde{s}\}$, $\tilde{r} \simeq \tilde{s}$. This proves the lemma.

Thus the class of $\tilde{k}$ is independent of the choice of map k and hence independent of the choice of representative f of α and g of β.

DEFINITION 2.2. The GWP of $\alpha = \{f\} \in \pi(\Sigma A, X)$ and $\beta = \{g\} \in \pi(\Sigma B, X)$ is defined to be $[\alpha, \beta] = \{\tilde{k}\} \in \pi(\Sigma(A \divideontimes B), X)$.

We consider next the second definition of the GWP. Here we represent $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ by maps $f : TA, A \to X, *$ and $g : TB, B \to X, *$ respectively, where T denotes the cone (§ 1) and A and B are polyhedra. In $TA \times TB$ consider the subspace $Q = TA \times B \cup A \times TB$ and define $h : Q \to X$ by

$$h((a, t), b) \ = f(a, t)$$

$$h(a, (b, u)) = g(b, u)$$

where $a \in A$, $b \in B$ and $t, u \in I$. Now there is a map ν from the join $A * B$ to Q defined by

$$\begin{aligned} \nu(a, b, t) &= (a, (b, 1 - 2t)) \quad 0 \leqq t \leqq \frac{1}{2} \\ &= ((a, 2t - 1), b) \quad \frac{1}{2} \leqq t \leqq 1 . \end{aligned}$$

The map ν is a homeomorphism, for the maps $\lambda_1 : A \times TB \to A * B$ and $\lambda_2 : TA \times B \to A * B$ defined by $\lambda_1(a, (b, u)) = (a, b, (1 - u)/2)$ and $\lambda_2((a, u), b) = (a, b, (1 + u)/2)$ determine a map $Q \to A * B$ which is, the inverse of ν (cf. [3; Theorem 2.4]). Consider next the subspace of $A * B$ consisting of all points $(a, *, t)$ and all points $(*, b, u)$. This is a contractible space consisting of two cones with a cone generator in common. When it is factored out we clearly obtain the space $\Sigma(A \divideontimes B)$. Since A and B are polyhedra the projection $\mu' : A * B \to \Sigma(A \divideontimes B)$ is a homotopy equivalence [13; p. 238]. Let $\mu : \Sigma(A \divideontimes B) \to A * B$ denote the homotopy inverse of μ'. Thus the map $h : Q \to X$ gives rise to a map $h\nu\mu : \Sigma(A \divideontimes B) \to X$. It is easily seen that the homotopy class of $h\nu\mu$ does not depend on the representatives f and g of α and β. Thus we have the second definition.

DEFINITION 2.3. The GWP of $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ is $[\alpha, \beta]' = \{h\nu\mu\} \in \pi(\Sigma(A \divideontimes B), X)$. Except for minor modifications, this is the absolute version of a definition given by D. E. Cohen [3].[2]

We remark that Definitions 2.2. and 2.3 can be extended to include the case A and B are not polyhedra. To do this for 2.2 we consider the mapping cylinder M of the inclusion map $A \vee B \to A \times B$. The GWP is then given by a map $\tilde{k} : \Sigma(M/A \vee B) \to X$. For Definition 2.3 we simply regard the GWP to be represented by $h\nu : A * B \to X$. We shall, however, only consider the case when A and B are polyhedra.

Next we show that Definitions 2.2 and 2.3 are identical.

THEOREM 2.4. *For all* $\alpha \in \pi(\Sigma A, X)$ *and* $\beta \in \pi(\Sigma B, X)$,

$$[\alpha, \beta] = [\alpha, \beta]' .$$

Proof. We have the diagram

$$\begin{tikzcd} & A * B \arrow[dl, "\theta"'] \arrow[dd, shift right, "\mu'"'] \arrow[dr, "h\nu"] & \\ \Sigma(A \times B) \arrow[dr, "\Sigma q"'] & & X \\ & \Sigma(A \divideontimes B) \arrow[uu, shift right, "\mu"'] \arrow[ur, "\tilde{k}"'] & \end{tikzcd}$$

[2] That is, Cohen's definition is stated for the more general carrier theory of Spanier and Whitehead. We have given the absolute case in 2.3 in order to simplify the notation and to emphasize the duality with the products of § 6.

where θ is the projection obtained by squeezing all points of the form $(a, b, 0)$, $(a, b, 1)$, or $(*, *, t)$ in $A * B$ to the base point. All other maps are as before. We must show

$$h\nu\mu \simeq \tilde{k} : \pi(A \divideontimes B) \to X .$$

Since $\mu'\mu \simeq 1$ and $\Sigma q\theta = \mu'$ it suffices to prove that

$$h\nu \simeq \tilde{k}\Sigma q\theta : A * B \to X .$$

But $\tilde{k}\Sigma q = k \simeq k'$, where $k' = (f'^{-1} \cdot g'^{-1}) \cdot (f' \cdot g')$, and so it suffices to prove that the maps $r = h\nu$ and $s = k'\theta$ of $A * B$ to X are homotopic. To this end we first define a deformation of $A * B$ by stretching $A \times B \times [1/4, 3/4]$ in $A * B$ to all of $A * B$. Thus we define $\psi_u : A * B \to A * B$ by

$$\begin{aligned} \psi_u(a, b, t) &= (a, b, (1-u)t) & 0 \leqq t \leqq \frac{1}{4} \\ &= (a, b, ut - \frac{u}{2} + t) & \frac{1}{4} \leqq t \leqq \frac{3}{4} \\ &= (a, b, (1-u)t + u) & \frac{3}{4} \leqq t \leqq 1 . \end{aligned}$$

Now $\psi_0 = 1$, the identity map, and so $r \simeq r_1$ where $r_1 = r\psi_1 : A * B \to X$. Explicitly,

$$\begin{aligned} r_1(a, b, t) &= * & 0 \leqq t \leqq \frac{1}{4} \\ &= g(b,\ 2 - 4t) & \frac{1}{4} \leqq t \leqq \frac{1}{2} \\ &= f(a,\ 4t - 2) & \frac{1}{2} \leqq t \leqq \frac{3}{4} \\ &= * & \frac{3}{4} \leqq t \leqq 1 . \end{aligned}$$

Next we define a homotopy $\phi_u : A * B \to X$ with $\phi_0 = r_1$ and $\phi_1 = s$. Let

$$\begin{aligned} \phi_u(a, b, t) &= f(a, (1 - 4t)u) & 0 \leqq t \leqq \frac{1}{4} \\ &= g(b, 2 - 4t) & \frac{1}{4} \leqq t \leqq \frac{1}{2} \\ &= f(a, 4t - 2) & \frac{1}{2} \leqq t \leqq \frac{3}{4} \\ &= g(b, (4t - 3)u) & \frac{3}{4} \leqq t \leqq 1 . \end{aligned}$$

Thus $r \simeq s$ as asserted and so $[\alpha, \beta] = [\alpha, \beta]'$.

This theorem is a generalization of a result due to Samelson [**11**; p. 750].

3. **Properties of the GWP.** In this section we derive some properties of the GWP from Definition 2.2. In all cases Lemma 2.1 and facts about commutators will enter into the proofs. We shall denote the commutator $(x^{-1}y^{-1})(xy)$ of two group elements (or maps on suspensions) x and y by (x, y). We shall, however, usually denote the group operation in $\pi(\Sigma R, S)$ additively, for all spaces R and S. The notation of § 2 will be used throughout.

PROPOSITION 3.1. If X is an H-space [3] then $[\alpha, \beta] = 0$ for all $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

Proof. For then the group $\pi(\Sigma(A \times B), X)$ is abelian by Proposition 1.1 and so the commutator map $k' = (f', g')$ of Definition 2.2 is nullhomotopic. Thus by Lemma 2.1 $\tilde{k} \simeq 0$, i.e., $[\alpha, \beta] = 0$.

PROPOSITION 3.2. If $\Sigma_* : \pi(\Sigma(A \circledast B), X) \to \pi(\Sigma^2(A \circledast B), \Sigma X)$ is the suspension homomorphism (§ 1), then $\Sigma_*[\alpha, \beta] = 0$ for all $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

Proof. [4] The group $\pi(\Sigma^2(A \times B), \Sigma X)$ is abelian by Proposition 1.1 and so $\Sigma k' \simeq (\Sigma f', \Sigma g')$ is nullhomotopic. Thus $\Sigma\tilde{k}\Sigma^2 q \simeq 0$, where $q : A \times B \to A \circledast B$ is the projection. By applying the transformation κ (see Proposition 1.2) we obtain $\kappa(\Sigma\tilde{k})\circ\Sigma q \simeq 0 : \Sigma(A \times B) \to \Omega\Sigma X$. From Lemma 2.1 we infer that $\kappa(\Sigma\tilde{k}) \simeq 0$ and so $\Sigma\tilde{k} \simeq 0$, i.e., $\Sigma_*[\alpha, \beta] = 0$.

PROPOSITION 3.3. (Anti-commutativity) For all $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$,

$$[\beta, \alpha] = -(\Sigma\sigma)^*[\alpha, \beta]$$

where $\sigma : B \circledast A \to A \circledast B$ is induced by the map $B \times A \to A \times B$ which sends (b, a) to (a, b).

This proof is an easy consequence of Lemma 2.1 and the commutator rule $(x, y)^{-1} = (y, x)$ and is thus left to the reader.

Next we prove that the GWP is additive in each variable. The proof makes use of the following theorem of G. W. Whitehead (see [**12**; Theorem 2.10] or [**2**; § 6] for a proof).

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[3] An H-space is a space with a continuous multiplication for which the base point is a two-sided homotopy unit.

[4] Cf. Cohen's proof, [**3**; pp. 238-240].

THEOREM (*Whitehead*). *If P is a polyhedron which has Lusternik-Schnirelmann category $\leqq n$ (briefly, cat $P \leqq n$)*[5] *and S is any space, then all n-fold commutators in the group $\pi(P, \Omega S)$ reduce to the identity. (Recall that an n-fold commutator is defined inductively as follows. A 1-fold commutator is just an element. An n-fold commutator is the commutator of an n_1-fold commutator with an n_2-fold commutator, where $n_1 + n_2 = n$.)*

Now we prove

PROPOSITION 3.4. (Bi-additivity) If A and B are suspensions then

(i) $[\alpha + \bar{\alpha}, \beta] = [\alpha, \beta] + [\bar{\alpha}, \beta]$

(ii) $[\alpha, \beta + \bar{\beta}] = [\alpha, \beta] + [\alpha, \bar{\beta}]$

for all $\alpha, \bar{\alpha} \in \pi(\Sigma A, X)$ and $\beta, \bar{\beta} \in \pi(\Sigma B, X)$.

Proof. We only prove (i) since (i) and Proposition 3.3 imply (ii). By a theorem of Bassi [**5**; Theorem 9]

$$\text{cat}\ (A \times B) \leqq \text{cat}\, A + \text{cat}\, B - 1\ .$$

Since A and B are suspensions, each is of category $\leqq 2$. Thus cat $(A \times B) \leqq 3$[6] and so by Whitehead's theorem all 3-fold commutators in $\pi(A \times B, \Omega X)$ reduce to the identity. By applying the isomorphism $\pi(\Sigma(A \times B), X) \approx \pi(A \times B, \Omega X)$ of Proposition 1.2 we see that this is true of the group $\pi(\Sigma(A \times B), X)$. However, if π is any group in which all 3-fold commutators are trivial then $(ab, c) = (a, c)(b, c)$ for all $a, b, c \in \pi$ (by [**14**; p. 60] or direct verification). Thus the previous equality holds for the group $\pi(\Sigma(A \times B), X)$. Now $(f + \bar{f})' = (f + \bar{f})\Sigma p_A = f\Sigma p_A + \bar{f}\Sigma p_A = f' + \bar{f}'$, where $\bar{f}$ represents $\bar{\alpha} \in \pi(\Sigma A, X)$, and so $((f + \bar{f})', g') = (f' + \bar{f}', g')$. But we have just seen that $(f' + \bar{f}', g') \simeq (f', g') + (\bar{f}', g')$. It then follows, without difficulty, that $[\alpha + \bar{\alpha}, \beta] = [\alpha, \beta] + [\bar{\alpha}, \beta]$.

We remark that it is possible to prove an appropriate Jacobi identity for GWPs of elements $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ and $\gamma \in \pi(\Sigma C, X)$, when A, B and C are suspensions. The proof, like the preceding one, is a generalization of G. W. Whitehead's argument in [**12**]. It is based on Whitehead's theorem above and the following algebraic fact: if π is a group in which all 4-fold commutators are trivial then $(a, (b, c))(b, (c, a))(c, (a, b)) = 1$, for all $a, b, c \in \pi$ [**14**; pp. 63–64]. A proof of the Jacobi identity from Definition 2.3 appears in [**3**; 5.8].

4. The product of two suspensions. In this section we derive a formula which relates the homotopy type of the product of two

[5] The standard reference on category is [**5**].

[6] The fact that cat $(A \times B) \leqq 3$ when A and B are suspensions is proved in Corollary 4.5.

reduced suspensions to the GWP map. We adopt the following notation. For any space R, RS is the quotient space of $R \times I$ by the relations $(r, 0) \sim (r', 0)$ and $(r, 1) \sim (r', 1)$ for all $r, r' \in R$. The base point is $(*, 0)$. There are obvious maps $t_R : TR \to SR$ and $s_R : SR \to \Sigma R$. We recall from Definition 2.3 that $Q = TA \times B \cup A \times TB \subset TA \times TB$ and that $\nu : A * B \to Q$ is the canonical homeomorphism. We let $h : Q \to \Sigma A \vee \Sigma B$ be the map of Definition 2.3 determined by the inclusion maps $i_1 : \Sigma A \to \Sigma A \vee \Sigma B$ and $i_2 : \Sigma B \to \Sigma A \vee \Sigma B$, i.e., $h((a, t), b) = ((a, t), *)$ and $h(a, (b, u)) = (*, (b, u))$. Throughout this section and the next we assume that either the polyhedron A or the polyhedron B is compact. It is possible, by complicating the argument, to get rid of this assumption. However it enters only in the proof of the following

LEMMA 4.1. *There is a map $F : C(A * B), A * B \to \Sigma A \times \Sigma B, \Sigma A \vee \Sigma B$ such that F induces isomorphisms of homology groups and $F | A * B = h\nu : A * B \to \Sigma A \vee \Sigma B$.*

Proof. Note that the map ν can be extended to a map $N : T(A * B), A * B \to TA \times TB, Q$ by setting

$$\begin{aligned} N((a, b, t), u) &= (a, u), (b, 1 - 2t(1 - u)) && 0 \leqq t \leqq \frac{1}{2} \\ &= (a, 1 - 2(1 - t)(1 - u)), (b, u) && \frac{1}{2} \leqq t \leqq 1 . \end{aligned}$$

This definition is due to D. E. Cohen who showed [3; Theorem 2.4] that N is a homeomorphism when A or B is compact. Next we observe that $t_A \times t_B : TA \times TB, Q \to SA \times SB, SA \vee SB$ is an identification map since A or B is compact [3; Lemma 1.6]. This implies $t_A \times t_B | (TA \times TB) - Q$ is a homeomorphism onto $(SA \times SB) - (SA \vee SB)$. Hence $t_A \times t_B$ induces isomorphisms of (relative) homology groups. Now the map $s_A \times s_B : SA \times SB, SA \vee SB \to \Sigma A \times \Sigma B, \Sigma A \vee \Sigma B$ induces homology homorphisms since s_A and s_B are homotopy equivalences [10; Hilfs. 5]. Thus the composition of the maps

$$\begin{aligned} T(A * B)A * B \xrightarrow{N} TA \times TB, Q \xrightarrow{t_A \times t_B} \\ SA \times SB, SA \vee SB \xrightarrow{s_A \times s_B} \Sigma A \times \Sigma B, \Sigma A \vee \Sigma B \end{aligned}$$

induces isomorphisms of homology groups. Furthermore this composite restricted to $A * B$ is the map $(s_A \vee s_B) \circ (t_A \times t_B) | Q \circ \nu = h\nu$. Finally we observe that $N\left(\left(*, *, \frac{1}{2}\right), u\right) = ((*, u)), (*, u))$[7] for all $u \in I$. Thus if $V : T(A * B), A * B \to C(A * B), A * B$ is the identification map, there exists a map $F : C(A * B), A * B \to \Sigma A \times \Sigma B, \Sigma A \vee \Sigma B$ such that

[7] Recall that $(*, *, 1/2)$ is the base point of $A * B$.

$FV = (s_A \times s_B)(t_A \times t_B)N$. F is the desired map.

If $f: R \to S$ is any map then $S \cup_f CR$ or C_f denotes the quotient space of $S \vee CR$ by the relation $(*, (r, 0)) \sim (fr, *)$. This is called the *space obtained* from S *by attaching a cone* on R by means of f. The symbol " $\simeq$ " shall signify same homotopy type.

THEOREM 4.2. $\Sigma A \times \Sigma B \simeq (\Sigma A \vee \Sigma B) \cup_{h\nu} C(A * B)$[8]

Proof. The map F: $C(A * B) \to \Sigma A \times \Sigma B$ of Lemma 4.1 and the inclusion j: $\Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$ give rise to a map

$$G:\ C_{h\nu} = (\Sigma A \vee \Sigma B) \cup_{h\nu} C(A * B) \to \Sigma A \times \Sigma B.$$

Let H: $C(A * B) \to C_{h\nu}$ denote the composition of injection and projection, $C(A * B) \to (\Sigma A \vee \Sigma B) \vee C(A * B) \to C_{h\nu}$. Then there is a diagram

$$\begin{tikzcd}
A * B \arrow[r, "h\nu"] \arrow[dd] \arrow[drr, "h\nu"] & \Sigma A \vee \Sigma B \arrow[dr, "1"] \arrow[dd] & \\
& & \Sigma A \vee \Sigma B \arrow[dd, "j"] \\
C(A * B) \arrow[r, "H"] \arrow[drr, "F"'] & C_{h\nu} \arrow[dr, "G"] & \\
& & \Sigma A \times \Sigma B
\end{tikzcd}$$

where the unmarked arrows denote inclusion maps. All squares and triangles are commutative. We claim that G induces isomorphisms of homology groups. By the exactness of the homology sequence of a pair it suffices to prove that G: $C_{h\nu}, \Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B, \Sigma A \vee \Sigma B$ induces homology isomorphisms. However, the composition of this map with H: $C(A * B), A * B \to C_{h\nu}, \Sigma A \vee \Sigma B$ is the map F. Hence by Lemma 4.1 it suffices to prove that H induces homology isomorphisms. But clearly there is a commutative diagram.

$$\begin{array}{ccc}
C(A * B),\ A * B & \xrightarrow{H} & C_{h\nu},\ \Sigma A \vee \Sigma B \\
\big\downarrow Q & & \big\downarrow Q' \\
\dfrac{C(A * B)}{A * B},\ * & \xrightarrow{1} & \dfrac{C(A * B)}{A * B} = \dfrac{C_{h\nu}}{\Sigma A \vee \Sigma B},\ *
\end{array}$$

where Q and Q' are the projections. Since A and B are polyhedra, Q and Q' induce homology isomorphisms. Thus H, and consequently, G: $C_{h\nu} \to \Sigma A \times \Sigma B$ induce homology isomorphisms. We complete the proof by remarking that, since A and B are polyhedra (i.e., connected,

[8] This formula appears on p. 200 of [**8**].

locally-finite CW-complexes), $\Sigma A \times \Sigma B$ and $C_{h\nu}$ are simply connected (the latter, e.g., by van Kampen's Theorem, [9 ; p. 666]). Thus a theorem of J. H. C. Whitehead [6 ; p. 113] guarantees that $G\colon C_{h\nu} \to \Sigma A \times \Sigma B$ is a homotopy equivalence.

We note for future use that $G|\Sigma A \vee \Sigma B = j\colon \Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$.

Now denote by $\tilde{k}\colon \Sigma(A \divideontimes B) \to \Sigma A \vee \Sigma B$ the map of Definition 2.2 which represents the GWP of the class of $i_1\colon \Sigma A \subset \Sigma A \vee \Sigma B$ and $i_2\colon \Sigma B \subset \Sigma A \vee \Sigma B$. We call $\tilde{k}$ the *GWP map*. An immediate consequence of Theorem 2.4, is

COROLLARY 4.3. $\Sigma A \times \Sigma B \simeq (\Sigma A \vee \Sigma B) \cup_{\tilde{k}} C\Sigma(A \divideontimes B) = C_{\tilde{k}}$.

COROLLARY 4.4. $j\tilde{k} \simeq 0 : \Sigma(A \divideontimes B) \to \Sigma A \times \Sigma B$.

By the first diagram of the preceding proof, $jh\nu$ factors through $C(A * B)$ and hence is nullhomotopic. Thus $j\tilde{k} \simeq 0$.

COROLLARY 4.5. $\operatorname{cat}(\Sigma A \times \Sigma B) \leqq 3$.

Since attaching a cone increases category by at most one and cat $(\Sigma A \vee \Sigma B) \leqq 2$, the corollary follows. It is of course, a very special case of Bassi's Theorem (see the proof of Proposition 3.4).

COROLLARY 4.6. *If* $p\colon \Sigma A \times \Sigma B \to \Sigma A \divideontimes \Sigma B$ *is the projection, then, for any space* X, *the following sequence is exact,*

$$\pi(\Sigma A \divideontimes \Sigma B, X) \xrightarrow{p^*} \pi(\Sigma A \times \Sigma B, X) \xrightarrow{j^*} \pi(\Sigma A \vee \Sigma B, X) \xrightarrow{\tilde{k}^*} \pi(\Sigma(A \divideontimes B), X) .$$

By Theorem 1.3 and Corollary 4.4 it suffices to show that for any map $l\colon \Sigma A \vee \Sigma B \to X$ with $l\tilde{k} \simeq 0$, there is a map $m\colon \Sigma A \times \Sigma B \to X$ such that $mj \simeq l$. But $l\tilde{k} \simeq 0$ evidently implies that l is extendible to a map $\tilde{l}\colon C_{\tilde{k}} \to X$. The composition of $\tilde{l}$ with the homotopy equivalence $\Sigma A \times \Sigma B \to C_{\tilde{k}}$ is the desired map m.

This corollary suggests a close relationship between the maps j and $\tilde{k}$. In the next section we shall see that this is so (Proposition 5.2).

5. The vanishing of the GWP and H-spaces. In this section we investigate various ways in which the GWP may be trivial and obtain equivalent conditions for each of them. We begin by considering a condition for the vanishing of the GWP of $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

PROPOSITION 5.1. $[\alpha, \beta] = 0$ if and only if there is a map $m\colon \Sigma A \times \Sigma B \to X$ such that $\{m \mid \Sigma A\} = \alpha$ and $\{m \mid \Sigma B\} = \beta$.

Proof. Let the maps f and g represent α and β and let

$l\colon \Sigma A \vee \Sigma B \to X$ be the map determined by f and g. Then it is evident that $l\tilde{k}$ represents $[\alpha, \beta]$. This observation, together with Corollary 4.6 establishes Proposition 5.1.

Next we consider conditions under which the GWP map is trivial.

PROPOSITION 5.2. $\tilde{k} \simeq 0 : \Sigma(A \mathbin{⨳} B) \to \Sigma A \vee \Sigma B$ if and only if $j\colon \Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B$ is a homotopy equivalence.

Proof. If $\tilde{k} \simeq 0$ then by Corollary 4.3 and the remark preceding it, there is a homotopy equivalence $K\colon (\Sigma A \vee \Sigma B) \vee \Sigma^2(A \mathbin{⨳} B) \to \Sigma A \times \Sigma B$ such that $K|\Sigma A \vee \Sigma B \simeq j$. Thus if $i\colon \Sigma A \vee \Sigma B \to (\Sigma A \vee \Sigma B) \vee \Sigma^2(A \mathbin{⨳} B)$ is the inclusion, $q\colon (\Sigma A \vee \Sigma B) \vee \Sigma^2(A \mathbin{⨳} B) \to \Sigma A \vee \Sigma B$ the projection and K' the homotopy inverse of K we have $qK'j \simeq qK'Ki \simeq qi = 1$. Hence j has a left homotopy inverse. It follows that the induced map on homotopy groups $j_*\colon \pi_r(\Sigma A \vee \Sigma B) \to \pi_r(\Sigma A \times \Sigma B)$, is a monomorphism for all r. But it is a standard result that j_* is an epimorphism [6; p. 43]. Thus j_* is an isomorphism and hence j is a homotopy equivalence [6; p. 107].

Conversely let j' be a homotopy inverse to j. Then since $j\tilde{k} \simeq 0$ by Corollary 4.4, $\tilde{k} \simeq j'j\tilde{k} \simeq 0$. We remark that the exact homology sequence of the pair $\Sigma A \times \Sigma B, \Sigma A \vee \Sigma B$ shows that j is an equivalence precisely when $H_n(\Sigma A \times \Sigma B, \Sigma A \vee \Sigma B) = 0$ for all n. By means of the Künneth formula this condition may be stated purely in terms of the homology groups of A and B. It is equivalent to asserting that $\Sigma A \mathbin{⨳} \Sigma B$ is contractible.

Next we investigate spaces in which all GWPs vanish, i.e., spaces X such that $[\alpha, \beta] = 0$ for all α, β and all A, B. By Proposition 3.1 we know that H-spaces are among such spaces but we shall see that the converse is false. First we prove

LEMMA 5.3. *If $\alpha, \beta \in \pi(\Sigma A, X)$ (i.e, $A = B$) and $d\colon A \to A \mathbin{⨳} A$ is the composition $A \to A \times A \to A \mathbin{⨳} A$ of diagonal map and projection then $(\Sigma d)^*[\alpha, \beta] = (\alpha, \beta)$, the commutator of α and β.*

The proof is a ready consequence of Definition 2.2.

PROPOSITION 5.4. All GWPs vanish in X if and only if $\pi(\Sigma P, X)$ $(\approx \pi(P, \Omega X))$ is abelian for all polyhedra P.

Proof. If $\pi(\Sigma P, X)$ is abelian for all P then $\pi(\Sigma(A \times B), X)$ is abelian. Using the notation of Definition 2.2. we see that this implies that the commutator of f' and g', $k' \simeq 0 : \Sigma(A \times B) \to X$. Thus $[\alpha, \beta] = 0$. The preceding lemma establishes the opposite implication.

In [2] Berstein and Ganea introduce a numerical invariant of homotopy type, nil ΩX, for any space X. In particular, nil $\Omega X \leqq 1$ is the assertion that the commutator map $\Omega X \times \Omega X \to \Omega X$ is nullhomotopic. It is easy to verify that nil $\Omega X \leqq 1$ implies $\pi(P, \Omega X)$ is abelian for all spaces P. Hence Proposition 5.4 shows that nil $\Omega X \leqq 1$ implies that all GWPs in X vanish. However, on pp. 112–113 of [2] Berstein and Ganea have constructed a space X with nil $\Omega X \leqq 1$ which is not an H-space. This shows the existence of spaces in which all GWPs vanish but which are not H-spaces. However, such spaces cannot be suspensions.

PROPOSITION 5.5. *Let ι denote the class of the identity map of ΣA. If the GWP $[\iota, \iota] = 0$, then ΣA is an H-space.* (This proposition is well-known when A is a sphere.)

Proof. Let j_1 and j_2 denote respectively the inclusion of ΣA into the first and second factors of $\Sigma A \times \Sigma A$. By definition ΣA is an H-space if there exists a map m: $\Sigma A \times \Sigma A \to \Sigma A$ such that $mj_1 \simeq 1$ and $mj_2 \simeq 1$. By Proposition 5.1 this occurs when $[\iota, \iota] = 0$.

6. The dual product. In this section we use the Eckmann-Hilton theory ([4] and [8]) to study a product which is dual to the GWP. Here, as in §2, we present two definitions of the dual product. In preparation for this we introduce some notation and recall some facts.

We noted in §1 that $\pi(R, \Omega S)$ has group structure for any spaces R and S. Explicitly, if $f, g : R \to \Omega S$ then define

$$\begin{aligned}(f\cdot g)(r)(t) &= f(r)(2t) & 0 &\leqq t \leqq \frac{1}{2}\\ &= g(r)(2t-1) & \frac{1}{2} &\leqq t \leqq 1\end{aligned}$$

and $f^{-1}(r)(t) = f(r)(1-t)$, where $r \in R$ and $t \in I$. This product and inverse induce group structure in $\pi(R, \Omega S)$. Also if $h : R \to S$ is any map, $\Omega h : \Omega R \to \Omega S$ is defined by $(\Omega h)(l) = h \circ l$. If R and S are subspaces of a space X, then $E(X; R, S)$ shall denote the space of paths in X which begin in R and end in S(i.e., maps $I, 0, 1 \to X, R, S$) with the compact open topology.

We say p: $E \to B$ is a *fibre map*, if for every space X and every homotopy f_t: $X \to B$ and every map g_0: $X \to E$ such that $pg_0 = f_0$, there exists a homotopy g_t: $X \to E$ of g_0 such that $pg_t = f_t$. The space $F = p^{-1}(*) \subset E$ is called the *fibre* and $F \to E \xrightarrow{p} B$ is called a *fibre sequence*. Now we show how, for any spaces A and B, the inclusion j: $A \vee B \to A \times B$ gives rise to a fibre map. Let $P(A, B) = E(A \times B; A \vee B, A \times B)$ and $A \flat B = E(A \times B; A \vee B, *)$ and consider the diagram

$$\begin{tikzcd}
A \flat B \arrow[r, "i"] \arrow[dr, "p_0"'] & P(A, B) \arrow[r, "p_1"] \arrow[d, shift right, "u'"'] & A \times B \\
& A \vee B \arrow[u, shift right, "u"'] \arrow[ur, "j"'] &
\end{tikzcd}$$

where $p_0(l) = l(0)$, $p_1(l) = l(1)$, $u'(l) = l(0)$, i is the inclusion and $u(x)(t) = x$ for $x \in A \vee B$ and all $t \in I$. The following facts are well-known (e.g., [8; pp. 16–20]) and not difficult to establish:

(i) $A \flat B \xrightarrow{i} P(A, B) \xrightarrow{p_1} A \times B$ is a fibre sequence

(ii) $u'i = p_0$ and $p_1 u = j$

(iii) $u'u = 1$ and $uu' \simeq 1$.

We now turn to the first definition of the dual product which is a pairing from $\pi(X, \Omega A)$ and $\pi(X, \Omega B)$ to $\pi(X, \Omega(A \flat B))$. Let $\{f\} = \alpha \in \pi(X, \Omega A)$ and $\{g\} = \beta \in \pi(X, \Omega B)$ where A, B and X are any topological spaces.[9] Letting i_A: $A \to A \vee B$ and i_B: $B \to A \vee B$ be the inclusion maps, we define $f' = (\Omega i_A)f$: $X \to \Omega(A \vee B)$ and $g' = (\Omega i_B)g$: $X \to \Omega(A \vee B)$ and then we define the commutator

$$k' = (f'^{-1} \cdot g'^{-1}) \cdot (f' \cdot g') : X \to \Omega(A \vee B) .$$

Since $\Omega(A \times B)$ is homeomorphic to $\Omega A \times \Omega B$ it follows that $(\Omega j)k' \simeq 0$: $X \to \Omega(A \times B)$, where j: $A \vee B \to A \times B$ is the injection. Using the notation of the preceding diagram, we obtain a commutative diagram

$$\begin{tikzcd}
& & \Omega P(A, B) \arrow[dd, "\Omega p_1"] \\
& \Omega(A \vee B) \arrow[ur, "\Omega u"] & \\
X \arrow[ur, "k'"] \arrow[rr, "\Omega j\, k'"] & & \Omega(A \times B) .
\end{tikzcd}$$

Since the loop functor Ω applied to a fibre sequence yields a fibre sequence, Ωp_1 is a fibre map with fibre $\Omega(A \flat B)$. Since $\Omega j\, k' \simeq 0$, there is a map k: $X \to \Omega P(A, B)$ such that $k \simeq \Omega u\, k'$ and $\Omega p_1\, k = 0$, the constant map. Thus k induces a map $\tilde{k}$: $X \to \Omega(A \flat B)$ such that $\Omega i\, \tilde{k} = k$, where i: $A \flat B \subset P(A, B)$. The following lemma shows that the class of $\tilde{k}$ is independent of choice of representative f and g of α and β.

LEMMA 6.1. *Given maps r, s: $X \to \Omega P(A, B)$ such that $\Omega p_1\, r = 0$ and $\Omega p_1\, s = 0$; then r and s induce maps $\tilde{r}, \tilde{s}$: $X \to \Omega(A \flat B)$ such that $\Omega i\, \tilde{r} = r$ and $\Omega i\, \tilde{s} = s$. If $r \simeq s$ then $\tilde{r} \simeq \tilde{s}$.*

[9] In this section A and B are not necessarily polyhedra.

The proof is dual to the proof of Lemma 2.1 with the fibre map $p_1 : P(A, B) \to A \times B$ here playing the role of the polyhedral inclusion $A \vee B \subset A \times B$ of 2.1. The exact sequence which is associated to any fibre sequence ([8; (4.5)] or [4; p. 2557]) takes the place of the exact sequence in 2.1. We omit details.

DEFINITION 6.2. The dual product of $\alpha \in \pi(X, \Omega A)$ and $\beta \in \pi(X, \Omega B)$ is $\lfloor \alpha, \beta \rfloor = \{\tilde{k}\} \in \pi(X, \Omega(A \flat B))$.

Next we prepare to give the second definition. For any space R, let $ER = E(R; R, *)$ and let q_R: $ER \to R$ be defined by $q_R(l) = l(0)$. Now for any spaces A and B let $\hat{Q}$ be the subspace of $(EA \vee B) \times (A \vee EB)$ consisting of pairs (x, y), $x \in EA \vee B$ and $y \in A \vee EB$, such that $(q_A \vee 1)(x) = (1 \vee q_B)(y)$ in $A \vee B$. We call the obvious projections χ_1: $\hat{Q} \to A \vee EB$ and χ_2: $\hat{Q} \to EA \vee B$. We also define the *cojoin* of A and B, $A \hat{*} B$, as $E(A \vee B; A, B)$. Let p_1 and p_2 be the projections of $EA \vee B$ onto EA and B respectively and let q_1 and q_2 be the projections of $A \vee EB$ onto A and EB respectively. Then there is a map $\hat{\nu} : \hat{Q} \to A \hat{*} B$ given by

$$\begin{aligned} \hat{\nu}(x)(t) &= (q_1\chi_1(x),\ q_2\chi_1(x)(1 - 2t)) & 0 \leqq t \leqq \tfrac{1}{2} \\ &= (p_1\chi_2(x)(2t - 1),\ p_2\chi_2(x)) & \tfrac{1}{2} \leqq t \leqq 1 \end{aligned}$$

where $x \in \hat{Q}$ and $t \in I$.

We are now able to give the second definition of the dual product. Since $\Omega A \subset EA$ and $\Omega B \subset EB$ we represent $\alpha \in \pi(X, \Omega A)$ by f: $X \to EA$ and $\beta \in \pi(X, \Omega B)$ by g: $X \to EB$. The maps f and g determine maps $X \to EA \vee B$ and $X \to A \vee EB$ respectively. These last two maps determine a map h: $X \to \hat{Q}$. Composing h with the map $\hat{\nu}$ of the preceding paragraph yields a map $\hat{\nu}h$: $X \to A \hat{*} B$. It is clear that the class of $\hat{\nu}h$ is independent of the representative f of α and g of β.

DEFINITION 6.3. The (second) dual product of $\alpha \in \pi(X, \Omega A)$ and $\beta \in \pi(X, \Omega B)$ is $[\alpha, \beta]' = \{\hat{\nu}h\} \in \pi(X, A \hat{*} B)$.

We make a few remarks regarding duality. We first observe that Definition 6.2 is an approximate, not a precise, dual of Definition 2.2. This is due to the fact that in Definition 2.2 we restricted our attention to the case when A and B were polyhedra so that the pair $A \times B, A \vee B$ would have the homotopy extension property. As we noted in the remark following Definition 2.3, by using the mapping cylinder M of the inclusion map $A \vee B \to A \times B$, a GWP in $\pi(\Sigma(M/A \vee B), X)$ is obtained for any spaces A and B. This GWP is precisely dual to Definition 6.2.[10] Regarding Definition 6.3, we note first of all that the cojoin and the

[10] However, there are advantages to considering $\Sigma(A \circledast B)$ instead of $\Sigma(M/A \vee B)$. Morever, the restriction to polyhedra is necessary at various places in §§ 3–5.

join are dual. Secondly, although $\hat{Q}$ is not dual to the space Q of Definition 2.3, it is possible to give a definition of Q (in terms of u.i. squares [8; §6]) which is dual to $\hat{Q}$. The map $\hat{\nu}: \hat{Q} \to A \hat{*} B$ is of course dual to ν: $A * B \to Q$. Thus if the second definition of the GWP is taken to be the class $\{h\nu\} \in \pi(A * B, X)$ (see the remark following Definition 2.3), then it would be the precise dual of 6.3.

Next we see that the two dual products are equivalent. Let p_0: $A \flat B \to A \vee B$ be defined by $p_0(l) = l(0)$ and let r: $\Omega(A \vee B) \to A \hat{*} B$ be the inclusion map. We set $\lambda = r\Omega p_0 : \Omega(A \flat B) \to A \hat{*} B$.[11] Then, for each space X, λ induces a map $\lambda_* : \pi(X, \Omega(A \flat B)) \to \pi(X, A \hat{*} B)$.

THEOREM 6.4. *For all* $\alpha \in \pi(X, \Omega A)$ *and* $\beta \in \pi(X, \Omega B)$,

$$\lambda_*[\alpha, \beta] = [\alpha, \beta]' .$$

The proof is essentially dual to the proof of Theorem 2.4 and hence is omitted.

Many results and proofs of the preceding sections can be dualized. However, many cannot since the dual products are not precise duals of the GWPs. It is left for the reader to determine which results of §§3–5 can be dualized and to supply the proofs. We shall close with an interesting question about the relationship between the dual product and the cup product.

Let A and B be Eilenberg-MacLane complexes of type $(G_1, p + 1)$ and $(G_2, q + 1)$ respectively. Let X be a polyhedron and let $H^n(X; G)$ denote the nth cohomology group of X with coefficients in G. Then it is well-known that there are natural identifications (i.e., group isomophisms), $\pi(X, \Omega A) = H^p(X; G_1)$ and $\pi(X, \Omega B) = H^q(X; G_2)$. Let C be an Eilenberg-MacLane complex of type $(G_1 \otimes G_2, p + q)$ and let "$\cup$" denote cup product. Then we conjecture that there is an element $\gamma \in \pi(\Omega(A \flat B), C)$ such that $\gamma \circ [\alpha, \beta] = \gamma_*[\alpha, \beta] = \alpha \cup \beta$, for all $\alpha \in \pi(X, \Omega A) = H^p(X; G_1)$ and $\beta \in \pi(X, \Omega B) = H^q(X; G_2)$. A proof of this conjecture would enable one to obtain information about cup products from facts about the dual product and commutators.

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[11] It can be shown (though we shall not do so) that the map λ is a singular homotopy equivalence.

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PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY