# ANALYTIC COMPOSITION KERNELS ON LIE GROUPS

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**Introduction.** In a previous paper [2], we have studied the structure of analytic distribution kernels, in rough terms, those which carry analytic functions on a manifold $G$ into analytic functions on $G$ (see definition in §6 below). We have shown that such kernels are analytic off the diagonal on $G \times G$ (the converse being not true, i.e., there exists kernels, analytic functions off the diagonal, which are not analytic distribution kernels) and we have given a condition in order that a kernel, which is an analytic function outside the diagonal, be an analytic kernel. As we pointed out, analytic kernels cannot be characterized in an analogous way as the so-called very regular kernels in the infinitely differentiable case. It has been proved ([5], tome I, 2nd edition; [2], § I, th. I) that a distribution kernel is very regular if and only if it is an infinitely differentiable function outside the diagonal.

Nevertheless, if we restrict ourselves to composition kernels (see §5 below) then those which are analytic can be characterized as being analytic functions off the diagonal.

In this paper we study the composition kernels on a general Lie group and characterize those which are analytic. In §1 to §4 we define and state some properties of the composition product of distributions on a Lie group $G$ that we shall use later on. Section 5 is devoted to define the composition kernels and to derive its main properties. In §6 we discuss and characterize the analytic composition kernels on Lie groups. We use there the results of our previous paper [2] and some techniques employed by L. Schwartz in [6], exposes 5 and 6, in the study of composition kernels on a Euclidean space, which have to be adapted to the present situation.

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**1. The composition product on a Lie group.** Let $G$ be a connected Lie group of dimension $n$. We denote by $\mathfrak{D}(G)$, $\mathscr{E}(G)$, $\mathscr{E}'(G)$ and $\mathfrak{D}'(G)$ the spaces of infinitely differentiable functions with compact support, infinitely differentiable functions, distributions with compact support, and distributions, respectively, on $G$ with their usual topologies. (Schwartz [5]).

For a fixed left invariant Haar measure $dx$ on $G$, we have the

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natural imbedding $f \in \mathcal{E}(G) \rightarrow \mu_f = fdx \in \mathfrak{D}'(G)$, where

$$\langle \mu_f, g \rangle = \int_G g(x) f(x) dx$$

for all $g \in \mathfrak{D}(G)$. We will usually denote by $f$ simply the distribution $\mu_f$.

If $S$ and $T$ are two distributions of $\mathfrak{D}'(G)$, their tensor product $S_x \otimes T_y$ is a well defined ([5], tome I, 2^e edition, pg. 106) distribution on $G \times G$. If $f \in \mathfrak{D}(G)$, the pairing

$$\langle S_x \otimes T_y, f(x \cdot y) \rangle$$

makes sense whenever the intersection of the support of $S_x \otimes T_y$ with the support of $f(x \cdot y)$ is a compact subset of $G \times G$. This is always the case when either $S$ or $T$ has compact support.

DEFINITION 1.1. *Let $S$ and $T$ be two distributions on $G$, one of them having compact support. The composition product of $S$ with $T$ will be the distribution $S*T$, defined by*

$$\langle S*T, f \rangle = \langle S_x \otimes T_y, f(x \cdot y) \rangle$$

*for all $f \in \mathfrak{D}(G)$.*

If $S$, $T$ and $R$ belong to $\mathfrak{D}'(G)$, then

$$(S*T)*R = S*(T*R)$$

is true if at least two of these distributions have compact support. The composition product on $G$ is not necessarily commutative, if $G$ is not commutative.

The following proposition is an easy consequence of [5], tome I, II^e édition, pg. 109, th IV.

PROPOSITION 1.1. *If $h$ is an infinitely differentiable function and $T$ is a distribution, one of them having compact support on $G$, then $T*h$ and $h*T$ are infinitely differentiable functions given by:*

$$(T*h)(x) = \langle T_y, h(y^{-1}x) \rangle$$

*and*

$$(h*T)(x) = \langle T_y, \Delta(y^{-1}) h(xy^{-1}) \rangle ,$$

*where $\Delta$ denotes the modular function of $G$.*

COROLLARY. *If $h$ and $g$ are infinitely differentiable functions, one of them having compact support on $G$, then we have*

$$(g*h)(x) = \int_G g(y) h(y^{-1}x)\, dy = \int_G g(xy^{-1}) h(y)\, dy$$

*and*

$$(h*g)(x) = \int_G \Delta(y^{-1}) g(y) h(xy^{-1})\, dy = \int_G \Delta(y^{-1}) g(y^{-1}x) h(y)\, dy .$$

**2. Support of the composition product.** In this section we transpose to the case of Lie groups the results of [6], exposé 5, théorème 1, Proposition 1 and corollaire.

First of all, it is easy to see that if $S$ and $T$ are two distributions such that the support of $S$ is $A$, the support of $T$ is a compact set $B$, then the support of $S*T$ is contained in $A\cdot B$, where $A\cdot B$ denotes the set all products $x\cdot y$ with $x\in A$ and $y\in B$.

If $M$ and $N$ are two subsets of $G$, let us denote by $M^{-1}N$ the set $\{x^{-1}\cdot y\colon x\in M, y\in N\}$.

Let $V$ be a compact neighborhood of $E\in G$ and $O$ an open set of $G$. We denote by $O_V$ the set

$$O_V = c(V\cdot c(O))\ ,$$

($c(A)$ will represent, in what follows, the complement of $A$ in $G$).

$O_V$ is obviously an open set, since $V$ being compact and $c(O)$ closed, $V\cdot c(O)$ is a closed set. Furthermore, $O_V$ is contained in $O$ because $c(O)\subset V\cdot c(O)$. If $V_1$ and $V_2$ are two compact neighborhoods of $e$ such that $V_1\subset V_2$ it is quite obvious that $O_{V_1}\supset O_{V_2}$.

If $V$ runs through a fundamental system of symmetric compact neighborhoods of $e$, the union of the open sets $O_V$ is $O$. In fact, if $x\in O$, it is possible to choose a symmetric compact neighborhood $V$ of $e$ such that $Vx\cap c(O)=\phi$. It follows that $x\notin V\cdot c(O)$; otherwise, $v^{-1}x=a$ with $v\in V$ and $a\in c(O)$. Since $V$ is symmetric, $v^{-1}\in V$ and then $a\in Vx$ which is a contradiction.

Let us define also

$$O'_V = c(c(O)\cdot V)\ .$$

This set has the same properties as the set $O_V$.

PROPOSITION 2.1. *If $S$ and $T$ are two distributions such that $T$ has compact support $B$ and $S$ is equal to zero on $OB^{-1}$ where $O$ is an open set, then $S*T$ is equal to zero on $O$.*

*Proof.* By the remark in the beginning of this section, the support of $S*T$ is contained in $c(OB^{-1})\cdot B$. If $x\in c(OB^{-1})$ any $y\in B$ we cannot have $xy\in O$, for then, $x\in Oy^{-1}\subset OB^{-1}$ which is a contradiction. We conclude that $c(OB^{-1})\cdot B\subset c(O)$ and $S*T$ is equal to zero on $O$, q.e.d.

COROLLARY 1. *Under the same hypothesis of the theorem, if $S$ is equal to zero on $(OB^{-1})_V$ then $S*T$ is equal to zero on $O_V$.*

*Proof.* The support of $S*T$ will be contained in the set

$$c[(OB^{-1})_V]\cdot B = V\cdot c(OB^{-1})\cdot B\ .$$

Since

$$c(OB^{-1})\cdot B \subset c(O)$$

it follows that

$$V\cdot c(OB^{-1})\cdot B \subset V\cdot c(O) = c(O_V)$$

which proves that $S*T$ is equal to zero on $O_V$, q.e.d.

Using the same method one can prove the following

COROLLARY 2. *Let $B$ be a compact and $V$ be a symmetric compact neighborhood of $e$. Let us suppose that the distribution $S$ is equal to zero on $OB^{-1}$ and that the support of the distribution $T$ is contained in $B.V$. Then, $S*T$ is equal to zero on $O'_V$.*

**3. Translations and vector fields.** Let us denote by $\sigma_s$ (resp. $\tau_s$) the left (resp. right) translation $x \to sx$ (resp. $x \to xs$) defined on $G$. If $f$ is any function defined on $G$ we define $\rho_s f$ and $\tau_s f$ to be the functions:

$$\sigma_s f(x) = f(sx) \text{ and } \tau_s f(x) = f(xs) .$$

Now if $T$ is a distribution on $G$ we define its left translation $\sigma_s T$, and its right translation, $\tau_s T$, by

$$\langle \sigma_s T, f\rangle = \langle T, \sigma_{s^{-1}} f\rangle$$

and

$$\langle \tau_s T, f\rangle = \langle T, \tau_{s^{-1}} f\rangle ,$$

If $X$ is a left invariant vector field on $G$ and $T \in \mathfrak{D}'(G)$, define $XT$ by

$$\langle XT, f\rangle = -\langle T, Xf\rangle , \quad \text{for all } f \in \mathfrak{D}(G) .$$

It is easy to show that

$$XT = T*X\delta ,$$

for all $T \in \mathfrak{D}'(G)$, where $\delta$ denotes the Dirac measure at the identity $e$ of $G$. It follows that

$$X(S*T) = S*XT .$$

Analogously, if $Y$ is a right invariant vector field we have:

$$YT = Y\delta*T \text{ and } Y(S*T) = YS*T .$$

We remark that these results can be extended to left (resp. right) invariant partial differential operators ([3]).

**4. Linear transformations which commute with the composition product.** Let $S$ be any fixed distribution of $\mathfrak{D}'(G)$ and consider the two linear continuous maps:

$$g \in \mathfrak{D}(G) \to S*g \in \mathfrak{D}'(G) \text{ and } g \in \mathfrak{D}(G) \to g*S \in \mathfrak{D}'(G)$$

We know that the distributions $S*g$ and $g*S$ belong to $\mathscr{E}(G)$ (proposition 1.1) and these two maps are continuous from $\mathfrak{D}(G)$ into $\mathscr{E}(G)$ ([5], tome II, pg. 23, th. XII). Moreover, they can be extended to continuous maps from $\mathscr{E}'(G)$ into $\mathfrak{D}'(G)$ ([5], tome II, pg. 13, th. V).

It is easy to show that the first one of the above maps (as well as its extension) commutes with the right translations, while the second (as well as its extension) commutes with left translations. More precisely, we have the following result:

PROPOSITION 4.1. *Every linear continuous map $L: \mathscr{E}'(G) \to \mathfrak{D}'(G)$ which commutes with the right translations on $G$ is of the form*

$$L(g) = S*g$$

*where $S = L(\delta)$ is a distribution of $\mathfrak{D}'(G)$.*

The proof which follows the same line as [5], tome II, th. X, is left to the reader. In the same order of ideas, one can prove the following

PROPOSITION 4.2. *A continuous linear map $L: \mathscr{E}'(G) \to \mathfrak{D}'(G)$ commutes with right translations if and only if it commutes with the left invariant vector fields on $G$.*

Analogous results can be stated for left translations and right invariant vector fields.

**5. Composition kernels.** Let $S$ be a fixed distribution in $\mathfrak{D}'(G)$. To the continuous map

$$g \in \mathfrak{D}(G) \to S*g \in \mathfrak{D}'(G)$$

there corresponds by the kernel theorem of L. Schwartz ([7]) a unique kernel $S_{x,y} \in \mathfrak{D}'(G \times G)$ called the left composition kernel associated to the distribution $S$. In the same way to the continuous map

$$g \in \mathfrak{D}(G) \to S*g \in \mathfrak{D}'(G)$$

there corresponds the so-called right composition kernel $S'_{x,y}$.

One can easily prove the following result:

PROPOSITION 5.1. *The kernels $S_{x,y}$ and $S'_{x,y}$ are given by the relations*

$$\langle S_{x,y}, \Phi(x, y)\rangle = \left\langle S_x, \int_G \Phi(xy, y)dy\right\rangle$$
$$\langle S'_{x,y}, \Phi(x, y)\rangle = \left\langle S_y, \int_G \Phi(x, xy)dx\right\rangle ,$$

*respectively, for all* $\Phi \in \mathfrak{D}(G \times G)$.

In these two formulas, the right hand side denotes the distribution on $x$ (resp. $y$) acting on the infinitely differentiable function with compact support in $x$ (resp. $y$) obtained after integrating $\Phi(xy, y)$ (resp. $\Phi(x, xy)$) with respect to the $y$(resp. $x$) variable on $G$.

Composition kernels are regular, i.e. the two mappings defined in the beginning of this section carry $\mathfrak{D}(G)$ into $\mathscr{E}(G)$ and can be extended continuously to mappings from $\mathscr{E}'(G)$ into $\mathfrak{D}'(G)$, as we have already remarked in the preceding section.

Let $s \in G$, $\sigma_s$ and $\tau_s$ denote the left and right translations, respectively on $G$. If $K_{x,y}$ is any distribution kernel on $G \times G$, let us denote by $\sigma_s K_{x,y}$ and by $\tau_s K_{x,y}$ the kenels defined by

$$\langle \sigma_s K_{x,y}, \Phi(x, y)\rangle = \langle K_{x,y}, \Phi(s^{-1}x, s^{-1}y)\rangle$$
$$\langle \tau_s K_{x,y}, \Phi(x, y)\rangle = \langle K_{x,y}, \Phi(xs^{-1}, ys^{-1})\rangle ,$$

respectively, for all $\Phi \in \mathfrak{D}(G \times G)$.

DEFINITION 5.1. *The kernels $\sigma_s K_{x,y}$ and $\tau_s K_{x,y}$ be will be called the left, respectively, right translation (parallel to the diagonal of $G \times G$) of $K_{x,y}$ by s.*

DEFINITION 5.2. *A distribution kernel $K_{x,y}$ is said to be invariant with respect to the left (resp. right) translations, if*

$$\sigma_s K_{x,y} = K_{x,y} \text{ (resp. } \tau_s K_{x,y} = K_{x,y}) .$$

*for all* $s \in G$.

PROPOSITION 5.1. *A distribution kernel $K_{x,y}$ is a left (resp. right) composition kernel if and only if it is invariant with respect to right (resp. left) translations.*

*Proof.* If $K_{x,y}$ is a left composition kernel then there exists a distribution $S \in \mathfrak{D}'(G)$ such that we have

$$\langle K_{x,y}, f(x)\cdot g(y)\rangle = \langle S*g, f\rangle$$

for all $f, g \in \mathfrak{D}(G)$. We have to show that in this case $\tau_s K_{x,y} = K_{x,y}$. This follows easily from Definition 5.1 and the definition of the translation of a distribution (Section 3).

Conversely, if $\tau_s K_{x,y} = K_{x,y}$, denoting by $L_K$ the continuous may from $\mathfrak{D}(G)$ into $\mathfrak{D}'(G)$ defined by the kernel $K_{x,y}$, one can easily see that $L_K$ commutes with the right translations. Then, by proposition 4.1 it is of the type:

$$g \in \mathfrak{D}(G) \rightarrow S*g \in \mathfrak{D}'(G)$$

where $S$ is a well defined distribution of $G$. Then $K_{x,y}$ is the left composition kernel defined by $S$, q.e.d.

Let $X$ be a left invariant vector field on $G$ and $K_{x,y}$ a distribution kernel on $G \times G$.

DEFINITION 5.3. *The kernel* $X_x(K_{x,y})$ *is defined by*

$$\langle X_x(K_{x,y}), \Phi(x, y)\rangle = -\langle K_{x,y}, X_x(\Phi(x, y))\rangle$$

*for all* $\Phi \in \mathfrak{D}(G \times G)$.

Analogously one can define the kernel $X_y(K_{x,y})$.

PROPOSITION 5.2. *A distribution kernel* $K_{x,y}$ *is a left composition kernel if and only if*

$$(X_i)_x(K_{x,y}) = -(X_i)_y(K_{x,y})$$

*where* $X_i$, $1 \leqq i \leqq n$, *denotes a basis of the left invariant vector fields on* $G$.

*Proof.* Suppose that $K_{x,y}$ is a left composition kernel. We have:

$$L_K(g) = S*g \ , \quad g \in \mathfrak{D}(G) \ .$$

From Propositions 4.1 and 4.2 we have:

$$X_i(L_K(g)) = L_K(X_i(g)) \quad \text{for all } i = 1, \cdots, n \ .$$

Thus for all $f \in \mathfrak{D}(G)$:

$$\begin{aligned}\langle X_i(L_K(g(y))), f(x)\rangle &= -\langle L_K(g(y)), X_i f(x)\rangle \\ &= -\langle K_{x,y}, X_i f(x)\cdot g(y)\rangle = \langle (X_i)_x K_{x,y}, f(x)\cdot g(y)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle L_K(X_i(g(y))), f(x)\rangle &= \langle K_{x,y}, f(x)\cdot (X_{i,y})g(y)\rangle \\ &= -\langle (X_i)_y K_{x,y}, f(x)\cdot g(y)\rangle\end{aligned}$$

Hence it follows that $(X_i)_x K_{x,y} = -(X_i)_y K_{x,y}$.

Conversely, this relation means that the map $L_K$: $\mathfrak{D}(G) \rightarrow \mathfrak{D}'(G)$ commutes with the left invariant vector fields $X_i$, $1 \leqq i \leqq n$, which again by Propositions 4.1 and 4.2 implies that $K_{x,y}$ is a left composition kernel, q.e.d.

An analogous result can be stated for right composition kernels.

**6. Analytic composition kernels.** In this section we discuss the analytic composition kernels and give the characterization of these kernels. In what follows we shall prove that the three conditions:

(a) *the distribution $S$ is an analytic function in the complement of $\{e\}$;*

(b) *the composition kernel (left or right) is an analytic function off the diagonal of $G \times G$;*

(c) *$S_{x,y}$ is analytic kernel (see below):*

are mutually equivalent.

First we recall the basic definition of [2], Section III: if $K_{x,y}$ is a kernel and $L_K$ the corresponding linear continuous mapping of $\mathfrak{D}(G)$ into $\mathfrak{D}'(G)$, then *$K_{x,y}$ is said to be an analytic kernel* if the following conditions are satisfied:

(i) *$K_{x,y}$ is very regular* ([5]),

(ii) *If $T \in \mathscr{E}'(G)$ then $L_k(T)$ is analytic on each open set on which $T$ is analytic.*

The equivalence of the three conditions above gives for the composition kernels a complete answer to a question studied in the general case in [2], Section III. It shows us that analytic composition kernels are those analytic off the diagonal, a property not true, in general, for analytic kernels.

Let us prove the equivalence between (a) and (b).

THEOREM 6.1. *The composition kernel $S_{x,y}$ is analytic in the complement of the diagonal of $G \times G$ if and only if the distribution $S$ is an analytic function in the complement of $\{e\}$.*

*Proof.* Suppose $S_{x,y}$ is an analytic function in the complement of the diagonal in $G \times G$ and let $H(x, y)$ be the restriction of $S_{x,y}$ to this complement. It is easy to check, on one hand, that $L_H(\delta) = H(x, e)$ (here, $L_H$ denotes the continuous map from $\mathfrak{D}(G)$ into $\mathfrak{D}'(G)$ defined by the distribution kernel $H(x, y)$) and, on the other hand, that $L_H(\delta) = L_S(\delta) = S$ in the complement of the set $\{e\}$. Since, by our hypothesis, $H(x, e)$ is an analytic function in the complement of $\{e\}$ the statement follows.

Conversely, suppose $S$ is analytic in the complement of $\{e\}$. Let us denote by $h(x)$ this function and by $H(x, y)$ the function $h(z \cdot y^{-1})$ which is, obviously, analytic in the complement of the diagonal in $G \times G$. We want to prove that $S_{x,y}$ coincides with $H(x, y)$ in the complement of the diagonal in $G \times G$, which proves the theorem. It suffices to show that for all $f, g \in \mathfrak{D}(G)$ with disjoint supports we have:

$$\langle S_{x,y}, f(x)\cdot g(y)\rangle = \langle H(x, y), f(x)\cdot g(y)\rangle . \tag{1}$$

Since $S_{x,y}$ is a left composition kernel, we have:

$$\langle S_{x,y}, f(x)\cdot g(y)\rangle = \langle S*g, f\rangle . \tag{2}$$

On the other hand,

$$\begin{aligned}\langle H(x, y), f(x), g(y)\rangle &= \iint_{G\times G} h(x\cdot y^{-1}) f(x)g(y)dxdy \\ &= \int_G f(x)dx \int_G h(x\cdot y^{-1})g(y)dy .\end{aligned}$$

By the corollary of Proposition 1.1, the integral in $y$ is $(h*g)(x)$, so we have:

$$\langle H(x, y), f(x)\cdot g(y)\rangle = \langle h*g, f\rangle . \tag{3}$$

Let $U$ and $V$ be two open sets such that $U$ contains the support of $g$ and $U \cap V = \phi$, which is always possible by our assumptions on $f$ and $g$. It follows, denoting by $K$ the support of $g$, that the open set $U\cdot K^{-1}$ does not contain $e$ so, in this open set, $S$ and $h$ coincide. By Proposition 2.1, we conclude that $S*g$ and $h*g$ coincide in $U$, thus we have

$$\langle S*g, f\rangle = \langle h*g, f\rangle$$

and then (2) and (3) imply (1). Q.E.D.

Before proving that conditions (b) and (c) are equivalent, we need to establish some results concerning analytic functions on Lie groups. We are going to show that in order to prove that a function $f$ is analytic at a point $x_0 \in G$, it suffices to obtain the usual bounds for the absolute value of $Z^p f$ at $x_0$, where $Z^p = Z_1^{p_1}, Z_2^{p_2} \cdots Z_n^{p_n}$, with $p_1, p_2, \cdots, p_n$ nonnegative integers, and $Z_i$, $1 \leqq i \leqq n$, denotes a basis of left (or right) invariant vector fields in $G$. More precisely:

Lemma 6.1. *Let $(U, x_1, \cdots, x_n)$ be a local coordinate system on $G$ and let $Z_i$, $1 \leqq i \leqq n$, be a basis of left (or right) invariant vector fields. A function $f$ is analytic on $U$ if and only if to each compact subset $k \subset U$ there corresponds a constant $C > 0$ such that*

$$|Z^p f(x)| \leqq C^{|p|}\,|p|!$$

*for all $x \in k$.*

The proof of Lemma 6.1 will be based on the following general lemma and its corollary below.

LEMMA 6.2. *Let $F$ be a finite set of analytic functions on $(U, x_1, \cdots, x_n)$ and let $D$ denote any one of the partial derivatives $\partial/\partial x_i, 1 \leqq i \leqq n$ on $U$. Let $g_1, g_2, \cdots, g_m$ be $m$ arbitrary functions (not necessarily distinct) chosen from the set $F$. Then, to each compact subset $k$ of $U$ there corresponds a constant $K > 0$, independent of $m$, such that*

$$(1) \qquad |D^p[g_m D(\cdots g_2 D(g_1))]| \leqq \left\{\frac{d^{|p|}}{dx^{|p|}}\left(\frac{1}{1-Kx}\frac{d}{dx}\right)^{m-1}\left(\frac{1}{1-Kx}\right)\right\}_{x=0}$$

*uniformly on $k$, for all $p = (p_1, p_2, \cdots, p_n)$. (On the right side $x$ denotes a real variable).*

*Proof.* Since $F$ is a finite set of analytic functions on $(U, x_1, \cdots, x_n)$ to each compact subset $k$ of $U$, there corresponds a constant $K > 0$ such that:

$$|D^p(g)| \leqq K^{|p|} p\,!$$

uniformly on $k$, for all $g \in F$. By remarking that:

$$\left\{\frac{d^{|p|}}{dx^{|p|}}\left(\frac{1}{1-Kx}\right)\right\}_{x=0} = K^{|p|}\cdot|p|\,!$$

and that $p! = p_1! \cdots p_n! \leqq (p_1 + \cdots + p_n)!$, we may conclude that

$$(2) \qquad |D^p(g_j)| \leqq \left\{\frac{d^{|p|}}{dx^{|p|}}\left(\frac{1}{1-Kx}\right)\right\}_{x=0}, \quad 1 \leqq j \leqq m\,,$$

uniformly on $k$.

Let us proceed by induction on $m$. The case $m = 1$ follows trivially from (2) setting $j = 1$. Suppose, then, that relation (1) is verified for $m - 1$. We can write

$$D^p[g_m D(\cdots g_2 D(g_1))] = \sum_{r+s=p} D^r(g_m)\, D^s[D(g_{m-1} D(\cdots g_2 D(g_1)))]\,.$$

Using relation (1) which is by assumption true for $m - 1$ and relation (2) and remarking that, in the last summand, the number of terms containing $D^r(g_m) = D^{r_1} \cdots D^{r_n}(g_m)$ with degree $|r| = r_1 + \cdots + r_n$ is precisely $|p|!/|r|!|s|!$ we obtain the majoration:

$$|D^p[g_m D(\cdots g_2 D(g_1))]|$$
$$\leqq \left\{\sum_{|r|+|s|=|p|} \frac{|p|!}{|r|!|s|!}\frac{d^{|r|}}{dx^{|r|}}\left(\frac{1}{1-Kx}\right)\frac{d^{|s|+1}}{dx^{|s|+1}}\right.$$

$$\cdot\Big[\Big(\frac{1}{1-Kx}\frac{d}{dx}\Big)^{m-2}\Big(\frac{1}{1-Kx}\Big)\Big]\Big\}_{x=0}$$

$$=\Big\{\sum_{|r|+|s|=|p|}\frac{|p|!}{|r|!|s|!}\frac{d^{|r|}}{dx^{|r|}}\Big(\frac{1}{1-Kx}\Big)\frac{d^{|s|}}{dx^{|s|}}$$

$$\cdot\Big[\frac{d}{dx}\Big(\Big(\frac{1}{1-Kx}\frac{d}{dx}\Big)^{m-2}\Big(\frac{1}{1-Kx}\Big)\Big)\Big\}_{x=0}$$

$$=\Big\{\frac{d^{|p|}}{dx^{|p|}}\Big(\frac{1}{1-Kx}\frac{d}{dx}\Big)^{m-1}\Big(\frac{1}{1-Kx}\Big)\Big\}_{x=0}$$

uniformly on $k$, which establishes the lemma.

COROLLARY 1. *If the functions $g_1, g_2, \cdots, g_m$ satisfy the hypothesis of Lemma 6.2, then we have*

(3) $$|g_m D(\cdots g_2 D(g_1))| \leqq (2K)^{m-1}(m-1)\,!$$

*uniformly on $k$.*

*Proof.* It follows from Lemma 6.2 that the left term of (3) is majorized by

$$\Big\{\Big(\frac{1}{1-Kx}\frac{d}{dx}\Big)^{m-1}\Big(\frac{1}{1-Kx}\Big)\Big\}_{x=0}$$

uniformly on the compact subset $k$. Thus we have to evaluate the above expression at $x=0$. By setting

$$x=\frac{1-\sqrt{1-2Ky}}{K}$$

we reduce the problem to evaluating the function

$$\frac{d^{m-1}}{dy^{m-1}}(\sqrt{1-2Ky})$$

at $y=0$. An easy calculation gives us:

$$\Big\{\frac{d^{m-1}}{dy^{m-1}}(\sqrt{1-2Ky})\Big\}_{x=0}=\frac{1.3\cdots(2m-3)}{2^{m-1}}(2K)^{m-1}\leqq(2K)^{m-1}\cdot(m-1)\,!$$

and the corollary is proved.

Without going into the proof (which would be a repetition of the arguments above) let us state a slight variant of Lemma 6.2 and corollary 1 which will be more convenient for our purposes.

LEMMA (6.2)$'$. *Let $F$ be a finite set of functions on $G$ and let $Z$ denote any one of the elements $Z_i$, $1\leqq i\leqq n$, of a basis of left (or right) invariant vector fields on $G$. Let $f$ be any function on $G$ and suppose* *that to each compact subset $k$ of $U$, there corresponds two constants $K > 0$ and $M > 0$ such that*

$$|Z^p(g)| \leqq K^{|p|}p! ,$$

*for all $g \in F$, and*

$$|Z^p(f)| \leqq MK^{|p|}\cdot p! ,$$

*both uniformly on $k$. Then if $g_1, \cdots, g_m$ are $m$ arbitrary functions (not necessarily distinct) chosen from $F$, we have, for all $p = (p_1, \cdots, p_n)$:*

$$|Z^p[g_m Z(\cdots g_1 Z(f))]| \leqq \left\{\frac{d^{|p|}}{dx^{|p|}}\left(\frac{1}{1-Kx}\frac{d}{dx}\right)^m \cdot \left(\frac{M}{1-Kx}\right)\right\}_{x=0}$$

*uniformly on $k$.*

COROLLARY 1′. *Under conditions of Lemma (6.2)′ we have*

$$|g_m Z(\cdots g_1 Z(f))| \leqq M\cdot(2K)^m\cdot m\ !$$

*uniformly on $k$.*

*Proof of Lemma* 6.1. Suppose that $f$ is an analytic function on $U$. As we know, each vactor field $Z_i$ can be represented as a linear combination with analytic coefficients in $U$ of the partial derivatives $\partial/\partial x_i$, $1 \leqq i \leqq n$:

$$Z_i = \sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j} .$$

Now, if $p = (p_1, p_2, \cdots, p_n)$ and if $m$ denotes the sum $|p| = p_1 + p_2 + \cdots, p_n$ then it is easy to see that we can represent $Z^p f$ by

$$Z^p f = \sum_{j_1,\ldots,j_m=1}^{n} a_{i_m j_m}\frac{\partial}{\partial x_{j_m}}\left(\cdots\cdots a_{i_1 j_1}\frac{\partial}{\partial x_{j_1}}(f)\right) .$$

From the corollary of Lemma 6.2, it follows that:

$$\left|a_{i_m j_m}\frac{\partial}{\partial x_{j_m}}\left(\cdots a_{i_1 j_1}\frac{\partial}{\partial x_{j_1}}(f)\right)\right| \leqq (2K)^m\cdot m\ !$$

uniformly on a given compact subset $k$ of $U$. Hence we obtain the following majoration

$$|Z^p f(x)| \leqq n^m\cdot(2K)^m m\ !$$

for all $x \in k$. Now, setting $C = 2Kn$, the "only if" part of Lemma 6.1 is proved.

Conversely, suppose that $f$ verifies condition of Lemma 6.1. To prove that $f$ is analytic on $U$, we have to state the usual bounds for the absolute value of $D^p f(x)$ on compact subsets of $U$. Since $U$ is by hypothesis a local coordinate system and $Z_i$, $1 \leqq i \leqq n$, a basis of left (or right) invariant vector fields in $G$ we can write

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} b_{ij}(x) Z_j$$

where the functions $b_{ij}(x)$ are analytic on $U$. Thus we can represent $D^p f$ by the sum

$$D^p f = \sum_{j_1, \cdots, j_m = 1}^{n} b_{i_m j_m} Z_{j_m}(\cdots b_{i_1 j_1} Z_{j_1}(f)) .$$

Since $f$ verifies condition of Lemma 6.1, then to each compact subset $k \subset U$ there corresponds a constant $C > 0$ such that

$$|Z^p f(x)| \leqq C^{|p|} |p| \,!$$

for all $x \in k$. On the other hand, the functions $b_{ij}$ being analytic on $U$ we may conclude from that we have just proved above that

$$|Z^p b_{ij}(x)| \leqq C^{|p|} |p| \,!$$

for all $x \in k$. We are then under conditions of Lemma(6.2)′ and the conclusion follows from Corollary 1′.

THEOREM 6.2. *The composition kernel $S_{x,y}$ is analytic if and only if $S$ is an analytic function in the complement of* $\{e\}$.

*Proof*. Suppose that $S_{x,y}$ is an analytic distribution kernel. As we have shown ([2], Section II, thm. 2), $S_{x,y}$ is an analytic function outside the diagonal of $G \times G$. From Theorem 6.1, it follows that $S$ is an analytic function in the complement of $\{e\}$.

Conversely, suppose that the distribution $S$ is an analytic function in the complement of $\{e\}$. As we have remarked (section 5), the kernel $S_{x,y}$ is a regular kernel, and by Theorem 6.1, it is an analytic function outside the diagonal of $G \times G$. To conclude that $S_{x,y}$ is analytically very regular, all we have to prove ([2], Section III, Theorem 3, corollary) is that for all $f \in \mathfrak{D}(G)$, $S * f$ is analytic in every open subset on which $f$ is analytic.

Suppose, then, $f$ analytic on an open set $\Omega$; all we have to prove is that if $\omega$ is any relatively compact open subset of $\Omega$ such that $\bar{\omega} \subset \Omega$, then $S * f$ is analytic on $\omega$. This subset $\omega$ being fixed, let $\alpha \in \mathfrak{D}(\Omega)$ such that $\alpha$ is equal to 1 on an open neighborhood of $\bar{\omega}$. We can write

$$f = \alpha f + (1 - \alpha) f$$

and

$$S*f = S*\alpha f + S*(1-\alpha)f .$$

Let us prove that both the summands on the right of the last expression are analytic on $\omega$. The easiest part is the analyticity of $S*(1-\alpha)f$. From our choice of $\alpha$, $(1-\alpha)f$ is an infinitely differentiable function with compact support, which is zero on $\omega$. We can derive our conclusion from the more general result:

PROPOSITION 6.1. *Suppose that $S$ is a distribution which is analytic in the complement of $\{e\}$. For all distribution $T$ with compact support in $G$, $S*T$ is analytic in the complement of the support of $T$.*

*Proof.* We may suppose without loss of generality that the support $B$ of $T$ is contained in a suitable coordinate system. Using the same argument as in [5], tome I, 2$^e$ édition, théoremè, pg. 91, XXVI, one a conclude that $T$ can be represented in the following way:

$$T = \sum_p X^p g_p ,$$

where the $g_p$'s are continuous functions with compact support contained in an arbitrary neighborhood of $B$, sey $BV$ (here $V$ denotes symmetric compact neighborhood of $e$), $X^p = X_1^{p_1}X_2^{p_2}\cdots X_n^{p_n}$, with $p_1, p_2, \cdots, p_n$ nonnegative integers and $X_1, X_2, \cdots, X_n$ a basis of the left invariant vector fields in $G$.

Let now $\beta$ be a fixed but arbitrary compact open set, which closure is contained in the complement of $B$ and suppose that $\bar{\beta}$ is contained in a suitable coordinate system. All we have to prove is that $S*T = \sum_p S*X^p g_p$ is analytic in $\beta$.

Since the support of $X^p g_p$ is contained in $BV$, by Proposition 2.1, Corollary 2, the values of each summand $S*X^p g_p$ (hence of $S*T$) on $\beta'_V$ depend only of the values of $S$ on $\beta\cdot B^{-1}$. In this open set $S$ is, by our hypothesis, analytic. Let us prove then that each summand $S*X^p g_p$ (hence $S*T$) is analytic on $\beta'_V$, from which, since $\beta$ is the union of all $\beta'_V$ when $V$ runs through a fundamental system of symmetric compact neighborhoods of $e$ in $G$, it will follow that $S*T$ will be analytic on $\beta$.

By Lemma 6.1, it suffices to state the usual bounds for the absolute values of $Y^q(S*X^p g_p)$ on compact subsets of $\beta'_V$ where, here, $Y^q = Y_1^{q_1}\cdots Y_n^{q_n}$ with $q_1, \cdots, q_n$ nonnegative integers and $Y_i, 1 \leqq i \leqq n$, a basis of the right invariant vector fields in $G$. We have, using the remarks following Proposition 3.2:

$$Y^q(S*X^p g_p) = Y^q S*X^p g_p = X^p(Y^q S*g_p) .$$

On the other hand, each $X_i$ can be written as a linear combination of $Y_j, 1 \leqq j \leqq n$, with coefficients which are analytic functions on $\beta$.

We can write $X^p = \sum_{|r| \leqq |p|} a_r(x) Y^r$ with $a_r(x)$ analytic on $\beta$, and then

$$Y^q(S*X^p g_p) = \sum_{|r| \leqq |p|} a_r(x) (Y^{q+r} S*g_p) .$$

Now, for each compact subset $k'$ of $\beta'_V$,

$$k = \{xy^{-1}: x \in k', y \in BV\}$$

is obviously, a compact subset of $\beta B^{-1}$ where $S$ is analytic. There exist, then, two constants $M$, depending on $k'$, and $N$, depending on $k$, such that:

$$|a_r(x)| \leqq M \text{ for all } x \in k' \text{ and all } r$$

and

$$|Y^{q+r} S(xy^{-1})| \leqq N^{|q+r|}(q+r)! \text{ on } k .$$

If we denote by $P$ a bound of the continuous functions $g_p$ on $BV$, we have, on $k'$, the following majoration for each of the above summands:

$$|a_r(x) (Y^{q+r} S*g_p)| \leqq P.M.N.^{|q+r|}(q+r) !$$

Thus:

$$|Y^q(S*X^p g_p)(x)| \leqq C^{|q|+|p|}(q+|p|) !$$

for all $x \in k'$, $C$ being a suitable constant which proves that $S*X^p g_p$ is analytic on $\beta'_V$ and so the proposition is proved. It follows from Proposition 6.1 that $S*(1-\alpha)f$ is analytic in $\omega$.

Let us prove now that $S*\alpha f$ is analytic on $\omega$. As before, we are going to show that the system of derivatives $X^p(S*\alpha f)$ does not grow faster than $p!C^{|p|}$ on an arbitrary compact subset of $\omega$. As we shall see some technical difficulties arise from the fact that, now, the support of $\alpha f$ intersects $\omega$. If $|p| = m$, we may write $X^p(S*\alpha f)$ as $X_{i_m} \cdots X_{i_2} X_{i_1}$ $(S*\alpha f)$. An easy induction argument gives us the following relation:

$$(1) \quad X_{i_m} \cdots X_{i_2} X_{i_1} (S*\alpha f)$$
$$= S*\alpha X_{i_m} \cdots X_{i_2} X_{i_1}(f) + \sum_{j=1}^{m} X_{i_m} \cdots X_{i_{j+1}} [S*X_{i_j}(\alpha) X_{i_{j-1}} \cdots X_{i_2} X_{i_1}(f)] .$$

First of all, let us obtain the bounds on each term of the above right summand. Writing the left invariant vector field $X_i$, $1 \leqq i \leqq n$, as a linear combination of the right invariant vector fields $Y_k$, $1 \leqq k \leqq n$, with analytic coefficients $a_{ik}(x)$ on $\omega$ and substituting on each of those terms we get:

$$(2) \quad X_{i_m} \cdots X_{i_{j+1}}[S*X_{i_j}(\alpha) X_{i_{j-1}} \cdots X_{i_2} X_{i_1}(f)]$$
$$= \sum_{k_m, \ldots, k_{j+1}=1}^{n} a_{i_m k_m} Y_{k_m}(a_{i_{m-1} k_{m-1}} \cdots (a_{i_{j+1} k_{j+1}} Y_{k_{j+1}}$$
$$\cdot [S*X_{i_j}(\alpha) X_{i_{j-1}} \cdots X_{i_2} X_{i_1}(f)]))$$

Recalling that we have chosen $\alpha \in \mathfrak{D}(\Omega)$ equal to 1 on an open neighborhood of $\bar{\omega}$, then $X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)$ has a compact support $\sigma$ disjoint from $\omega$ and so by Proposition 6.1, $S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)$ is an analytic function on $\omega$. As we have proved in Proposition 2.1, the values of $S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)$ on $\omega$ depends only on the value of $S$ on $\omega\sigma^{-1}$, thus the bounds of this composition product on a compact subset $k$ of $\omega$, amounts to bounds of $X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)$ on $\sigma$ and of $S$ on a compact subset $k'$ (corresponding to $k$) of $\omega\sigma^{-1}$. The function $f$ is analytic on $\omega$, so there exists a constant $M > 0$, depending on $k$, such that

$$|X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)| \leqq M^{j-1}(j-1)\,!$$

uniformly on $k$. On the other hand, $\alpha$ is an infinitely differentiable function with compact support, so we can find a constant $B > 0$ such that

$$\int |X_i(\alpha)|\, dx \leqq B$$

for all $1 \leqq i \leqq n$. Finally, $S$ being analytic in $\omega\sigma^{-1}$ (because $\omega$ is disjoint of $\sigma$ and $S$ is, by hypothesis, analytic on the complement of $\{e\}$), there exists a constant $k > 0$, depending on $k'$, such that

$$|Y^p(S)| \leqq K^{|p|}\cdot p\,!$$

uniformly on $k'$. From these inequalities it follows that

$$|Y^p(S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f))| \leqq BM^{j-1}(j-1)!\, K^{|p|}p\,!$$

uniformly on $k$.

Now, applying Lemma (6.2)′ and Corollary 1′ to the set of functions $F = \{a_{ij}(x), 1 \leqq i, j \leqq n\}$ and to the function $S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)$, which we can do because all the functions are analytic in $\omega$, we obtain for each term of (2) the inequality:

$$\begin{aligned} &|a_{i_m k_m} Y_{k_m}(\cdots a_{i_{j+1}k_{j+1}} Y_{k_{j+1}}[S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)]| \\ &\quad \leqq BM^{j-1}\cdot(j-1)!\,(2K)^{m-j}(m-j)\,! \end{aligned}$$

uniformly on $k$. Then we obtain for (2) the following majoration:

$$\begin{aligned} &|X_{i_m} \cdots X_{i_{j+1}}[S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)]| \\ &\quad \leqq n^{m-j}(2K)^{m-j}(m-j)!\,\cdot B\cdot M^{j-1}(j-1)\,! \end{aligned}$$

and, finally, choosing a suitable constant $C > 0$, we obtain for the summand which appear in (1), the majoration:

$$(3) \qquad \left|\sum_{j=1}^{m} X_{i_m} \cdots X_{i_{j+1}}[S * X_{i_j}(\alpha)X_{i_{j-1}} \cdots X_{i_2}X_{i_1}(f)]\right| \leqq C^m \cdot m!, \text{ on } k\,.$$

Let us consider, in the last step of our proof, the first term $S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)$ of the expression (1). The function $\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)$ has a compact support contained in the support $s$ of $\alpha$. If $V$ and $W$ are two symmetric compact neighborhoods of $e$ in $G$, such that $(\omega s^{-1})_V \subset (\omega s^{-1})_W$ and $\gamma$ is an infinitely differentiable function with compact support contained in $(\omega s^{-1})_W$ and equal to 1 on an open neighborhood of $(\omega s^{-1})_V$, the distribution $\gamma S$ coincides with $S$ on $(\omega s^{-1})_V$, hence, from Proposition 2.1, Corollary 1, it follows that $\gamma S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)$ coincides with $S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)$ on $\omega_V$.

Since the distribution $\gamma S$ has compact support, by the some remark in the beginning of the proof of Proposition 6.1, we can write:

$$\gamma S = \sum_q Y^q g_q$$

where the $g_p$'s are continuous functions with compact support contained in an arbitrary open neighborhood of the support of $\gamma S$, say $(\omega s^{-1})_W$. Then, we have:

$$\begin{aligned}\gamma S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f) &= \sum_q (Y^q g_q)*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)\\ &= \sum_q Y^q(g_q*\alpha X_{i_m}\cdots X_{i_2}X_{i1}(f))\end{aligned}$$

which coincides with $S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f)$ on $\omega_V$.

Again if we write each vector field $Y_j$, $1 \leqq j \leqq n$, as a linear combination of the basis $X_i$, $1 \leqq i \leqq n$, of left invariant vector fields, with analytic coefficients on $\omega$, we obtain:

$$Y^q = \sum_p b_{qp}(x)X^p$$

where the functions $b_{qp}(x)$ are analytic on $\omega$. Substituting $Y^q$ in the above summand, we get:

$$\gamma S*\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f) = \sum_{p,q} b_{qp}(x)\,(g_q*X^p(\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f))) \tag{4}$$

Now, remarking that:

(i) the number of the analytic functions $b_{qp}$ is finite and depends only on the above expression of $\gamma S$, and so all these functions are uniformly bounded on every compact subset of $\omega_V \subset \omega$;

(ii) by the same argument of (i), one can find a positive constant which bounds all the absolute values of the continuous functions with compact support $g_p$;

(iii) the functions $X^r\alpha$ are infinitely differentiable with compact support contained in $s$ and the number of factors $X^r\alpha$ which appear developing $X^p(\alpha X_{i_m}\cdots X_{i_2}X_{i_1}(f))$ is finite and depends only on the expression of $\gamma S$, so one can find a positive constant with bounds the absolute values of $X^r\alpha$ on $s$;

(iv) the function $f$ is analytic on $\Omega$ and $s$ is a compact subset of $\Omega$; then it is possible to find a constant $N > 0$, such that:

$$|b_{qp}(x)(g*X^p(\alpha X_{i_m} \cdots X_{i_1}(f)))| \leqq N^{m+|p|}(m + |p|)\,!$$

on a compact subset of $\omega_V$. Denoting by $r$ the maximum of the numbers $|p|$ and by $t$ the number of summands of (4), we obtain:

$$|\gamma S*\alpha X_{i_m} \cdots X_{i_2}X_{i_1}(f)| \leqq tN^{m+r}(m + r)!\,.$$

Hence, by choosing a suitable constant $C_1$, we get:

$$|\gamma S*\alpha X_{i_m} \cdots X_{i_2}X_{i_1}(f)| \leqq C_1^m \cdot m\,!$$

on a compact subset of $\omega_V$. This inequality combined with (3) shows us that $S*\alpha f$ is analytic on $\omega_V$, hence on $\omega$, and Theorem 6.2 is proved.

Theorems 6.1 and 6.2 state, for composition kernels, that the property of being analytically very regular is equivalent to that of being analytic outside the diagonal. This gives us an affirmative answer in this case to the question studied in [2] for more general analytic kernels.

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