

REPRODUCING KERNELS AND ORTHOGONAL KERNELS FOR ANALYTIC DIFFERENTIALS ON RIEMANN SURFACES

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Introduction. In the case of plane regions of finite connectivity, there exist (Bergman [2]) some kernel functions which possess reproducing or orthogonal properties with respect to the space of analytic exact differentials. Such Bergman kernels can be defined in a constructive way, and are related to some derivatives of the Neumann function for the region. Starting with the Green's function, one may construct the corresponding Bergman kernels for the space of analytic differentials in the region.

On the other hand, by Hilbert space methods one may prove the existence of corresponding kernels on arbitrary Riemann surfaces (Ahlfors and Sario [1]) — these methods are of the nonconstructive type.

In this paper we shall construct actually such kernels for the space of analytic differentials on an arbitrary Riemann surface W . We shall first establish the relationship between the kernels and some principal functions in the case of a compact bordered \bar{W} . The principal functions solve some well defined boundary value problems on \bar{W} . By a canonical exhaustion by regular regions (with compact bordered closures) the results are extended to an open Riemann surface W .

Chapter I is preliminary in nature and contains important theorems as well as definitions used in the sequel. It is a brief survey of the theory of principal functions and of the theory of differentials on Riemann surfaces (Ahlfors and Sario [1]).

In Chapter II, we introduce the concept of abstract reproducing and orthogonal kernels (Bergman [2], Schiffer [5]). We prove a uniqueness theorem for the reproducing kernel corresponding to a closed subspace Γ_α of Γ_α , the space of analytic differentials on a Riemann surface W , and for the related orthogonal kernels with a given analytic singularity. We then determine some functionals which are extremalized by reproducing and orthogonal kernels. In particular, the reproducing kernel $h_0(z, \zeta)dz$ for Γ_α minimizes the expression

$$\|adz\|^2 - 2\text{Re}a(\zeta)$$

among all differentials $adz \in \Gamma_\alpha$, and the kernel $h_0(z, \zeta)dz$ with singularity

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$dz/(z - \zeta)^2$ orthogonal to the class Γ_{ae} of analytic exact differentials minimizes the functional

$$\frac{i}{2} \int_{\beta} f \bar{d}f$$

among all differentials of functions analytic on W except for the singularity $-(1/(z - \zeta))$.

Such extremal properties suggest some close relationship between the kernels and the principal functions on W .

In Chapter III, the case of planar surfaces is investigated in details. First we study Γ_{ae} . It is shown that for a compact bordered surface \bar{W} , with principal analytic functions P_0 and P_1 corresponding to the singularity $1/(z - \zeta)$ the following relations hold:

$$\begin{aligned} \frac{d}{dz} \left(\frac{P_1 - P_0}{2} \right) dz &= -\tilde{k}_0 dz \\ \frac{d}{dz} \left(\frac{P_1 + P_0}{2} \right) dz &= -\tilde{h}_0 dz \end{aligned}$$

where $\tilde{k}_0 dz$ is the reproducing kernel for Γ_{ae} , and $\tilde{h}_0 dz$ is the orthogonal kernel corresponding to the singularity $dz/(z - \zeta)^2$.

The preceding properties are shown there to hold for arbitrary planar Riemann surfaces.

Moreover, analogous properties for other kind of kernels enable us to prove relations between the principal functions P_{im} corresponding to the singularity $1/(z - \zeta)^m$, $m > 1$ and the principal functions P_i corresponding to the singularity $1/(z - \zeta)$.

We then complete the study of the kernels for Γ_a . We show the relationship between Bergman kernels and principal functions in the compact bordered case, and then construct directly reproducing and orthogonal kernels for the orthogonal complement Γ_{am} of Γ_{ae} .

In Chapter IV we extend the result of Chapter III to Riemann surfaces of nonzero genus. In particular, starting from the harmonic principal functions p_0 and p_1 for a nonplanar compact bordered \bar{W} , we construct reproducing and orthogonal kernels for Γ_{ae} . For an open Riemann surface W , the corresponding kernels are constructed, using a canonical exhaustion of W by regular regions, with compact bordered closures. To obtain the kernel for Γ_a , it is then only necessary to get kernels for Γ_{as} , the space of analytic Schottky differentials, which is the orthogonal complement of Γ_{ae} . We construct the kernels in the compact bordered case by considering the double \hat{W} of \bar{W} and use the theory of differentials on closed surfaces. By a limiting process (canonical exhaustion) we extend our results to an open Riemann surface W .

Chapter I. PRELIMINARIES

1. Principal Functions.

1A. We summarize here the theory of principal functions on open Riemann surfaces. For a more complete discussion we refer to [1]. Let W be an open Riemann surface. At a finite number of points $\zeta_j \in W$ there are given singularities of the form:

$$s^{(j)} = Re \sum_{n=1}^{\infty} b_n^{(j)}(z - \zeta_j)^{-n} + c^{(j)} \log |z - \zeta_j|$$

where the $c^{(j)}$ are real and subject to the condition $\sum_j c^{(j)} = 0$. Then there exist functions harmonic on W except for the singularity $s^{(j)}$ at ζ_j . The "principal" functions are defined in the following way: if $\{\Omega_n\}$ is a canonical exhaustion of W , then p_0 is the limit of such functions on Ω_n which have vanishing normal derivatives on the boundary $\beta(\Omega_n)$ of Ω_n and if \mathcal{P} is a partition of the ideal boundary β of W , then $(\mathcal{P})_{p_1}$ is the limit of such functions on Ω_n , which are constant on each part of $\beta(\Omega_n)$ associated with \mathcal{P} and whose conjugate periods along each part is zero. The functions p_0 and $(\mathcal{P})_{p_1}$ are unique save for an additive constant.

1B. Let p be a function harmonic on W except for a finite number of singularities $s^{(j)}$. Near ζ_j , $p - s^{(j)}$ has the following expansion:

$$p - s^{(j)} = a_0^{(j)} + a_1^{(j)}(z - \zeta_j) + a_2^{(j)}(z - \zeta_j)^2 + \dots$$

We introduce the notation

$$C(p) = 2\pi Re \left\{ \sum_j [c^{(j)} a_0^{(j)} - \sum_{n=1}^{\infty} n b_n^{(j)} a_n^{(j)}] \right\}$$

and set C_0, C_1 for $C(p_0), C(p_1)$ respectively

Let Ω be a regular subregion of W with boundary $\beta(\Omega)$, containing all ζ_j in its inside. We set

$$B_\Omega(p) = \int_{\beta(\Omega)} p dp^*$$

and define $D_\Omega(p - \sum_j s^{(j)})$ as the Dirichlet integral of $p - \sum_j s^{(j)}$ over Ω .

Now let

$$B(p) = \lim_{\Omega \rightarrow W} \int_{\beta(\Omega)} p dp^*$$

and

$$D\left(p - \sum s^{(j)}\right) = \lim_{\Omega \rightarrow W} D_{\Omega}\left(p - \sum_j s^{(j)}\right).$$

We have the following

THEOREM. *The function $(p_0 + p_1)/2$ minimizes the functional $B(p)$ in the class of all functions p with singularities $\sum_j s^{(j)}$ whose conjugate periods vanish over all dividing cycles associated with the partition \mathcal{P} . (Here p_1 stands for $\mathcal{P}(p_1)$). Explicitly the formula:*

$$B(p) = \frac{1}{4}(C_0 - C_1) + D\left(p - \frac{p_0 + p_1}{2}\right)$$

which is valid for such functions shows that the minimum is $(C_0 - C_1)/4$ and that the deviation from the minimum is measured by $D(p - ((p_0 + p_1)/2))$.

Let now $q = p_0 - p_1$. The harmonic function q satisfies the following theorem:

THEOREM. *For regular admissible functions the expression*

$$D(u) + 4\pi(u(\zeta_1) - u(\zeta_2))$$

attains its minimum when $u = q = p_0 - p_1$. Here p_0 and p_1 are the principal functions corresponding to the singularity $\log|z - \zeta_1| - \log|z - \zeta_2|$. Similarly,

$$D(u) - \frac{4\pi}{(n-1)!} \operatorname{Re} \frac{\partial^n u}{\partial z^n} \Big|_{z=\zeta}$$

is minimized by the function $q = p_0 - p_1$ which corresponds to the singularity $\operatorname{Re}(1/(z - \zeta)^n)$.

In both cases the minimum is $-D(q)$ and the deviation from the minimum is $D(u - q)$.

1C. We shall make use of the following result

THEOREM. *Let Ω and Ω' be two regular subregions of a Riemann surface W , such that $\Omega' \supset \Omega$; let*

$$p_{i\Omega} \quad \text{and} \quad p_{i\Omega'} \quad i = (0, 1)$$

be the principal functions corresponding respectively to Ω and Ω' ; let

$$D_{\Omega}(p_{i\Omega'} - p_{i\Omega}) \quad i = (0, 1)$$

be the Dirichlet integral of $p_{i\Omega'} - p_{i\Omega}$ extended over Ω , then

$$\lim_{\varrho \rightarrow W} D_{\varrho}(p_{i\varrho'} - p_{i\varrho}) = 0 \quad i = (0, 1) .$$

1D. Previous results may be specialized to the case of planar Riemann surfaces. By definition, every cycle on a planar surface W is dividing. For this reason it is natural to consider the canonical partition \mathcal{Q} for the boundary β . Accordingly, the admissible functions, in the planar case, have all conjugate periods equal to zero. This means that $P = p + ip^*$ is an analytic (single-valued) function. The functions p are supposed to have a given singularity s and if p^* is to be single valued, we require that s has likewise a single-valued conjugate function s^* . We then shall not consider singularities with logarithmic terms. We denote by P_0 and P_1 the analytic functions which correspond to the principal functions p_0 and p_1 . They are determined up to additive complex constants. If we denote by E the area of the complement of the range of P and by a the coefficient in the expansion.

$$P = \frac{1}{z - \zeta} + a(z - \zeta) + \dots ,$$

we have the following result:

THEOREM. *The function $(P_0 + P_1)/2$ maximizes E in the class of all normalized univalent mappings. In this case $E = (\pi/2)[a(P_0) - a(P_1)]$.*

1E. Consider now the analytic function $Q = P_0 - P_1$. The properties of Q are summarized in the following theorem.

THEOREM. *The function Q minimizes the expression $D(U) - 4\pi \operatorname{Re}a(U)$ in the class of all analytic functions $U = u + iu^*$ on W . Moreover, $a(Q)$ is nonnegative and $2\pi a(Q) = D(Q) = E$ where E denotes the complementary area associated with the mapping $P_0 + P_1$.*

2. Differentials on Riemann surfaces.

2A. We shall suppose the reader acquainted with the definitions and orthogonal decompositions of the Hilbert space Γ of square integrable differentials on an open Riemann surface. (See, for instance, [1] Chapter V). We merely recall some results about some subspaces of Γ and state the theorems on singular differentials which we shall use below.

2B. Schottky differentials.

Let \bar{W} be a bordered Riemann surface, and \hat{W} its double. The whole class of differentials on W which have a harmonic extension to \hat{W} can be shown to be identical with the direct sum $\Gamma_{h_0}(W) + \Gamma_{h_0}^*(W)$. Such differentials are called Schottky differentials.

If now one considers, on an arbitrary open surface all differentials which can be approximated in the sense of the norm by Schottky differentials, i.e. by differentials in $\Gamma_{h_0}(\Omega) + \Gamma_{h_0}^*(\Omega)$ (Ω being a regular subregion of W), the corresponding subspace Γ_s can be shown to be identical with the closure of $\Gamma_{h_0} + \Gamma_{h_0}^*$.

The consideration of Schottky differentials leads to the following decomposition:

THEOREM. *On any Riemann surface*

$$\Gamma_h = \Gamma_s + \Gamma_{ae} + \overline{\Gamma_{ae}}$$

where Γ_{ae} is the space of exact analytic differentials, $\overline{\Gamma_{ae}}$ the space of exact antianalytic differentials.

2C. Analytic differentials.

If our attention is restricted to analytic differentials, writing $\Gamma_a = \Gamma_a \cap \Gamma_s$ we obtain the following decomposition:

$$\Gamma_a = \Gamma_{ae} + \Gamma_{as}.$$

In the case of a compact bordered \bar{W} , the following result holds: An analytic Schottky differential on \bar{W} can be written as the sum of an analytic differential which is real along the boundary and one which is purely imaginary along the boundary.

On an open surface W , an analytic Schottky differential ϕ is by definition a limit of analytic Schottky differentials $\omega_{a_s\Omega}$. It follows that:

$$\phi = \lim_{\Omega \rightarrow W} (\phi_{1\Omega} + i\phi_{2\Omega})$$

where $\phi_{1\Omega}$ and $\phi_{2\Omega}$ are real along $\beta(\Omega)$.

The following theorem states an important decomposition property of analytic differentials:

THEOREM. *Any analytic differential ϕ_a has a unique representation in the form:*

$$\phi_a = \phi_{ase} + \omega_{hm} + i\omega_{hm}^*$$

where ϕ_{ase} is semi exact analytic and ω_{hm} is a harmonic measure.

If we call a differential of the form $\omega_{hm} + i\omega_{hm}^*$ an analytic measure and Γ_{am} the space of analytic measures we can write:

$$\Gamma_a = \Gamma_{ase} + \Gamma_{am}.$$

2D. Singular differentials.

Consider a point p_0 on a Riemann surface W . We say that a differential θ_0 defines a singularity at p_0 if it is defined in a punctured neighborhood of p_0 . We speak of an analytic singularity if θ_0 is an analytic differential in a punctured neighborhood of p_0 .

A differential of the second kind is a differential whose singularities are poles with vanishing residues. We can consider one pole at a time, hence it is sufficient to study the case of a single pole

$$\frac{dz}{(z - \zeta)^{m+2}} \quad m \geq 0$$

where z is a local variable with range $|z| < 1$ and ζ the value of z at p_0 . The following theorem summarizes the study of such differentials.

THEOREM. *With every singularity of the form $(z - \zeta)^{-m-2}dz$, $m \geq 0$, one can associate two differentials*

$$\begin{aligned} \phi_m &= h_m(z, \zeta)dz \quad \text{analytic except for the given singularity} \\ \psi_m &= k_m(z, \zeta)dz \quad \text{analytic.} \end{aligned}$$

The differentials ϕ_m and ψ_m are connected by

$$\begin{aligned} h_m(z, \zeta) &= \frac{1}{(m + 1)!} \frac{d^m}{dz^m} h_0(z, \zeta) \\ \overline{k_m(z, \zeta)} &= \frac{1}{(m + 1)!} \frac{d^m}{d\zeta^m} \overline{k_0(z, \zeta)} \end{aligned}$$

and h_0, k_0 satisfy symmetry relations:

$$h_0(z, \zeta) = h_0(\zeta, z), \quad k_0(z, \zeta) = \overline{k_0(\zeta, z)}$$

The differentials ψ_m have the reproducing property:

$$(a, \psi_m) = \frac{2\pi}{(m + 1)!} \alpha^{(m)}(\zeta)$$

for all $\alpha = adz \in \Gamma_a$ while the ϕ_m satisfy $(\alpha, \phi_m) = 0$, provided that the inner product is interpreted as a Cauchy limit.

2E. Differentials and chains.

In the Laurent development

$$\sum_{n=1}^{\infty} b_n(z - \zeta)^{-n} dz$$

of a singular differential θ , b_1 is called the residue at ζ . It can be shown that in order that there exists a closed differential θ with compact support and given singularities, it is necessary that the sum of the

residues be 0. The simplest case is represented by a singularity $(1/(z - \zeta_1) - 1/(z - \zeta_2))dz$ where ζ_1 and ζ_2 are points in the same parametric disc A mapped on $|z| < 1$. Results of the study of such differentials are summarized in the following theorem

THEOREM. *To each finite chain c one can assign two differentials*

$$\phi(c) = h(z, c)dz$$

analytic except for simple poles with residues equal to the coefficients in ∂c and

$$\psi(c) = k(z, c)dz$$

analytic. One has:

$$h(z, c) = \int_c h_0(z, \zeta)d\zeta$$

$$k(z, c) = \int_c k_0(z, \zeta)d\bar{\zeta}$$

and $(\alpha, \psi) = -(\alpha, \phi) = 2\pi \int_c \alpha$ for analytic α .

2F. Differentials and periods.

If c is a finite cycle, $\phi(c), \psi(c)$ are all regular. Results about this case are stated below:

THEOREM. *If c is a finite cycle $\phi(c) = -\psi(c)$ and the reproducing differential $\sigma(c)$ corresponding to the cycle c is $\sigma(c) = (1/\pi)Im\phi(c)$. The periods of $\sigma(c)$ are integers.*

2G. One can strengthen Theorems 2D and 2E by requiring the associated differentials to be semi-exact. In that case we denote them by $\tilde{\phi}_m, \tilde{\psi}_m, \tilde{\phi}(c), \tilde{\psi}(c)$.

It can be shown that $\tilde{\phi}$ and $\tilde{\psi}$ have the same reproducing properties as ϕ and ψ , but only with respect to semi-exact analytic differentials. Since $\tilde{\phi}_m$ and $\tilde{\psi}_m, \tilde{\phi}(c)$ and $\tilde{\psi}(c)$ are themselves semi exact, Theorems 2D, 2E remain valid with modified notations, except that the differentials α must be assumed semi exact.

2H. Remark on integration by parts.

Let $\omega \in \Gamma, df \in \Gamma_e$. If $\partial\bar{W}$ denotes the boundary of a bordered surface a useful formula for integration by parts is:

$$\int_W (df)\omega = \int_{\partial\bar{W}} f\omega - \int_W fd\omega .$$

We plan to extend such a formula to differentials with singularities. Let ω be a closed differential with an analytic singularity at $z = \zeta$. Let h be an admissible local homeomorphism representing a neighborhood of ζ onto the disk $|z| < 1$. Let $|z| < r < 1$ be a disk whose inverse image under h will be called Δ . We denote by α the boundary of Δ . The previous formula for integration by parts yields:

$$\int_{W-\Delta} (df)\omega = \int_{\partial\bar{W}-\alpha} f\omega .$$

But now in Δ , if we consider $\int_{\Delta} (df)\omega$ as a Cauchy limit, the following relation holds:

$$\int_{\Delta} (df)\omega = \int_{\alpha} f\omega$$

and by addition we get the formula

$$\int_W (df)\omega = \int_{\partial\bar{W}} f\omega .$$

Notice that on $W, \partial\bar{W} - \alpha$ bounds the 2-chain $W - \Delta$, which implies that $\partial\bar{W}$ and α are homologous on W . The last integral depends only on the homology class of $\partial\bar{W}$, hence can be transferred to α .

Chapter II. GENERAL DEFINITIONS AND FUNDAMENTAL PROPERTIES OF REPRODUCING AND ORTHOGONAL KERNELS

We shall define here some particular kernels and derive some of their characteristic extremal properties.

1. General Properties.

1A. DEFINITION. Let Γ_{α} be a subspace of Γ_{α} . A differential $k_0(z, \zeta)dz$ defined globally will be called a reproducing kernel for Γ_{α} if

1. $k_0(z, \zeta)dz \in \Gamma_{\alpha}$.
2. for $a(z)dz \in \Gamma_{\alpha}$, the inner product $(a(z)dz, k_0(z, \zeta)dz) = a(\zeta)$.

More generally we shall consider m -kernels according to the following definition:

DEFINITION. A globally defined differential $k_m(z, \zeta)dz$ is an m -kernel for $\Gamma_{\alpha} \subset \Gamma_{\alpha}$ whenever:

1. $k_m(z, \zeta) \in \Gamma_{\alpha}$
2. for $a(z)dz \in \Gamma_{\alpha}$, $(a(z)dz, k_m(z, \zeta)dz) = a^{(m)}(\zeta)$ where $a^{(m)}(\zeta) = \left. \frac{d^m a(z)}{dz^m} \right|_{z=\zeta}$.

1B. We introduce now some kernels with singularities. Given an analytic singularity $t(z, \zeta)dz$, one may consider differentials which are elements of Γ_α except for the singularity $t(z, \zeta)dz$.

DEFINITION. A differential $h_t(z, \zeta)dz$ is an orthogonal kernel for $\Gamma_\alpha \subset \Gamma_\alpha$, corresponding to the analytic singularity $t(z, \zeta)dz$ if the following requirements are fulfilled.

1. $h_t(z, \zeta)$ is an element of Γ_α except for the singularity $t(z, \zeta)dz$
2. for $a(z)dz \in \Gamma_\alpha$, $(a(z)dz, h_t(z, \zeta)dz) = 0$.

In particular we shall call $h_m(z, \zeta)dz$, the orthogonal kernel corresponding to the singularity $dz/(z - \zeta)^{m+2}$ for $m \geq 0$.

1C. Now we can easily prove uniqueness theorems.

THEOREM. *When there exists a reproducing kernel for Γ_α , it is unique.*

Proof. Let $k_0(z, \zeta)dz$ and $k'_0(z, \zeta)dz$ be two different reproducing kernels for Γ_α . Then

$$\begin{aligned} 0 &\leq \|k_0(z, \zeta)dz - k'_0(z, \zeta)dz\|^2 \\ &= (k_0(z, \zeta)dz - k'_0(z, \zeta)dz, k_0(z, \zeta)dz) \\ &\quad - (k_0(z, \zeta)dz - k'_0(z, \zeta)dz, k'_0(z, \zeta)dz) \\ &= k_0(\zeta, \zeta)dz - k'_0(\zeta, \zeta)dz - k_0(\zeta, \zeta)dz + k'_0(\zeta, \zeta)dz = 0. \end{aligned}$$

Therefore $k_0(z, \zeta)dz = k'_0(z, \zeta)dz$.

1D. An analogous argument proves the next statement.

THEOREM. *When there exists an m -kernel for Γ_α it is unique.*

1E. Correspondingly the following uniqueness theorem holds for $h_t(z, \zeta)dz$.

THEOREM. *When there exists for Γ_α an orthogonal kernel $h_t(z, \zeta)dz$ with singularity $t(z, \zeta)dz$, it is unique.*

Proof. If $h_t dz$ and $h'_t dz$ are two different orthogonal kernels corresponding to the same singularity, $h_t dz - h'_t dz$ is regular and orthogonal to Γ_α . But $h_t dz - h'_t dz \in \Gamma_\alpha$. Therefore $h_t dz - h'_t dz = 0$, which proves uniqueness.

2. Extremal Properties.

2A. We shall prove some extremal properties which characterize reproducing and orthogonal kernels.

THEOREM. *Among all $a(z)dz \in \Gamma_\alpha$, such that $a^{(m)}(\zeta) = 1$, $(k_m(z, \zeta)/k_m^{(m)}(\zeta, \zeta))dz$ has minimal norm.*

Proof. By Schwarz's inequality:

$$|(a(z)dz, k_m(z, \zeta)dz)|^2 \leq \|adz\|^2 \|k_m(z, \zeta)dz\|^2 .$$

But

$$\begin{aligned} (a(z)dz, k_m(z, \zeta)dz) &= a^{(m)}(\zeta) = 1 \\ \|k_m dz\|^2 &= (k_m(z, \zeta)dz, k_m(z, \zeta)dz) = k_m^{(m)}(\zeta, \zeta) . \end{aligned}$$

Therefore: $\|a(z)dz\|^2 \geq 1/k_m^{(m)}(\zeta, \zeta)$ with equality only for $a(z)dz = \lambda k_m(z, \zeta)dz$. Due to the normalization, $\lambda = k_m^{(m)}(\zeta, \zeta)$ and therefore $(k_m(z, \zeta)/k_m^{(m)}(\zeta, \zeta))dz$ has minimal norm among all $a(z)dz \in \Gamma_\alpha$ such that $a^{(m)}(\zeta) = 1$.

2B. The following theorem introduces a functional which is extremalized by $k_m(z, \zeta)dz$.

THEOREM. *Among all differentials $a(z)dz \in \Gamma_\alpha$, $k_m(z, \zeta)$ minimizes the expression $\|adz\|^2 - 2\text{Re}a^{(m)}(\zeta)$. The minimum is $-k_m^{(m)}(\zeta, \zeta)$ and the deviation from the minimum is measured by $\|adz - k_m dz\|^2$.*

Proof. Let $a(z)dz$ be an element of Γ_α , $k_m(z, \zeta)dz$ be the m kernel for Γ_α . Then:

$$\begin{aligned} \|a(z)dz - k_m(z, \zeta)dz\|^2 &= \|adz\|^2 + \|k_m dz\|^2 - (a(z)dz, k_m(z, \zeta)dz) \\ &\quad - (k_m(z, \zeta)dz, a(z)dz) . \end{aligned}$$

The last equation can be rewritten as:

$$\|a(z)dz\|^2 - 2\text{Re}(a(z)dz, k_m(z, \zeta)dz) = \|adz - k_m dz\|^2 - \|k_m dz\|^2 .$$

But $\text{Re}(a(z)dz, k_m(z, \zeta)dz) = \text{Re}a^{(m)}(\zeta)$. Therefore:

$$\|a(z)dz\|^2 - 2\text{Re}a^{(m)}(\zeta) = \|adz - k_m dz\|^2 - \|k_m dz\|^2$$

which establishes the extremal property. The minimum is

$$-||k_m dz||^2 = -(k_m(z, \zeta)dz, k_m(z, \zeta)dz) = -k_m^{(m)}(\zeta, \zeta) .$$

2C. An important extremal property can be proved for the kernel $h_m(z, \zeta)dz$ orthogonal to the class Γ_{ae} , among the class of differentials of functions analytic on W except for the singularity $-1/(m + 1)(z - \zeta)^{m+1}$, $m \geq 0$. Let us call $\Gamma_{ae}^{(m)}$ such a class. Clearly $h_m(z, \zeta)dz$ is the differential of such a function $H_m(z, \zeta)$. In other words $h_m(z, \zeta)dz \in \Gamma_{ae}^{(m)}$.

Let \bar{W} be now a compact bordered Riemann Surface. Define a surface W' obtained by deleting from W the inverse image Δ of the disk $|z| < r < 1$, where $|z| < 1$ is the image of Δ . We shall suppose that ζ lies in Δ .

If α denotes the boundary of Δ , and β the boundary of \bar{W} we shall write for f analytic on W' :

$$B(f) = \frac{i}{2} \int_{\beta} f \bar{d}f$$

$$A(f) = \frac{i}{2} \int_{\beta} f \bar{d}f$$

Applying Green's formula to W' we get

$$||df||^2 = 2(B(f) - A(f))$$

We are now ready to prove the following theorem.

THEOREM. *The kernel $h_m(z, \zeta)dz$ orthogonal to the class Γ_{ae} minimizes the functional $B(a)$ among all differentials $da = a'(z)dz$ in $\Gamma_{ae}^{(m)}$.*

Proof. We first consider the compact bordered case. Let W' be the surface obtained by the method mentioned above from a compact bordered \bar{W} . Let $h_m dz = dH_m$ be the orthogonal kernel for the class Γ_{ae} , corresponding to the singularity $dz/(z - \zeta)^{m+2}$ $m \geq 0$. For each $da = a'(z)dz$ we have:

$$||a'dz - h_m dz||_{W'}^2 = ||a'dz||_{W'}^2 + ||h_m dz||_{W'}^2 - 2Re(a'dz, h_m dz)_{W'} .$$

But

$$||a'dz||_{W'}^2 = 2(B(a) - A(a))$$

$$||h_m dz||_{W'}^2 = 2(B(H_m) - A(H_m)) .$$

Therefore:

$$||a'dz - h_m dz||_{W'}$$

$$= ||a'dz||_{W'}^2 - ||h_m dz||_{W'}^2 - 2Re(a'dz - h_m dz, h_m dz)_{W'}$$

$$= 2[B(a) - A(a)] - 2[B(H_m) - A(H_m)] - 2Re(a'dz - h_m dz, h_m dz)_{W'} .$$

But as $r \rightarrow 0$, $A(a) \rightarrow 0$ and $A(H_m) \rightarrow 0$ as Cauchy limits. Moreover, by orthogonality $((a' - h_m)dz, h_m dz)_{\bar{W}} = 0$. Therefore:

$$\begin{aligned} \|a'dz - h_m dz\|_{\bar{W}}^2 &= 2B(a) - 2B(H_m) - 2\text{Re}(a'dz - h_m dz, h_m dz)_{\bar{W}} \\ &= 2B(a) - 2B(H_m) \end{aligned}$$

by the orthogonal property of $h_m dz$ over Γ_{ae} . Hence

$$B(a) = B(H_m) + \frac{1}{2} \|a'dz - h_m dz\|_{\bar{W}}^2$$

which proves the theorem.

To extend the extremal property to open Riemann surfaces W , we exhaust W by canonical regions Ω .

For $\Omega' \supset \Omega$, $H_{m\Omega'}$ is a competing function hence:

$$B_{\Omega}(H_{m\Omega'}) = B_{\Omega}(H_{m\Omega}) + \frac{1}{2} \|dH_{m\Omega'} - dH_{m\Omega}\|_{\Omega}^2$$

and clearly $B_{\Omega'}(H_{m\Omega'}) \geq B_{\Omega}(H_{m\Omega'})$. Therefore

$$B_{\Omega'}(H_{m\Omega'}) \geq B_{\Omega}(H_{m\Omega})$$

which shows that $B_{\Omega}(H_{m\Omega})$ is nondecreasing. If there exists a function P on W , analytic except for the singularity $-1/(m + 1)(z - \zeta)^{m+1}$ and such that $\lim_{\Omega \rightarrow W} B_{\Omega}(P) < \infty$, then the functional $B_{\Omega}(H_{m\Omega})$ has a finite limit when $\Omega \rightarrow W$. Consequently $(1/2)\|dH_{m\Omega'} - dH_{m\Omega}\|_{\Omega}^2 = D_{\Omega}(H_{m\Omega'} - H_{m\Omega})$, where D_{Ω} is the Dirichlet integral extended over Ω , has limit zero when $\Omega \rightarrow W$. It follows in the customary way that there exists then in $\Gamma_{ae}^{(m)}$ an orthogonal kernel $dH_m = \lim_{\Omega \rightarrow W} dH_{m\Omega}$.

In the case of a planar surface P_0 or P_1 can play the part of such a function P .

2D. We shall now consider particular types of kernels corresponding to the set of chains on a Riemann surface.

Let first ζ_1 and ζ_2 be two points on a Riemann surface, and c a path joining ζ_1 to ζ_2 . We say that $k(z, c)dz$ is a reproducing kernel attached to the path c for the class Γ_a whenever:

1. $k(z, c)dz \in \Gamma_a$
2. for any $a(z)dz \in \Gamma_a$, $(a(z)dz, k(z, c)dz) = \int_c a dz$.

The uniqueness proof is analogous to the proof of Theorem 1C. We

shall prove an extremal property for $k(z, c)dz$.

THEOREM. *The kernel $k(z, c)dz$ attached to the path c , which is reproducing for the class Γ_α minimizes the functional*

$$\|a(z)dz\|^2 - 2\operatorname{Re} \int_c adz$$

over the class Γ_α .

Proof. Let $a(z)dz$ be an element of Γ_α . Then:

$$\begin{aligned} & \|a(z)dz - k(z, c)dz\|^2 \\ &= \|a(z)dz\|^2 + \|k(z, c)dz\|^2 - 2\operatorname{Re}(a(z)dz, k(z, c)dz) \\ &= \|a(z)dz\|^2 + \|k(z, c)dz\|^2 - 2\operatorname{Re} \int_c adz. \end{aligned}$$

Therefore $\|a(z)dz\|^2 - 2\operatorname{Re} \int_c adz = \|a(z)dz - k(z, c)dz\|^2 - \|k(z, c)dz\|^2$ which proves the theorem. Notice that $\|k(z, c)dz\|^2 = \int_c k(z, c)dz$, which shows that $\int_c k(z, c)dz$ is real.

2E. Let us now suppose that ζ_1 and ζ_2 are located in the same parametric disk. Consider the functions analytic on W cut along c , except for the singularity $s = \log(z - \zeta_1)/(z - \zeta_2)$. The singularity for their differentials is $dz[1/(z - \zeta_1) - 1/(z - \zeta_2)]$.

Assume that among these there exists a differential dH_c which has the following property: for any $adz \in \Gamma_{\alpha c}$, $(adz, dH_c) = \int_c adz$.

The differential dH_c has an important extremal property.

THEOREM. *The differential dH_c minimizes the functional*

$$B(a) - \operatorname{Re} \int_c a'dz - ds$$

among the class $\Gamma_{\alpha c}(c)$ of differentials of functions a analytic on W cut along c , except for the singularity

$$s = \log \frac{z - \zeta_1}{z - \zeta_2}.$$

Proof. Let us first consider a compact bordered Riemann surface \bar{W} . Let α be a path around c , bounding a region Δ . Let $W' = \bar{W} - \Delta$. For any $da = a'(z) \in \Gamma_{\alpha c}(c)$ one can write:

$$\begin{aligned} & \|a'(z)dz - dH_c\|_W^2, \\ & = \|a'(z)dz\|_W^2 - \|dH_c\|_W^2 - 2\operatorname{Re}(a'(z)dz - dH_c, dH_c)_W. \end{aligned}$$

Using the same method as in the proof of Theorem 2C this equality can be transformed into

$$\begin{aligned} \|a'(z)dz - dH_c\|_W^2 & = 2B(a) - 2B(H_c) - 2\operatorname{Re}(a'(z)dz - dH_c, dH_c)_{\bar{W}} \\ & = 2B(a) - 2B(H_c) - 2\operatorname{Re} \int_c a'(z)dz - dH_c \end{aligned}$$

or

$$\begin{aligned} B(a) - \operatorname{Re} \int_c a'dz - ds & = B(H_c) - \operatorname{Re} \int_c dH_c - ds \\ & \quad + \frac{1}{2} \|a'(z)dz - dH_c\|_W^2 \end{aligned}$$

which completes the proof.

To extend the property to open Riemann surfaces, we may use the method outlined in the proof of Theorem 2C. For canonical regions Ω , the functional

$$B_\Omega(a) - \operatorname{Re} \int_c a'dz - ds$$

is easily seen to be nondecreasing when Ω increases. If there exists on W cut along c a function analytic except for the singularity s , then the functional has a finite limit and H_{c_Ω} has a limit H_c when Ω tends to W . By linearity one can extend the property to any finite chain.

Chapter III. PLANAR RIEMANN SURFACES

We shall now restrict our attention to planar Riemann surfaces W . We shall establish the connection between the principal functions for W and the reproducing and orthogonal kernels, with our main interest devoted to the class Γ_{ae} .

1. Reproducing and Orthogonal Kernels For Γ_{ae} .

1A. In the case $\Gamma_a = \Gamma_{ae}$, we shall prove the following theorem:

THEOREM. *Let W be a planar Riemann surface. If P_0 and P_1 denote the analytic principal functions on W corresponding to the singularity $1/(z - \zeta)$ and if $(\tilde{k}_0(z, \zeta)/2\pi)dz$ and $(\tilde{h}_0(z, \zeta)/2\pi)dz$ are respectively the reproducing and orthogonal (singularity $(1/2\pi)(z - \zeta)^2$) kernels for Γ_{ae} on W , then*

$$\frac{d}{dz} \frac{P_1 - P_0}{2} dz = -\tilde{k}_0(z, \zeta) dz$$

$$\frac{d}{dz} \frac{P_1 + P_0}{2} dz = -\tilde{h}_0(z, \zeta) dz .$$

1B. First Proof. (By extremal property) Let $a(z)dz$ be an element of Γ_{ae} . Theorem 2B shows that

$$\|a(z)dz\|^2 - 2\operatorname{Re}a(\zeta) = \left\| a(z)dz - \frac{\tilde{k}_0(z, \zeta)}{2\pi} dz \right\|^2 - \left\| \frac{\tilde{k}_0(z, \zeta)}{2\pi} dz \right\|^2$$

or

$$\begin{aligned} & \|4\pi a(z)dz\|^2 - 8\pi \operatorname{Re}(4\pi a(\zeta)) \\ &= \|4\pi a(z)dz - 2\tilde{k}_0(z, \zeta)dz\|^2 - \|2\tilde{k}_0(z, \zeta)dz\|^2 . \end{aligned}$$

If we put $4\pi a(z)dz = b(z)dz$, we get:

$$\|b(z)dz\|^2 - 8\pi \operatorname{Re}b(\zeta) = \|b(z)dz - 2\tilde{k}_0(z, \zeta)dz\|^2 - \|2\tilde{k}_0(z, \zeta)dz\|^2 .$$

But the analytic function $P_0 - P_1 = Q$ corresponding to the singularity $1/(z - \zeta)$ minimize the expression

$$D(U) - 4\pi \operatorname{Re} \frac{dU}{dz} \Big|_{z=\zeta} = D(U - Q) - D(Q)$$

among all analytic functions on W . In terms of the differential dU , the expression becomes:

$$\frac{1}{2} \|dU\|^2 - 4\pi \operatorname{Re} \frac{dU}{dz} \Big|_{z=\zeta} = \frac{1}{2} \|dU - dQ\|^2 - \frac{1}{2} \|dQ\|^2$$

or

$$\|dU\|^2 - 8\pi \operatorname{Re} \frac{dU}{dz} \Big|_{z=\zeta} = \|dU - dQ\|^2 - \|dQ\|^2$$

which shows that $(dQ/dz)dz$ minimizes the same functional as $2k_0(z, \zeta)dz$ over Γ_{ae} . Therefore

$$\frac{d}{dz} \frac{(P_1 - P_0)}{2} dz = -\tilde{k}_0(z, \zeta) dz .$$

An analogous proof will show,

$$\frac{d}{dz} \frac{P_1 + P_0}{2} dz = -\tilde{h}_0(z, \zeta) dz .$$

1C. Second Proof. (By canonical exhaustion)

Let \bar{W} be a planar compact bordered Riemann surface, with border β . We shall first prove a lemma on the boundary behavior of the differentials of $P_0 - P_1$ and $P_0 + P_1$.

LEMMA. *Along β*

$$\overline{\frac{d(P_0 - P_1)}{dz} dz} = \frac{d(P_0 + P_1)}{dz} dz .$$

Proof.

$$\begin{aligned} \frac{1}{2} \frac{dP_0}{dz} &= \frac{\partial p_0}{\partial x} - i \frac{\partial p_0}{\partial y} \\ \frac{1}{2} \frac{dP_1}{dz} &= \frac{\partial p_1}{\partial x} - i \frac{\partial p_1}{\partial y} . \end{aligned}$$

For a special choice of the local variable $dz = dx$ along β and then by definition

$$\begin{aligned} \frac{\partial p_0}{\partial y} &= 0 \\ \frac{\partial p_1}{\partial x} &= 0 . \end{aligned}$$

Therefore the equality in the lemma corresponds to

$$\overline{\left(\frac{\partial p_0}{\partial x} + i \frac{\partial p_1}{\partial y}\right) dx} = \left(\frac{\partial p_0}{\partial x} - i \frac{\partial p_1}{\partial x}\right) dx .$$

We may notice that the proof actually shows that for any compact bordered \bar{W}

$$\left(\frac{\partial p_0}{\partial z} - i \frac{\partial p_1}{\partial z}\right) dz = \overline{\left(\frac{\partial p_0}{\partial z} + i \frac{\partial p_1}{\partial z}\right) dz} \quad \text{along } \beta .$$

We shall now show directly that

$$-\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz$$

is the reproducing kernel for Γ_{ee} on \bar{W} . Let $da = a'(z)dz$ be an element of Γ_{ee} , then

$$\left(a'(z)dz, \frac{d(P_1 - P_0)}{dz} dz\right) = -\frac{1}{i} \int_{\bar{W}} a'(z) \overline{\frac{d(P_1 - P_0)}{dz} dz} dz \bar{dz} .$$

But we know that

$$\int_{\bar{W}} \omega df = \int_{\beta} \omega f - \int_{\bar{W}} f d\omega .$$

If ω is closed, then

$$\int_{\bar{\omega}} \omega df = \int_{\beta} \omega f .$$

Therefore:

$$\begin{aligned} \left(a'(z) dz, \frac{d(P_1 - P_0)}{dz} dz \right) &= -\frac{1}{i} \int_{\beta} a(z) \frac{d \overline{(P_1 - P_0)}}{dz} d\bar{z} \\ &= \frac{1}{i} \int_{\beta} (az) \frac{d(P_1 + P_0)}{dz} dz \end{aligned}$$

by the property proved in the lemma. Let α be the boundary of a circle of radius r about $z = \zeta$, r being small enough for α to lie in the same parametric disk as $z = \zeta$; then:

$$\frac{1}{i} \int_{\beta} a(z) \frac{d(P_1 + P_0)}{dz} dz = \frac{1}{i} \int_{\alpha} a(z) \frac{d(P_1 + P_0)}{dz} dz .$$

In a neighborhood of $z = \zeta$, $(P_1 + P_0)/2$ may be expressed as:

$$\frac{P_1 + P_0}{2} = \frac{1}{z - \zeta} + l(z, \zeta)$$

where $l(z, \zeta)$ is analytic in z . Therefore

$$\frac{1}{i} \int_{\alpha} a(z) \frac{d(P_1 + P_0)}{dz} dz = -\frac{1}{i} \int_{\alpha} \frac{a(z)}{(z - \zeta)^2} dz + \frac{1}{i} \int_{\alpha} a(z) \frac{dl(z, \zeta)}{dz} dz .$$

The second integral tends to zero as r tends to zero, for the integrand is analytic at ζ . The first one has the value $-2\pi a'(\zeta)$. We conclude that

$$\left(a'(z) dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right) = a'(\zeta)$$

which proves the reproducing property.

The same method can be used to prove that $(-1/2\pi)(d/dz)((P_1 + P_0)/2) dz$ is the orthogonal kernel for Γ_{ae} , corresponding to the singularity $dz/2\pi(z - \zeta)^2$. For $a'(z) dz \in \Gamma_{ae}$, we compute the following inner product:

$$\begin{aligned} \left(a'(z) dz, \frac{d(P_1 + P_0)}{dz} dz \right) &= -\frac{1}{i} \int_{\bar{\omega}} a'(z) \frac{d \overline{(P_1 + P_0)}}{dz} dz d\bar{z} \\ &= -\frac{1}{i} \int_{\beta} a(z) \frac{d \overline{(P_1 + P_0)}}{dz} dz \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{i} \int_{\beta} a(z) \frac{d(P_1 - P_0)}{dz} \frac{dz}{2} \\ &= -\frac{1}{i} \int_{\alpha} a(z) \frac{d(P_1 - P_0)}{dz} \frac{dz}{2}. \end{aligned}$$

The last integral tends to zero when r tends to zero for the integrand is analytic at $z = \zeta$. Therefore

$$\left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 + P_0)}{dz} dz \right) = 0.$$

1D. To extend the property to open Riemann surfaces W , we shall consider a canonical exhaustion of W by regular regions Ω

Let $da = a'(z)dz$ be an exact analytic differential on W ; we denote by

$P_{i\Omega}$ $i = (0, 1)$ the analytic principal functions for the region Ω and by

P_i $i = (0, 1)$ the analytic principal functions corresponding to the surface W . For each $\Omega \subset W$, we proved that:

$$\left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_{1\Omega} - P_{0\Omega})}{dz} dz \right)_{\Omega} = a'(\zeta)$$

We now consider:

$$\begin{aligned} &\left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_W - a'(\zeta) \\ &= \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_W - \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_{1\Omega} - P_{0\Omega})}{dz} dz \right)_{\Omega} \\ &= \left(a'(z)dz, -\frac{1}{2\pi} \frac{d[(P_1 - P_{1\Omega}) - (P_0 - P_{0\Omega})]}{dz} dz \right)_{\Omega} \\ &\quad + \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_{W-\Omega}. \end{aligned}$$

Therefore:

$$\begin{aligned} &\left| \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_W - a'(\zeta) \right| \\ &\leq \left| \left(a'(z)dz, -\frac{1}{2\pi} \frac{d[(P_1 - P_{1\Omega}) - (P_0 - P_{0\Omega})]}{dz} dz \right)_{\Omega} \right| \\ &\quad + \left| \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_{W-\Omega} \right|. \end{aligned}$$

By Schwarz's inequality:

$$\begin{aligned} & \left| \left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_{W-\Omega} \right| \\ & \leq \| a'(z)dz \|_{W-\Omega} \left\| -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right\|_{W-\Omega} \end{aligned}$$

both norms in the right hand side tend to zero when $\Omega \rightarrow W$ because $a'(z)dz$ has finite norm and

$$\left\| -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right\|_{W-\Omega} = \sqrt{2} D_{W-\Omega} \left[\frac{p_1 - p_0}{4\pi} \right]^{1/2}$$

and $p_1 - p_0$ has finite Dirichlet integral.

As to the first expression in the right hand side of the inequality:

$$\begin{aligned} & \left| \left(a'(z)dz, -\frac{1}{2\pi} \frac{d[(P_1 - P_{1\Omega}) - (P_0 - P_{0\Omega})]}{dz} dz \right)_{\Omega} \right| \\ & \leq \| a'(z)dz \|_{\Omega} \left\| -\frac{1}{2\pi} \frac{d[(P_1 - P_{1\Omega}) - (P_0 - P_{0\Omega})]}{dz} dz \right\|_{\Omega}; \end{aligned}$$

$\| a'(z)dz \|_{\Omega}$ has a finite limit when $\Omega \rightarrow W$ and

$$\left\| -\frac{1}{2\pi} \frac{d[(P_1 - P_{1\Omega}) - (P_0 - P_{0\Omega})]}{dz} dz \right\|_{\Omega} = 2D_{\Omega}^{1/2} \frac{1}{4\pi} [(p_1 - p_{1\Omega}) - (p_0 - p_{0\Omega})].$$

Moreover by the triangle inequality

$$\begin{aligned} & D_{\Omega}^{1/2} \left(\frac{1}{4\pi} [(p_1 - p_{1\Omega}) - (p_0 - p_{0\Omega})] \right) \\ & \leq D_{\Omega}^{1/2} \left(\frac{1}{4\pi} (p_1 - p_{1\Omega}) \right) + D_{\Omega}^{1/2} \left(\frac{1}{4\pi} (p_0 - p_{0\Omega}) \right). \end{aligned}$$

But Theorem 1F, Ch. I shows that

$$\lim_{\Omega \rightarrow W} D_{\Omega}(p_i - p_{i\Omega}) = 0 \qquad i = 0, 1.$$

We conclude that

$$\left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 - P_0)}{dz} dz \right)_W = a'(\zeta).$$

The same pattern of proof applies to show that

$$\left(a'(z)dz, -\frac{1}{2\pi} \frac{d(P_1 + P_0)}{dz} dz \right)_W = 0$$

which proves that $(-1/2\pi)(d/dz)((P_1 + P_0)/2)dz$ is the kernel orthogonal

to Γ_{ae} corresponding to the singularity $1/2\pi(z - \zeta)^2$.

By the uniqueness property we have proved that:

$$\frac{d}{dz} \frac{(P_1 - P_0)}{2} dz = -\tilde{k}_0(z, \zeta) dz$$

$$\frac{d}{dz} \frac{(P_1 + P_0)}{2} dz = -\tilde{h}_0(z, \zeta) dz .$$

1E. We now investigate the relationship between the principal functions corresponding to the singularity $(1/(z - \zeta)^{m+1}) m > 0$ and the kernels $\tilde{k}_m(z, \zeta) dz$ and $\tilde{h}_m(z, \zeta) dz$ (singularity $(1/(z - \zeta)^{m+2}) m > 0$). In the statement of the next theorem we call P_{0m} and P_{1m} the analytic principal functions corresponding to the singularity $1/(z - \zeta)^{m+1}$.

THEOREM.

$$\frac{1}{m + 1} \frac{d}{dz} \frac{(P_{1m} - P_{0m})}{2} dz = -\tilde{k}_m(z, \zeta) dz$$

$$\frac{1}{m + 1} \frac{d}{dz} \frac{(P_{1m} + P_{0m})}{2} dz = -\tilde{h}_m(z, \zeta) dz .$$

Proof. We shall prove that both sides in the first equation minimize the same functional over the class Γ_{ae} . We know by Theorem 2B, Ch. II that

$$\frac{(m + 1)!}{2\pi} \tilde{k}_m(z, \zeta) dz$$

minimizes

$$\| |a(z) dz|^2 - 2\text{Re}a^{(m)}(\zeta) \text{ over } \Gamma_{ae} ,$$

and by Theorem 1E, Ch. I that

$$Q_m = P_{0m} - P_{1m}$$

minimizes

$$D(U) - \frac{4\pi}{m!} \text{Re} \frac{d^{m+1}U}{dz^{m+1}} \Big|_{z=\zeta} .$$

The last expression can be written as:

$$2\|dU\|^2 - \frac{2\pi}{m!} \text{Re} \frac{d^m}{dz^m} \left(\frac{dU}{dz} \right) \Big|_{z=\zeta} .$$

Therefore $(m!/2\pi)(dQ_m/dz)dz$ minimizes the same expression as $((m + 1)!/2\pi)\tilde{k}_m(z, \zeta)dz$ over Γ_{ae} . Then

$$\frac{1}{m + 1} \frac{d(P_{1m} - P_{0m})}{dz} dz = -\tilde{k}_m(z, \zeta)dz .$$

To prove the second equality, one can show by the method outlined in the second proof of Theorem 1A that

$$\frac{1}{m + 1} \frac{d(P_{1m} + P_{0m})}{dz} dz$$

is the orthogonal kernel for Γ_{ae} corresponding to the singularity $dz/(z - \zeta)^{m+2}$, $m > 0$; then by the uniqueness theorem it follows that:

$$\frac{1}{m + 1} \frac{d(P_{1m} + P_{0m})}{dz} dz = -\tilde{h}_m(z, \zeta)dz .$$

1F. We now may use our knowledge about $\tilde{k}_m(z, \zeta)dz$ and $\tilde{h}_m(z, \zeta)dz$ to obtain new results about the functions P_{0m} and P_{1m} .

THEOREM. *Between the derivatives of P_{im} and P_i , $i = 0, 1$, the following relations hold.*

$$\begin{aligned} \frac{d}{dz}(P_{1m} + P_{0m}) &= \frac{1}{m!} \frac{d}{dz} \frac{d^m}{dz^m}(P_1 + P_0) \\ \frac{d}{d\bar{z}}(\overline{P_{1m} - P_{0m}}) &= \frac{1}{m!} \frac{d}{d\bar{z}} \frac{d^m}{d\zeta^m}(\overline{P_1 - P_0}) \end{aligned}$$

moreover:

$$\begin{aligned} \frac{d}{dz}(P_1 + P_0) &\text{ is symmetric in } z \text{ and } \zeta \\ \frac{d}{d\bar{z}}(P_1 - P_0) &\text{ is conjugate symmetric in } z \text{ and } \zeta. \end{aligned}$$

Proof. From the relations

$$\begin{aligned} \overline{\tilde{k}_m(z, \zeta)} &= \frac{1}{(m + 1)!} \frac{d^m}{d\zeta^m} \overline{\tilde{k}_0(z, \zeta)} \\ \tilde{h}_m(z, \zeta) &= \frac{1}{(m + 1)!} \frac{d^m}{dz^m} \tilde{h}_0(z, \zeta) , \end{aligned}$$

it follows that

$$\frac{1}{m + 1} \frac{d}{d\bar{z}}(\overline{P_{1m} - P_{0m}}) = \frac{1}{(m + 1)!} \frac{d}{d\bar{z}} \frac{d^m}{d\zeta^m}(\overline{P_1 - P_0})$$

or

$$\frac{d}{d\bar{z}}(P_{1m} - P_{0m}) = \frac{1}{m!} \frac{d}{d\bar{z}} \frac{d^m}{d\zeta^m} \overline{(P_1 - P_0)} .$$

Similarly

$$\frac{d}{dz}(P_{1m} + P_{0m}) = \frac{1}{m!} \frac{d^m}{dz^m} (P_1 + P_0)$$

and one gets the first set of relations in the theorem. To get the second set, we recall that:

$$\begin{aligned} \tilde{h}_0(z, \zeta) &= \tilde{h}_0(\zeta, z) \\ \tilde{k}_0(z, \zeta) &= \overline{\tilde{k}_0(\zeta, z)} . \end{aligned}$$

It follows that

$$\frac{d}{dz}(P_1 + P_0) \text{ is symmetric in } z \text{ and } \zeta$$

and

$$\frac{d}{dz}(P_1 - P_0) \text{ is conjugate symmetric in } z \text{ and } \zeta .$$

1G. To complete our study, we shall consider the case of the singularity $(1/(z - \zeta_1) - 1/(z - \zeta_2))dz$, where ζ_1 and ζ_2 lie in the same parametric disk.

Let P_{c_0} and P_{c_1} be the principal functions corresponding to the singularity $\log(z - \zeta_1)/(z - \zeta_2)$. P_{c_0} and P_{c_1} are analytic on W cut along a path c joining ζ_1 to ζ_2 . We shall prove the following results:

1H. THEOREM. *The derivatives of P_{c_0} , P_{c_1} and the kernels $\tilde{k}(z, c)dz$, $\tilde{h}(z, c)dz$ are connected by the relations:*

$$\begin{aligned} \frac{d}{dz} \left(\frac{P_{c_1} - P_{c_0}}{2} \right) dz &= \tilde{k}(z, c) dz \\ \frac{d}{dz} \left(\frac{P_{c_1} + P_{c_0}}{2} \right) dz &= \tilde{h}(z, c) dz . \end{aligned}$$

Proof. $(\tilde{k}(z, c)/2\pi)dz$ is known to minimize the functional

$$\| |a'(z)dz|^2 + 2Re(a(\zeta_2) - a(\zeta_1))$$

among all $da = a'(z)dz$ in Γ_{ae} .

But $P_{c_0} - P_{c_1} = Q_c$ minimizes

$$D(U) + 4\pi Re(U(\zeta_1) - U(\zeta_2)) \text{ over the class of analytic}$$

functions U on W .

The last expression can be written as:

$$\frac{1}{2} [\|dU\|^2 - 8\pi \operatorname{Re}(U(\zeta_2) - U(\zeta_1))] .$$

By comparison of the two functionals it follows that

$$\frac{d}{dz} \left(\frac{P_{c_1} - P_{c_0}}{2} \right) dz = \tilde{k}(z, c) dz .$$

To get the second relation mentioned in the theorem, one can either follow the method outlined in Theorem 1C, Ch III, second proof, where α is now a path around the cut c in some parametric disk containing ζ_1 and ζ_2 , or notice that $\tilde{h}(z, c)dz$ and $(d/dz)((P_c + P_{c_0})/2)dz$ minimize the functional:

$$B(\alpha) - \operatorname{Re} \int_c \alpha' dz - ds$$

over the class $\Gamma_{\alpha c}(c)$ of differential of functions α , analytic on W cut along c , except for the singularity $s = \log((z - \zeta_1)/(z - \zeta_2))$.

II. We prove next a theorem establishing a relation between the derivatives of P_{c_0} and P_{c_1} , and the principal functions P_0 and P_1 .

THEOREM. *Let P_{c_0} and P_{c_1} be the principal functions on W cut along c corresponding to the singularity $\log((z - \zeta_1)/(z - \zeta_2))$. If we denote by $P_0(z, \zeta)$, $P_1(z, \zeta)$ the principal functions on W corresponding to the singularity $1/(z - \zeta)$ with $P_0 = p_0 + ip_0^*$, $P_1 = p_1 + ip_1^*$ then*

$$\begin{aligned} \frac{d}{dz} P_{c_1} &= p_1(\zeta_2, z) - p_1(\zeta_1, z) + i(p_0^*(\zeta_2, z) - p_0^*(\zeta_1, z)) \\ \frac{d}{dz} P_{c_0} &= i(p_1^*(\zeta_2, z) - p_1^*(\zeta_1, z)) + p_0(\zeta_2, z) - p_0(\zeta_1, z) . \end{aligned}$$

Proof. We recall the equalities:

$$\begin{aligned} \tilde{k}(z, c) &= \int_c \tilde{k}_0(z, \zeta) \bar{d}\zeta \\ \tilde{h}(z, c) &= \int_c \tilde{h}_0(z, \zeta) d\zeta , \end{aligned}$$

and the symmetry relations:

$$\tilde{k}_0(z, \zeta) = \overline{\tilde{k}_0(\zeta, z)} \quad \tilde{h}_0(z, \zeta) = \tilde{h}_0(\zeta, z) .$$

Therefore:

$$\begin{aligned} \frac{d}{dz}(P_{c_1} - P_{c_0}) &= \int_c \frac{d}{dz} [P_1(z, \zeta) - P_0(z, \zeta)] d\bar{\zeta} \\ &= \int_c \frac{d}{d\zeta} [P_1(\zeta, z) - P_0(\zeta, z)] d\zeta \\ &= P_1(\zeta_2, z) - P_1(\zeta_1, z) - P_0(\zeta_2, z) + P_0(\zeta_1, z) \\ \frac{d}{dz}(P_{c_1} + P_{c_0}) &= \int_c \frac{d}{dz} [P_1(z, \zeta) + P_0(z, \zeta)] d\zeta \\ &= \int_c \frac{d}{d\zeta} [P_1(\zeta, z) + P_0(\zeta, z)] d\zeta \\ &= P_1(\zeta_2, z) - P_1(\zeta_1, z) + P_0(\zeta_2, z) - P_0(\zeta_1, z) \end{aligned}$$

by addition and subtraction we obtain

$$\begin{aligned} \frac{dP_{c_1}}{dz} &= p_1(\zeta_2, z) - p_1(\zeta_1, z) + ip_0^*(\zeta_2, z) - ip_0^*(\zeta_1, z) \\ \frac{dP_{c_0}}{dz} &= ip_1^*(\zeta_2, z) - ip_1^*(\zeta_1, z) + p_0(\zeta_2, z) - p_0(\zeta_1, z) . \end{aligned}$$

2. Reproducing and Orthogonal Kernels for Γ_a .

2A. We shall now again restrict ourselves to the compact bordered case, and investigate the relationship between the principal functions and Green's and Neumann functions [Schiffer [5], Bergman [2]].

Let \bar{W} be a planar compact bordered Riemann surface with k contours $\beta_i, i = 1 \dots k$. We recall the definition of the Green's function $g(z, \zeta)$ for \bar{W} .

DEFINITION. The Green's function $g(z, \zeta)$ for \bar{W} is defined in the following way:

- (a) $g(z, \zeta)$ is harmonic on \bar{W} except at $z = \zeta$
- (b) $g(z, \zeta) + \log |z - \zeta|$ is harmonic at $z = \zeta$
- (c) $g(z, \zeta) = 0$ on $\beta = \bigcup_{i=1}^k \beta_i$.

The function $g(z, \zeta)$ possesses a harmonic conjugate $g^*(z, \zeta)$ and one may construct the function

$$G(z, \zeta) = g + ig^*$$

G is harmonic in ζ , analytic in z ; it has a logarithmic pole with coefficient 1 at ζ and is determined up to an additive constant. It is not single valued.

2B. We introduce the harmonic measure of a boundary component β_i .

DEFINITION. The harmonic measure of the boundary component β_i with respect to W is the harmonic function $\omega_i(z)$ taking the following boundary values:

$$\begin{aligned} \omega_i(z) &= 0 && \text{on } \beta_j \quad j \neq i \\ \omega_i(z) &= 1 && \text{on } \beta_i . \end{aligned}$$

The period of $G(z, \zeta)$ along β_i is easily seen to be $2\pi i \omega_i(\zeta)$.

We may associate to the harmonic measure an analytic function:

$$w_i(z) = \omega_i(z) + i\omega_i^*(z) .$$

The period of $w_i(z)$ around β_j will be denoted by $-2\pi i P_{ji}$. It is easy to show that:

1. $P_{ij} = P_{ji}$.
2. The $k \times k$ matrix (P_{ij}) is positive semi-definite.
3. The $(k - 1) \times (k - 1)$ matrix $(P_{ij})_{1,2,\dots,k-1}$ is positive definite.

2C. Let us now introduce the function

$$F(z, \zeta) = \exp(-G(z, \zeta))$$

$F(z, \zeta)$ is not in general single-valued. To get a single valued function we shall add to $G(z, \zeta)$ an appropriate linear combination of functions $w_i(z)$; to do so, we solve the system:

$$\sum_{j=1}^{k-1} P_{ij} u_j(\zeta) = \omega_i(\zeta) \quad (i = 1 \dots k - 1)$$

in the form

$$u_j(\zeta) = \sum_{i=1}^{k-1} \Pi_{ji} \omega_i(\zeta) \quad \text{with} \quad (\Pi_{ji})_{1,\dots,k-1} = (P_{ij})_{1,\dots,k-1}^{-1} .$$

The function $w(z, \zeta) = \sum_{j=1}^{k-1} u_j(\zeta) w_j(z)$ is analytic in z and has period

$$-2\pi i \sum_{j=1}^{k-1} P_{ij} u_j(\zeta) = -2\pi i \omega_i(\zeta) \quad \text{along } \beta_i .$$

Therefore

$$\log f(z, \zeta) = -[G(z, \zeta) + \sum_{j=1}^{k-1} u_j(\zeta) w_j(z)]$$

has period zero along $\beta_i, i = 1 \dots k - 1$. Its period along β_k is $2\pi i$.

Therefore

$$f(z, \zeta) = \exp \left\{ -[G(z, \zeta) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w_i(\zeta) w_j(z)] \right\}$$

is single valued on \bar{W} . It has a simple zero at $z = \zeta$, and from the known boundary behavior of $g(z, \zeta)$ and $w_j(z)$,

$$\log |f(z, \zeta)| = \operatorname{Re}\{\log f(z, \zeta)\} = \begin{cases} 0 & \text{on } \beta_k \\ -u_i(\zeta) & \text{on } \beta_i, i < k. \end{cases}$$

It follows, by differentiation with respect to ζ ($\zeta = \xi + i\eta$) that $\operatorname{Re}(\partial/\partial\xi) \log f(z, \zeta) = \lambda_i(\zeta)$ on β_i , where $\lambda_i(\zeta)$ is a constant and that $\Psi(z, \zeta) = -(\partial/\partial\xi) \log f(z, \zeta)$ has a constant real part on each β_i . Moreover, it has at $z = \zeta$ a simple pole with residue 1. Except for an additive constant, it coincides then with the principal function P_1 . In the same way $\Phi(z, \zeta) = -(1/i)(\partial/\partial\eta) \log f(z, \zeta)$ has a constant imaginary part on each β_i . At $z = \zeta$ it has a single pole with residue 1. Except for an additive constant it coincides then with the principal function P .

2D. We now define the Neumann function $N(z, \zeta)$ by the following properties:

- (a) $N(z, \zeta)$ is harmonic for $z \in \bar{W}$ except at the point $z = \zeta$
- (b) $N(z, \zeta) + \log |z - \zeta|$ is harmonic at $z = \zeta$
- (c) $(\partial/\partial n)N(z, \zeta) = (2\pi/L)$ for $z \in \beta$ where L is the total length of β

N is defined up to an additive constant depending on ζ . To fix N completely we require that:

(d) $\int_{\beta} N dz = 0$.

Between the derivatives of N and g the following relations hold:

$$\begin{aligned} \frac{\partial^2 N(z, \zeta)}{\partial z \partial \zeta} &= \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta} + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) w'_j(\zeta) \\ \frac{\partial^2 N(z, \zeta)}{\partial z \partial \bar{\zeta}} &= -\frac{\partial^2 g(z, \zeta)}{\partial z \partial \bar{\zeta}} - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) \overline{w'_j(\zeta)}. \end{aligned}$$

We now fix our attention on the function

$$m(z, \zeta) = -\log |f(z, \zeta)| = -\frac{1}{2} \log f(z, \zeta) - \frac{1}{2} \log \overline{f(z, \zeta)}$$

which is harmonic in z and constant on each β_i . It has a logarithmic pole with coefficient -1 .

It is a consequence of the definition that

$$m(z, \zeta) = g(z, \zeta) + \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w_i(z) w_j(\zeta).$$

Therefore:

$$\frac{\partial^2 N}{\partial z \partial \zeta} = \frac{\partial^2 m}{\partial z \partial \zeta}$$

$$\frac{\partial^2 \overline{N}}{\partial z \partial \overline{\zeta}} = \frac{\partial^2 \overline{m}}{\partial z \partial \overline{\zeta}}$$

2E. We shall consider reproducing and orthogonal kernels for the space Γ_a on \overline{W} . In terms of the inner product we used for Γ_a , the expressions for the Bergman Kernel corresponding to Γ_a are

$$K(z, \overline{\zeta})dz = -\frac{1}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta} dz$$

$$L(z, \zeta)dz = -\frac{1}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta} dz .$$

From the properties of the Green's function it can easily be derived that for $a(z)dz \in \Gamma_a$

$$(a(z)dz, K(z, \overline{\zeta})dz) = a(\zeta)$$

$$(a(z)dz, L(z, \zeta)dz) = 0$$

$K(z, \overline{\zeta})dz$ is then the reproducing kernel for Γ_a , $L(z, \zeta)dz$ is then the orthogonal kernel for Γ_a corresponding to the singularity $dz/2\pi(z - \zeta)^2$. It is known that

$$L(z, \zeta) = L(\zeta, z) , \quad K(z, \overline{\zeta}) = \overline{K(\zeta, \overline{z})}$$

and for $z \in \beta, \zeta \in \overline{W}$

$$L(z, \zeta)dz = -\overline{K(z, \overline{\zeta})}dz .$$

2E. For Γ_{a_e} one defines in a similar way

$$K_0(z, \overline{\zeta})dz = \frac{1}{\pi} \frac{\partial^2 N(z, \zeta)}{\partial z \partial \overline{\zeta}}$$

$$L_0(z, \zeta)dz = -\frac{1}{\pi} \frac{\partial^2 N(z, \zeta)}{\partial z \partial \zeta}$$

which, from the properties of the Neumann functions can be shown to be the corresponding reproducing and orthogonal kernels.

We recall that we had, up to additive constants,

$$P_1 = -\frac{\partial}{\partial \xi} \log f(z, \zeta)$$

$$P_0 = -\frac{1}{i} \frac{\partial}{\partial \eta} \log f(z, \zeta) .$$

Therefore

$$\frac{P_1 + P_0}{2} = -\frac{\partial}{\partial \zeta} \log f(z, \zeta)$$

$$\frac{P_1 - P_0}{2} = -\frac{\partial}{\partial \bar{\zeta}} \log f(z, \zeta) .$$

If we now take the derivatives with respect to z :

$$\begin{aligned} \frac{\partial}{\partial z} \left(\frac{P_1 + P_0}{2} \right) dz &= -\frac{\partial^2}{\partial z \partial \zeta} \log f(z, \zeta) dz = \frac{\partial^2 m(z, \zeta)}{\partial z \partial \zeta} dz \\ &= \frac{\partial^2 N}{\partial z \partial \zeta} dz = -\pi L_0(z, \zeta) dz \\ \frac{\partial}{\partial z} \left(\frac{P_1 - P_0}{2} \right) dz &= -\frac{\partial^2}{\partial z \partial \bar{\zeta}} \log f(z, \zeta) dz = \frac{\partial^2 m(z, \zeta)}{\partial z \partial \bar{\zeta}} dz \\ &= \frac{\partial^2 N}{\partial z \partial \bar{\zeta}} dz = \pi K_0(z, \bar{\zeta}) dz . \end{aligned}$$

But

$$\frac{\partial^2 N}{\partial z \partial \zeta} = \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta} + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) w'_j(\zeta)$$

$$\frac{\partial^2 N}{\partial z \partial \bar{\zeta}} = -\frac{\partial^2 g(z, \zeta)}{\partial z \partial \bar{\zeta}} - \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) \overline{w'_j(\zeta)}$$

and

$$K(z, \bar{\zeta}) dz = -\frac{1}{\pi} \frac{\partial^2 g}{\partial z \partial \bar{\zeta}} dz$$

$$L(z, \zeta) dz = -\frac{1}{\pi} \frac{\partial^2 g}{\partial z \partial \zeta} dz .$$

Consequently:

$$\begin{aligned} \frac{\partial}{\partial z} \left(\frac{P_1 + P_0}{2} \right) dz &= -\pi L(z, \zeta) dz + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) w'_j(\zeta) dz \\ \frac{\partial}{\partial z} \left(\frac{P_1 - P_0}{2} \right) dz &= -\pi K(z, \bar{\zeta}) dz + \frac{1}{4} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} w'_i(z) \overline{w'_j(\zeta)} dz . \end{aligned}$$

2G. We now shall study in a more detailed way the orthogonal complement of Γ_{ae} . We recall that

$$\Gamma_a = \Gamma_{ase} \dot{+} \Gamma_{am} .$$

In our planar case, $\Gamma_{ase} = \Gamma_{ae}$, and Γ_a has then the decomposition

$$\Gamma_a = \Gamma_{ae} \dot{+} \Gamma_{am} .$$

2H. The following lemma is useful for the study of reproducing

kernels.

LEMMA. *If $\Gamma_a = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$, if there exists a reproducing kernel $k dz$ for Γ_a , and if $k dz$ has the decomposition*

$$k dz = \sum_{i=1}^n k_i dz \quad \text{where } k_i dz \in \Gamma_i$$

then $k_i dz$ is the reproducing kernel for Γ_i .

Proof. Let $a_i dz \in \Gamma_i$. By the reproducing property of $k dz$ on Γ_a , $(a_i dz, k dz) = a_i(\zeta)$. But

$$\begin{aligned} (a_i dz, k dz) &= \left(a_i dz, \sum_{j=1}^n k_j dz \right) = \sum_{j=1}^n (a_i dz, k_j dz) \\ &= (a_i dz, k_i dz) \end{aligned}$$

because

$$(a_i dz, k_j dz) = 0 \quad \text{for } i \neq j.$$

This proves

$$(a_i dz, k_i dz) = a_i(\zeta).$$

2I. We shall prove a somewhat related proposition for the orthogonal kernel.

LEMMA. *Let $\Gamma_a = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$.*

Let sdz be a singular differential which is in Γ_k , $1 \leq k \leq n$ except for some analytic singularity θ .

Let us suppose there exists an orthogonal kernel hdz for Γ_a corresponding to the singularity θ .

If $hdz - sdz = \sum_{i=1}^n h_i dz$.

Then $sdz + h_k dz$ is the orthogonal kernel for Γ_k .

Proof. Let $a_k dz \in \Gamma_k$. hdz is orthogonal to Γ_a hence:

$$(a_k dz, hdz) = 0 = \left(a_k dz, sdz + \sum_{i=1}^n h_i dz \right) = (a_k dz, sdz + h_k dz).$$

But $sdz + h_k dz \in \Gamma_k$ except for the singularity, therefore

$$sdz + h_k dz$$

is the orthogonal kernel for Γ_k corresponding to the singularity θ .

Conversely, let us suppose we know the orthogonal kernel $sdz + h_k dz$ for Γ_k , we can extend it to an orthogonal kernel for Γ_a . Let

consequently $\Gamma_a = \Gamma_k + \Gamma_l$, and let us suppose there exists a *finite* basis for Γ_l . By the process of orthonormalization one can get an orthonormal basis

$$e_1 dz, e_2 dz, \dots, e_m dz \text{ for } \Gamma_l .$$

Any element of Γ_a has the following expression:

$$adz = a_k dz + \sum_{i=1}^m \mu_i e_i dz .$$

Let us suppose that the corresponding orthogonal kernel for Γ_a is

$$hdz = sdz + h_k dz + \sum_{i=1}^m \nu_i e_i dz .$$

Then

$$\begin{aligned} (adz, hdz) = 0 &= \left(a_k dz + \sum_{i=1}^m \mu_i e_i dz, sdz + h_k dz + \sum_{i=1}^m \nu_i e_i dz \right) \\ &= (a_k dz, sdz + h_k dz) + \left(a_k dz, \sum_{i=1}^m \nu_i e_i dz \right) \\ &\quad + \left(\sum_{i=1}^m \mu_i e_i dz, sdz + h_k dz \right) + \left(\sum_{i=1}^m \mu_i e_i dz, \sum_{i=1}^m \nu_i e_i dz \right) \\ &= \sum_{i=1}^m \mu_i \left[(e_i dz, sdz + h_k dz) + \left(e_i dz, \sum_{i=1}^m \nu_i e_i dz \right) \right] \\ &= \sum_{i=1}^m \mu_i [(e_i dz, sdz) + \nu_i (e_i dz, e_i dz)] . \end{aligned}$$

This has to be true for all μ_i , hence:

$$\nu_i = -(e_i dz, sdz) .$$

2J. We now shall apply the two preceding lemmas to the construction of the reproducing kernels for the orthogonal complement of Γ_{ae} in Γ_a , namely Γ_{am} , and of the orthogonal kernel for Γ_a corresponding to the singularity $dz/(z - \zeta)^2$. The method has an obvious extension to the construction of m kernels and of orthogonal kernels corresponding to $dz/(z - \zeta)^{m+2}$, $m > 0$.

2K. First let us construct the reproducing kernel $\hat{k}_0(z, \zeta) dz$ for Γ_{am} . The analytic measures for the $k - 1$ boundary contours β_i form a basis for Γ_{am} . Let $w'_i(z) dz$ be the corresponding differentials. We wish to construct a linear combination

$$\sum_{i=1}^{k-1} a_i(\zeta) w'_i(z) dz$$

which possesses the reproducing property on the elements of the basis:

$$\left(w'_j(z)dz, \sum_{i=1}^{k-1} a_i(\xi)w'_i(z)dz \right) = w'_j(\xi) \quad (j = 1 \cdots k)$$

or

$$\sum_{i=1}^{k-1} \overline{a_i(\xi)}(w'_j dz, w'_i dz) = w'_j(\xi) .$$

We have to compute

$$\begin{aligned} (w'_j dz, w'_i dz) &= (d\omega_j + id\omega_j^*, d\omega_i + id\omega_i^*) = 2(d\omega_j, d\omega_i) \\ &= 2 \int_{\bar{w}} d\omega_j d\omega_i^* = 2 \int_{\beta} \omega_j d\omega_i^* \\ &= -2 \int_{\beta} \omega_j \frac{\partial \omega_i}{\partial n} ds = -2 \int_{\beta_j} \frac{\partial \omega_i}{\partial n} ds = 4\pi P_{ij} . \end{aligned}$$

Therefore:

$$4\pi \sum_{i=1}^{k-1} \overline{a_i(\xi)} P_{ij} = w'_j(\xi)$$

and

$$\overline{a_i(\xi)} = \frac{1}{4\pi} \sum_{j=1}^{k-1} \Pi_{ij} w'_j(\xi)$$

which shows that the reproducing kernel $\hat{k}_0(z, \xi)dz$ for Γ_{am} is:

$$\hat{k}_0(z, \xi)dz = \frac{1}{4\pi} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} \overline{w'_j(\xi)} w'_i(z)dz .$$

2L. To find now the orthogonal kernel $\hat{h}(z, \xi)dz$ corresponding to the singularity $dz/(z - \xi)^2$ we look for an expression

$$\hat{h}(z, \xi)dz = \frac{dz}{(z - \xi)^2} + \sum_{i=1}^{k-1} b_i(\xi)w'_i(z)dz .$$

By definition, for $w'_j(z)dz \in \Gamma_{am}$

$$\left(w'_j(z)dz, \frac{dz}{(z - \xi)^2} + \sum_{i=1}^{k-1} b_i(\xi)w'_i(z)dz \right) = 0$$

or

$$\sum_{i=1}^{k-1} b_i(w'_j(z)dz, w'_i(z)dz) = - \left(w'_j(z)dz, \frac{dz}{(z - \xi)^2} \right) .$$

Now

$$\left(w'_j(z)dz, \frac{dz}{(z - \xi)^2} \right) = -\frac{1}{i} \int_{\bar{w}} \frac{w'_j(z)}{(\bar{z} - \bar{\xi})^2} dz \bar{d}\bar{z} .$$

Let α be the boundary of a circle of radius r around $z = \zeta$. Then

$$\begin{aligned} \left(w'_j dz, \frac{dz}{(z - \zeta)^2} \right) &= \lim_{r \rightarrow 0} \frac{1}{i} \int_{\beta - \alpha} \frac{w'_j(z) dz}{(\bar{z} - \bar{\zeta})} \\ &= \frac{1}{i} \int_{\beta} \frac{w'_j(z) dz}{(\bar{z} - \bar{\zeta})} - \lim_{r \rightarrow 0} \int_{\alpha} \frac{w'_j(z) dz}{(\bar{z} - \bar{\zeta})}. \end{aligned}$$

The last integral has limit zero, because on α , $\bar{z} - \bar{\zeta} = r^2/(z - \zeta)$. On β however, $w'_j(z)$ is imaginary. It follows then that on β

$$w'_j(z) dz = -\overline{w'_j(z) dz}.$$

Therefore

$$\left(w'_j dz, \frac{dz}{(z - \zeta)^2} \right) = -\frac{1}{i} \int_{\beta} \frac{\overline{w'_j(z) dz}}{(\bar{z} - \bar{\zeta})} = -i \int_{\beta} \frac{w'_j(z) dz}{(z - \zeta)}.$$

The integral can be transferred to α hence

$$\left(w'_j dz, \frac{dz}{(z - \zeta)^2} \right) = -i \int_{\alpha} \frac{w'_j(z) dz}{z - \zeta} = 2\pi \overline{w'_j(\zeta)}.$$

We find then that:

$$\sum_{i=1}^{k-1} b_i P_{ij} 4\pi = -2\pi \overline{w'_j(\zeta)}$$

or

$$b_i = -\frac{1}{2} \sum_{j=1}^{k-1} \Pi_{ij} \overline{w'_j(\zeta)}.$$

The orthogonal kernel has then the following expression:

$$\hat{h}_0(z, \zeta) dz = \frac{dz}{(z - \zeta)^2} - \frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} \Pi_{ij} \overline{w'_i(\zeta)} w'_j(z) dz.$$

The results obtained in this section may be extended to open Riemann surfaces, following the method which will be outlined in the next chapter.

Chapter IV. RIEMANN SURFACES OF NON-ZERO GENUS

1. Kernels For Γ_{ae} . We shall extend in this paragraph our results about reproducing and orthogonal kernels to non-planar open Riemann surfaces. We shall first construct the reproducing kernel for Γ_{ae} .

1A. We recall the orthogonal decomposition:

$$\Gamma_a = \Gamma_{ae} + \Gamma_{as}.$$

On an arbitrary surface W , the principal functions p_0 and p_1 have vanishing flux. Therefore the differentials $\partial p_0/\partial z$ and $\partial p_1/\partial z$ are analytic semi-exact. It follows that $(\partial/\partial z)((p_1 - p_0)/2)dz \in \Gamma_{ase}$. Therefore $-(1/\pi)(\partial/dz)((p_1 - p_0)/2)dz$ has a unique decomposition:

$$-\frac{1}{\pi} \frac{\partial (p_1 - p_0)}{\partial z} \frac{dz}{2} = \omega_e + \omega_s, \quad \text{where } \omega_e \in \Gamma_{ae}, \omega_s \in \Gamma_{as}.$$

Let us first consider a compact bordered surface \bar{W} . The differential ω_s has the same periods as $-(1/\pi)(\partial/\partial z)((p_1 - p_0)/2)dz$. The analytic Schottky differential ω_s may be written as:

$$\omega_s = \phi_1 + i\phi_2$$

where ϕ_1 and ϕ_2 are analytic Schottky differentials which are real on the boundary β of W . We shall prove the following result:

THEOREM. *Let p_1 and p_0 be the principal functions on \bar{W} corresponding to the singularity $1/(z - \zeta)$. Then*

$$\omega_e = -\frac{1}{\pi} \frac{\partial (p_1 - p_0)}{\partial z} \frac{dz}{2} - \omega_s$$

is the reproducing kernel for $\Gamma_{ae}(\bar{W})$.

Proof. Let $a'(z)dz \in \Gamma_{ae}$. We compute the inner product $(a'(z)dz, \omega_e)$:

$$\begin{aligned} (a'(z)dz, \omega_e) &= i \int_{\bar{W}} a'(z)dz \overline{\omega_e} \\ &= i \int_{\beta} a(z) \left[-\frac{1}{\pi} \frac{\partial \overline{(p_1 - p_0)}}{\partial z} \frac{dz}{2} - \overline{(\phi_1 + i\phi_2)} \right] \\ &= i \int_{\beta} a(z) \left[\frac{1}{\pi} \frac{\partial (p_1 + p_0)}{\partial z} \frac{dz}{2} - (\phi_1 - i\phi_2) \right]. \end{aligned}$$

The integral may be transferred to the boundary α of a circle of radius r around $z = \zeta$. We get:

$$(a'(z)dz, \omega_e) = i \int_{\alpha} a \left[\frac{1}{\pi} \frac{\partial (p_1 + p_0)}{\partial z} \frac{dz}{2} - (\phi_1 - i\phi_2) \right].$$

But in a neighborhood of $z = \zeta$,

$$\frac{\partial (p_1 + p_0)}{\partial z} \frac{dz}{2} = \frac{-dz}{2(z - \zeta)^2} + b(z)dz$$

where $b(z)dz$ is analytic. Therefore

$$(a'(z)dz, \omega_e) = -i \int_{\alpha} \frac{a(z)dz}{2\pi(z - \zeta)^2} + i \int_{\alpha} a(z) \frac{b(z)}{2\pi} dz - i \int_{\alpha} a(z)(\phi_1 - i\phi_2).$$

When r tends to zero the last two integrals tend to zero. The first has value $a'(\zeta)$. Therefore

$$(a'(z)dz, \omega_e) = a'(\zeta) ,$$

which completes the proof of the theorem.

1B. We describe now a procedure to construct ω_e . We suppose that $-(1/\pi)(\partial/\partial z)((p_1 - p_0)/2)dz$ is known; in particular its periods can be computed. The differential ω_s is an analytic Schottky differential on \bar{W} . It can be extended to the double \hat{W} of \bar{W} . The double \hat{W} is a closed surface, on which one can construct a unique analytic Schottky differential ω'_s with the same periods as $-(1/\pi)(\partial/\partial z)((p_1 - p_0)/2)dz$. The expression $-(1/\pi)(\partial/\partial z)((p_1 - p_0)/2)dz - \omega'_s$ has no period at all. It is analytic exact, and by the uniqueness of the decomposition $\omega'_s = \omega_s$. Knowing ω_s , we get $\omega_e = -(1/\pi)((p_1 - p_0)/2)dz - \omega_s$, the reproducing kernel for $\Gamma_{ae}(\bar{W})$.

1C. An analogous procedure can be applied to obtain orthogonal kernels with singularity $dz/(z - \zeta)^{m+2}$, $m \geq 0$. We shall show, for instance, how to get the kernel corresponding to $dz/(z - \zeta)^2$.

Accordingly, let us suppose there exists a differential sdz analytic exact except for the singularity $dz/(z - \zeta)^2$ on \bar{W} . We shall use the following decomposition for $-2(\partial/\partial z)(p_1 + p_0)/2dz$:

$$-2 \frac{\partial}{\partial z} \frac{(p_1 + p_0)}{2} dz - sdz = \omega_{1e} + \omega_{1s} \text{ with } \omega_{1e} \in \Gamma_{ae}(\bar{W})$$

$$\omega_{1s} \in \Gamma_{as}(\bar{W}) .$$

Let $a'dz \in \Gamma_{ae}(\bar{W})$, $\omega_{1s} = \psi_1 + i\psi_2$ where ψ_1 and ψ_2 are analytic Schottky differentials on \bar{W} , which are real on β . We show that $-2(\partial/\partial z)((p_1 + p_0)/2)dz - \omega_{1s}$ is the orthogonal kernel for \bar{W} corresponding to the singularity $dz/(z - \zeta)^2$. Thus:

$$\begin{aligned} & \left(a'dz, -2 \frac{\partial}{\partial z} \frac{(p_1 + p_0)}{2} dz - \omega_{1s} \right) \\ &= \lim_{r \rightarrow 0} i \int_{\beta - \alpha} a \left[-2 \frac{\partial}{\partial z} \frac{(p_1 + p_0)}{2} dz - \overline{(\psi_1 + i\psi_2)} \right] \\ &= i \int_{\beta} a \left[2 \frac{\partial}{\partial z} \frac{(p_1 - p_0)}{2} dz - (\psi_1 - i\psi_2) \right] \\ &\quad - \lim_{r \rightarrow 0} \int_{\alpha} -a \left[\frac{\partial}{\partial z} \frac{(p_1 + p_0)}{2} dz + \overline{(\psi_1 + i\psi_2)} \right] . \end{aligned}$$

Clearly, the last integral tends to zero when $r \rightarrow 0$. The first integral may be written as an inner product and we get:

$$\begin{aligned} & \left(a'dz, -2\frac{\partial}{\partial z} \frac{(p_1 + p_0)}{2} - \omega_{1s} \right) \\ &= \left(a'dz, 2\frac{\partial}{\partial z} \frac{(p_1 - p_0)}{2} dz - (\psi_1 - i\psi_2) \right) = 0. \end{aligned}$$

To actually construct the kernel, it suffices to compute the periods of $-2(\partial/\partial z)(p_1 + p_0)/2$ and obtain, as in the case of the reproducing kernel, the analytic Schottky differential which exhibits the same periods on \bar{W} .

1D. We wish now to extend the preceding results to the case of open Riemann surfaces. Let W be an arbitrary Riemann surface, $\{\Omega_n\}$ a canonical exhaustion of W . Each Ω_n is of finite genus and has a compact bordered closure.

Let $\Gamma_{ae}(\Omega_n)$ be the space of analytic differentials on Ω_n ; we call $k_{0\Omega_n}(z, \zeta)dz$ the reproducing kernel for $\Gamma_{ae}(\Omega_n)$, $h_{m\Omega_n}(z, \zeta)dz$ the orthogonal kernel for $\Gamma_{ae}(\Omega_n)$ corresponding to the singularity $dz/(z - \zeta)^{m+2}$, $m \geq 0$. We prove the following proposition.

1E. THEOREM. *If W is not of class \mathcal{O}_{AD} , there exists a reproducing kernel $k_0(z, \zeta)dz$ for $\Gamma_{ae}(W)$.*

Proof. Let $\Omega \subset \Omega' \subset W$, when Ω and Ω' are elements of $\{\Omega_n\}$. Let $\zeta \in \Omega$. We recall that a reproducing kernel is nonnegative at $z = \zeta$. Then:

$$\| \check{k}_{0\Omega'} dz - \check{k}_{0\Omega} dz \|_{\Omega}^2 = \| \check{k}_{0\Omega} dz \|_{\Omega}^2 + \| \check{k}_{0\Omega'} dz \|_{\Omega}^2 - 2\check{k}_{0\Omega'}(\zeta, \zeta)$$

or

$$\begin{aligned} \| \check{k}_{0\Omega'} dz - \check{k}_{0\Omega} dz \|_{\Omega}^2 &\leq \| \check{k}_{0\Omega} dz \|_{\Omega}^2 + \| \check{k}_{0\Omega'} dz \|_{\Omega}^2 - 2\check{k}_{0\Omega'}(\zeta, \zeta) \\ &= \check{k}_{0\Omega}(\zeta, \zeta) - \check{k}_{0\Omega'}(\zeta, \zeta). \end{aligned}$$

Therefore the sequence $\{\check{k}_{0\Omega_n}(\zeta, \zeta)\}$ is a nonincreasing sequence of nonnegative numbers. It has then a limit when $n \rightarrow \infty$. If $W \notin \mathcal{O}_{AD}$, this limit cannot be zero. In fact, if there exists on W an analytic function $f(z)$ of bounded norm, then $f'(z)dz \in \Gamma_{ae}(W)$ and, by Theorem 2A, Ch. II, if $f'(\zeta) \neq 0$.

$$\left\| \frac{k_{0\Omega}(z, \zeta)dz}{k_{0\Omega}(\zeta, \zeta)} \right\|_{\Omega}^2 \leq \left\| \frac{f'(z)dz}{f'(\zeta)} \right\|_{\Omega}^2$$

or

$$\frac{1}{k_{0\Omega}(\zeta, \zeta)} \leq \left\| \frac{f'(z)dz}{f'(\zeta)} \right\|_{\Omega}^2;$$

when $\Omega \rightarrow W$, the right hand side remains bounded. Hence the left

hand side has to remain bounded too and

$$\lim_{n \rightarrow \infty} \check{k}_{0\Omega_n}(\zeta, \zeta) \neq 0 .$$

But

$$\|\check{k}_{0\Omega'} dz - \check{k}_{0\Omega} dz\|_{\Omega''}^2 \leq \|\check{k}_{0\Omega'} dz - \check{k}_{0\Omega} dz\|_{\Omega}^2 \leq \check{k}_{0\Omega}(\zeta, \zeta) - \check{k}_{0\Omega'}(\zeta, \zeta) .$$

where Ω'' is fixed and $\Omega'' \subset \Omega \subset \Omega'$.

If Ω now tends to W , then

$$\lim_{\Omega, \Omega' \rightarrow W} \|\check{k}_{0\Omega'} dz - \check{k}_{0\Omega} dz\|_{\Omega''} = 0$$

and therefore $\check{k}_{0\Omega} dz$ has a limit $\check{k}_0 dz$ when $\Omega \rightarrow W$ and that limit is attained uniformly on every compact subset $E \subset W$.

Moreover, for each $\Omega \in \{\Omega_n\}$ the following decomposition holds:

$$\check{k}_{0\Omega} dz = -\frac{\partial}{\pi \partial z} \left(\frac{p_1 - p_0}{2} \right)_{\Omega} dz - \omega_{s\Omega} .$$

The preceding equation shows that when Ω tends to W , $\omega_{s\Omega}$ has a limit such that

$$\begin{aligned} \lim_{\Omega \rightarrow W} \omega_{s\Omega} &= \lim_{\Omega \rightarrow W} (-\check{k}_{0\Omega} dz) + \lim_{\Omega \rightarrow W} \left(-\frac{\partial}{\pi \partial z} \left(\frac{p_1 - p_0}{2} \right)_{\Omega} dz \right) \\ &= -\check{k}_0 dz - \frac{\partial}{\pi \partial z} \left(\frac{p_1 - p_0}{2} \right)_{\mathcal{W}} . \end{aligned}$$

It remains to show that $\check{k}_0 dz$ has the reproducing property. For $a(z) dz \in \Gamma_{ae}(W)$ we form:

$$(a(z) dz, \check{k}_0 dz - \check{k}_{0\Omega} dz)_{\Omega} = (a(z) dz, \check{k}_0 dz)_{\Omega} - a(\zeta) .$$

By Schwarz's inequality

$$|(a(z) dz, \check{k}_0 dz)_{\Omega} - a(\zeta)|^2 \leq \|a(z) dz\|_{\Omega}^2 \|\check{k}_0 dz - \check{k}_{0\Omega} dz\|_{\Omega}^2 ;$$

when Ω tends to W :

$$\lim_{\Omega \rightarrow W} |(a(z) dz, \check{k}_0 dz)_{\Omega} - a(\zeta)| = 0 .$$

Therefore

$$(a(z) dz, \check{k}_0 dz) = a(\zeta) .$$

1F. An analogous proof can be used to show the existence of orthogonal kernels $\check{h}_{0m} dz$ on W .

1G. We prove now that if $W \in \mathcal{O}_{AD}$, the reproducing kernel $\check{k}_{0\Omega} dz$

vanishes as $\Omega \rightarrow W$. If not,

$$\|\check{k}_0 dz\|^2 = \check{k}_0(\zeta, \zeta) \neq \infty$$

and the preceding proof can be carried over. But then $\check{k}_0 dz$ is the differential of an analytic function of finite Dirichlet norm, which contradicts the fact that $W \in \mathcal{O}_{AD}$.

2. Kernels for Γ_a .

2A. We shall use the following decomposition of Γ_a :

$$\Gamma_a = \Gamma_{ae} + \Gamma_{as} .$$

We have constructed a reproducing kernel for Γ_{ae} . To get the reproducing kernel for Γ_a , it remains to find the reproducing kernel for Γ_{as} .

2B. Let $\Omega \in \{\Omega_n\}$. We consider the double $\hat{\Omega}$ of Ω . On $\hat{\Omega}$ there exists a canonical homology basis $\{A_i, B_i\}$. One may define on $\hat{\Omega}$ analytic differentials α_k with period 1 on A_k , 0 on A_h for $h \neq k$, and “symmetric” on $\hat{\Omega}$ in the sense that they are analytic Schottky differentials on Ω .

These $\alpha_k = a_k dz$ form a basis for $\Gamma_{as}(\Omega)$. They are linearly independent because their periods are linearly independent, and every $a(z)dz \in \Gamma_{as}(\Omega)$ can be expressed as a linear combination of the α_k : if we subtract from $a(z)dz$ a linear combination of the α_k such that the difference has no A_n -periods, this difference can be extended by “symmetry” to $\hat{\Omega}$, and we would have on $\hat{\Omega}$ an analytic differential without A_n -period. Such a differential is necessarily zero.

2C. Let N be the total number of α_k . N is equal to g' , the genus of $\hat{\Omega}$. It depends on the genus g and on the number of contours c of Ω : $g' = 2g + c - 1$.

We form now a linear combination

$$\sum_{i=1}^N b_i a_i(z) dz$$

which has the reproducing property on $\Gamma_{as}(\Omega)$. It suffices to show that the property is valid for the α_k , ($k = 1 \dots N$). We want

$$\left(a_k(z) dz, \sum_{i=1}^N b_i(\zeta) a_i(z) dz \right)_\Omega = a_k(\zeta)$$

or

$$\sum_{i=1}^N \bar{b}_i (a_k dz, a_i dz)_\Omega = a_k(\zeta) .$$

Because the $a_i dz$ are a basis, the matrix (t_{kl}) such that

$$(a_k dz, a_l dz)_\Omega = t_{kl}$$

is non-singular.

We now may solve the system for the b_i

$$\sum_{i=1}^N \bar{b}_i(\zeta) t_{ki} = a_k(\zeta)$$

$$\bar{b}_i(\zeta) = \sum_{k=1}^N T_{ik} a_k(\zeta) \quad \text{where} \quad (T_{ik}) = (t_{ki})^{-1}$$

and we get the reproducing kernel:

$$k_{0\Omega}^*(z, \zeta) dz = \sum_{i=1}^N \sum_{k=1}^N \overline{T_{ik} a_k(\zeta)} a_i(z) dz .$$

2D. By the method of the preceding section it is now easy to show that $k_0^* dz = \lim_{\Omega \rightarrow W} k_{0\Omega}^*(z, \zeta) dz$ exists and possesses the reproducing property. We have constructed the kernel

$$k_0(z, \zeta) dz = \hat{k}_0(z, \zeta) dz + k_0^*(z, \zeta) dz$$

which has the reproducing property on $\Gamma_a(W)$.

By an obvious modification of the preceding argument one may construct m kernels $k_m dz$, and orthogonal kernels $h_m dz$ ($m \geq 0$) for $\Gamma_a(W)$.

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