ON THE ACTION OF SO(3) ON S^n

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1. Introduction. This paper contains some facts about the possible actions of the rotation group SO(3) on the n-sphere S^n. For some of the results the action is required to be differentiable, but for others this is not necessary.

We recall for an action of a compact Lie group, that a principal isotropy group is an isotropy group of the lowest possible dimension and that among these it is one with the fewest possible components. An orbit with such an isotropy group is called a principal orbit. For a compact Lie group acting on a cohomology manifold over Z, principal orbits form an open connected everywhere dense set. In fact the complement is a closed set of dimension at most $n-2$ [1, Chapter IX].

Two of the results to be proved are the following, where B is the set of points on orbits of dimension less than the highest dimension of any orbit: If SO(3) acts differentiably on S^n with three-dimensional principal orbits and if $\dim B < n-2$ then the principal isotropy group is the identity; if SO(3) acts differentiably on S^7, then some orbit has dimension less than 3.

Part of the motivation for our work was the attempt to discover whether the latter result is true for all n. If SO(3) does act differentiably on S^n with all orbits three-dimensional then, as far as rational coefficients are concerned, the sheaf generated by the orbits is constant and relative to these coefficients we obtain similar results to those for a fibering by S^3 (see [1, 3]); hence $n = 4k-1$. There cannot be such an action for $n=3$ and the result above shows there is none for $n=7$. The general case remains open. It is known [1, p. 187; 3] that if a compact connected Lie group G acts nontransitively on S^n with all orbits of the same dimension, then rank $G=1$, and every isotropy group is finite. There are only three such groups, the circle, SO(3), and SU(2), the simply connected covering of SO(3). The circle and SU(2) can act on S^{2k-1} and S^{4k-1} respectively with orbits of constant dimension, but no such example is known for SO(3). This suggests the question we have mentioned of whether or not SO(3) can act on S^n with every orbit 3-dimensional.

2. Cyclic isotropy groups. In this section no differentiability is assumed; in fact the space on which SO(3) acts is only required to be a generalized manifold; we assume all spaces to be strongly paracompact;

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i.e., all open sets are paracompact. We recall that principal orbits are everywhere dense and that a principal isotropy group has at least one fixed point on every orbit.

THEOREM 1. *Let $G = \mathrm{SO}(3)$ act on a space X which is a cohomology n-manifold over Z and a cohomology n-sphere over Z. If the principal isotropy group is a finite cyclic group then it must be the trivial group containing only the identity.*

Assume the theorem false. Let A be one of the principal isotropy group; it leaves at least one point fixed in every orbit. By assumption A is cyclic and nontrivial. Let p be a prime factor of the order of A and let A_1 be a cyclic subgroup of A where A_1 has order p.

We show first that p cannot be 2. Let a be an element of order 2 in G, and let $H = \{e, a, b, c\}$ be a subgroup of G isomorphic to $Z_2 \oplus Z_2$. Let T be a circle group in G, such that $A \subset T$ and $a \in T$; and let $N = T \cup bT$ be the normalizer of T.[1]

If $p = 2$, then, since all elements of order 2 of G are conjugate, and since a has a one-dimensional fixed point set in each three-dimensional orbit, we have

$$\dim F(a) = n - 2 .$$

It follows from a theorem of Borel [1, p. 175] that

$$\dim F(H) = n - 3 .$$

The subgroups of G, containing H, are:
(1) the icosahedral group I,
(2) the octahedral group C,
(3) the tetrahedral group S,
(4) the dihedral group D_{2k}, $k > 1$,
(5) $D_2 = H$,
(6) N,
(7) G.

In an orbit with any one of these as isotropy group, H has a finite number of fixed points. One concludes from known dimension relations for the singular set B [1, p. 118, 120] that H is contained in a principal isotropy group, contradicting the assumption that A is cyclic; and so $p \neq 2$. Since $A_1 \subset A$, we have $F(A) \subset F(A_1)$; by Smith theory $F(A_1)$ is a cohomology sphere over Z_p.

Let $x \in F(A_1)$, and suppose $G(x)$ is a principal orbit. Choose a slice K at x. Then A leaves all of K fixed, and also leaves TK and NK

[1] For any subset $E \subset G$ we write $F(E)$ for the set of points in X left fixed by all elements of E.

pointwise fixed. Now K is an $(n-3)$-cohomology manifold over Z; and TK, which is locally a product of K and T/A, is an $(n-2)$-cohomology manifold over Z. In a neighborhood of x the set TK coincides with $F(A)$, and also with $F(A_1)$, as one sees by considering the action in each orbit through K. It follows that $F(A_1)$ is of dimension $n-2$, and, since principal orbits are dense, that

$$F(A) = F(A_1) .$$

If x is a point of $F(A)$, and the isotropy group G_x is finite, then A is a normal subgroup of G_x [4]. Both $F(A)$ and $F(A) - F(G)$ are invariant under N.

Suppose x and gx (for some $g \in G$) are in $F(A) - F(G)$, so that

$$g^{-1}Ag \subset G_x .$$

A being odd cyclic, one finds, by considering all possible finite subgroups of G (those containing H, cf. above, the cyclic groups Z_n, and the odd dihedral groups D_{2k+1}) that

$$g^{-1}Ag = A ;$$

this means $g \in N$. Therefore $X - F(G)$ is partitioned into sets $g(FA) - F(G))$. If R is a small 2-cell in G, transversal to T, then $R\cdot[F(A) - F(G)]$ is homeomorphic to $R \times [F(A) - F(G)]$. Hence the partitioning is a fibering (in the sense of local product) of $X - F(G)$ with fiber $F(A) - F(G)$ and base $G/N = P^2$, the real projective plane. By [1, p. 120] we have $\dim F(G) \leqq n-4$, so that the fiber $F(A) - F(G)$ is connected. By [2, p. 230] we have (cohomology with closed supports on the left, compact supports on the right)

$$H^1(X - F(G); Z_2) \approx \operatorname{Hom}(H_c^{n-1}(X - F(G); Z_2), Z_2) .$$

But $H_c^{n-1}(X - F(G); Z_2) = 0$, from the exact sequence of $(X, F(G))$. The spectral sequence of the fibering is now in contradiction to the $H^1(P^2, Z_2) = Z_2$, and the theorem follows.

3. **Differentiable action with** $\dim B < n-2$. Richardson has noted [5] that the double suspension of $\mathrm{SO}(3)/I$ gives an example of SO(3) acting on a cohomology manifold over Z which is a cohomology 5-sphere over Z (and possibly is equal to S^5), with $\dim B = 1$ and with each principal orbit a Poincaré space. This example shows that differentiability is needed for Theorem 2.

THEOREM 2. *Let $G = \mathrm{SO}(3)$ act differentiably on the n-sphere S^n with principal orbits of dimension three. If the dimension of the singular set B is $< n-2$, then the principal isotropy group contains* *only the unit element.*

It follows from Theorem 1, without the restriction on B, that the principal isotropy group cannot be cyclic non-trivial. Assume Theorem 2 false. Then, since all finite subgroups of SO(3) of odd order are cyclic, the principal isotropy group must contain an element of order two; and as shown in the proof of Theorem 1, it must then contain a group isomorphic to H. It can therefore be only one of I, C, S, D_{2k}, $k > 1$ H.

For reference we list the one-dimensional integral homology of the quotients of SO(3) by its finite subgroups:

$$H_1(\mathrm{SO}(3)/Z_n) = Z_{2n} ;$$
$$H_1(\mathrm{SO}(3)/D_{2n}) = Z_2 \oplus Z_2 ;$$
$$H_1(\mathrm{SO}(3)/D_{2n+1}) = Z_4,\ H_1(\mathrm{SO}(3)/S) = Z_3 ;$$
$$H_1(\mathrm{SO}(3)/C) = Z_2 ;$$
$$H_1(\mathrm{SO}(3)/I) = 0 .$$

Case 1. The principal isotropy group is I.

In this case all orbits in $S^n - B$ are of type G/I, since I is a maximal finite subgroup of G; and on each such orbit I has exactly one fixed point. Therefore

$$S^n - B = (F(I) - B) \times G/I .$$

Since $\dim B \leqq n - 3$, we see that $U = S^n - B$ is simply connected. (Because of differentiability B can be represented as a C^1-complex in S^n. A singular 2-disc with boundary in $S^n - B$ can be deformed slightly, by simplicial approximation and shift to general position, so that it does not intersect B). Since G/I is not simply connected, this is a contradiction, and shows this case is impossible.

Case 2. The principal isotropy group is C.

The proof for this case is exactly the same as Case 1.

Case 3. The principal isotropy group is S.

The set $F(H)$ is an $(n-3)$-cohomology sphere mod 2, and $F(S) \subset F(H)$; we assume $H \subset S \subset C$. Since $F(S)$ is a manifold (because of differentiability) of dimension $n - 3$, it follows that

$$F(S) = F(H) .$$

The set $F(S) - B$ is invariant under C. Assume

$$x \in F(S) - B,\ gx \in F(S) - B .$$

Then

$$Sgx = gx,\ g^{-1}Sg \subset G_x \ .$$

In this case every isotropy group in $S^n - B$ is isomorphic either to S or to C. In either case we have

$$g^{-1}Sg = S$$

so $g \in C$. Hence $S^n - B$ is fibered by the sets $g[F(S) - B]$ with base G/C. To see that this is a fibering in the sense of a local product, note that if R is a small neighborhood of e in G, then $R \times [F(S) - B]$ is homeomorphic to $R[F(S) - B]$. All points of B are stationary under G, since all subgroups strictly between S and G are finite (of type C or I). Using the linear orthogonal behavior of G at points of B (Bochner's theorem), one sees easily that $\dim B < n - 4$; otherwise two-dimensional orbits would occur. But then the fiber $F(S) - B$ of $S^n - B$ is connected, and we get a contradiction from the homotopy sequence: $0 = \pi_1(S^n - B) \to \pi_1(G/C) \to \pi_0(F(S) - B) = 0$.

Case 4. The principal isotropy group is D_{2k}, $k > 1$.

Let Z_{2k} be the cyclic subgroup of D_{2k}. We know that $Z_2 \subset Z_{2k}$, and $F(Z_2)$ is an $(n-2)$-cohomology sphere mod 2. Now $F(Z_{2k})$ is an $(n-2)$-manifold in $F(Z_2)$ so

$$F(Z_{2k}) = F(Z_2)$$

and $F(Z_{2k})$ is a connected manifold.

The set $F(Z_{2k}) - B$ is invariant under N where N is the normalizer of Z_{2k}. Assume

$$x,\ gx \in F(Z_{2k}) - B \ .$$

Then

$$Z_{2k}gx = gx,\ g^{-1}Z_{2k}gx = x \ .$$

The isotropy group at x is either D_{2k} or D_{4k} and in either case we see that

$$g^{-1}Z_{2k}g = Z_{2k},\ g \in N \ .$$

Hence $S^n - B$ is fibered by the sets $g[F(Z_{2k}) - B]$ with base $G/N = P^2$. The set B satisfies $\dim B \leqq n - 4$. This is easily seen by considering the (linearized) action of the stability group Gy at a point $y \in B$ (cf. case 3); one should separate the cases $G_y = G$ and $G_y \approx T$ or N.

The fiber $F(Z_{2k}) - B$ is again connected, and we arrive at a contradiction as in case 3.

Case 5. The principal isotropy group is H.

The normalizer of H is C; any finite isotropy group must contain a normal subgroup isomorphic to H. Suppose

$$x,\, gx \in F(H) - B\,.$$

Then $Hgx = gx$ and $g^{-1}Hgx = x$, so that $g^{-1}Hg \subset G_x$. We use again the linear orthogonal discription of G_x. Since H and $g^{-1}Hg$ are principal isotropy groups, their fixed point sets are of dimension $n - 3$; and they act trivially perpendicularly to the orbit at x. It follows that they must coincide, since otherwise we would get too large a principal isotropy group at x. Consequently $g \in C$, and therefore $S^n - B$ is fibered by translates of $F(H) - B$, with G/C as base. (As in case 4, we have $\dim B \leqq n - 4$.) $S^n - B$ is connected. The operation of C on $F(H) - B$ must therefore permute the components of $F(H) - B$ transitively, and there are then at most 6 components (note that the subgroup H of C operates trivially). The fundamental group of G/C is of order 48, and the homotopy sequence of the fibering gives a contradiction.

This completes the proof of Theorem 2.

4. Relations between $F(a)$ and $F(T)$. As before, T is a circle group in SO(3), a the element of order 2 in T, $N = T \cup bT$ the normalizer of T.

THEOREM 3. *Let $G = \mathrm{SO}(3)$ act on S^n. If $F(a) \neq F(T)$, then $F(H) \neq \phi$ (the empty set).*

Let k be the greatest integer such that $F(Z_{2^k}) \neq F(T)$. By hypothesis $k \geqq 1$. We shall assume $F(H) = \phi$ and show that this leads to a contradiction.

Let a_1 be the generator of a cyclic group in T of order 2^k. Then $F(a_1)$ is a sphere mod 2. The group $N' = N/Z_{2^k}$ operates on $F(a_1)$, and we shall consider this action of N'.

LEMMA. *Let $x \in F(a_1) - F(T)$. Then N'_x is odd cyclic.*

Assume that G_x is cyclic. Then $N'_x = G_x/Z_{2^k}$ and hence N'_x is odd cyclic.

Assume next that $G_x = D_{2i+1}$ which is the only other possibility since $F(H) = \phi$. In this case $k = 1$. Hence the group Z_{2^k} is the group $\{e, a\}$. In this case we see that $N_x = \{e, a\}$ and that $N'_x = \{e\}$. This proves the above lemma.

We now consider the mod 2 sphere $F(a_1)$ with the action of N' and note that $H = Z_2 \oplus Z_2$ (more precisely a group in N' isomorphic to H) acts freely on

$$F(a_1) - F(T)\,.$$

There is then a spectral sequence for $H^*((F(a_1) - F(T))/H; Z_2)$ (cohomology with compact supports) whose E_2 term is

$$H^p(B_H; H^q((F(a_1) - F(T)); Z_2) .$$

We see from this sequence that H cannot act freely and this contradiction proves the theorem; note that $F(T)$ is also a sphere mod 2.

5. Action on S^7.

THEOREM 4. *Let $G = \mathrm{SO}(3)$ act differentiably on the 7-sphere S^7. Then some orbit has dimension less than three.*

Assume that we are given an action for which all orbits are three-dimensional; we shall arrive at a contradiction.

As before we let $H = \{e, a, b, c\}$ be a specific subgroup of G, isomorphic to $Z_2 \oplus Z_2$; we write T for the one-parameter subgroup of G, containing a, and N for the normalizer of T in G; here $N = T \cup bT$ and $bT = cT$.

We know (from § 3) that the principal isotropy group is e. This implies that $\dim F(a) = 3$ (if $\dim F(a) = 5$, then clearly a would have a fixed point in every orbit), and so $\dim F(H) = 1$, by Borel's theorem [1, p. 175]; of course, $F(H)$ is homeomorphic to the circle S^1. We note that $F(H) = F(a) \cap F(b)$.

The normalizer of H is the octahedral group, C; by general principles C maps $F(H)$ into itself, with H acting trivially. We have to consider separately the various ways in which the group C/H, isomorphic to the symmetric group $\mathscr{S}_3$ on three letters, can act on a circle, and to show that a contradiction arises in each case.

If x is any point of S^7, then the stability group G_x acts on the tangent space at x; we may assume this action to be orthogonal, and write K or K_x for a G_x-invariant complement to the tangent space of the orbit of x (this is the same as the tangent space to a differentiable slice).

We shall need the irreducible representations of C. There are 5 of them (see e.g. [7]); they are all real: $C^{(0)}$, the trivial representation of dimension 1; $C^{(1)}$, 1-dimensional nontrivial action "as Z_2", i.e., with the tetrahedral group S acting trivially; $C^{(2)}$, operation on the plane "as dihedral group D_3", i.e., with H acting trivially; $C^{(3)}$, the usual operation of C on 3-space; $C^{(4)}$, the conjugate operation (tensor product with $C^{(1)}$).

Case 1. Suppose C acts trivially on $F(H)$. Since C is a maximal finite subgroup, the stability group at all points of $F(H)$ is exactly C.

Let $x \in F(H)$; then in K_x there will be a one-dimensional subspace, on which C operates trivially, namely the tangent line of $F(H)$. Let L_x be a C-invariant complementary (three-dimensional) space in K_x. Then a must have a one-dimensional fixed space in L_x, since $F(a)$ has dimension 3 and a leaves already fixed a direction tangent to the orbit $G(x)$ and the direction tangent to $F(H)$. From the list of representations of C it follows, that C must act on L_x by $C^{(3)}$ or $C^{(4)}$; however $C^{(4)}$ is impossible, since no element of G_x can reverse the orientation of the tangent space to S^7 at x, G being connected. Let now a_1 be one of the two square roots of a in G; it lies in C, and we observe that the dimension of its fixed space in K_x is 2, so that $\dim F(a_1) = 3$. Since $F(a_1) \subset F(a)$, and both sets are manifolds, it follows that $F(a_1) = F(a)$. On the other hand let d be an element of order 2 in C, not in H, and let d_1 be a square root of d. Since $d_1 \cdot x \neq x$, we have $F(d_1) \neq F(d)$. But we can conjugate the pair (a, a_1) into (d, d_1) in G, and so we have a contradiction.

Case 2. C acts on $F(H)$ with exactly two fixed points ("reflection across a diameter"). Letting x be one of the two fixed points, we see that C operates by $C^{(1)}$ on the tangent line to $F(H)$ at x, and that it then must operate by $C^{(4)}$ in L_x. It follows that a_1 has no nontrivial fixed vector in K_x, so that all fixed points of a_1 near $G(x)$ are actually in $G(x)$ and form a circle. By the Smith theorem this is the full fixed point set of a_1 on S^7. But the second fixed point of C on $F(H)$ lies in another orbit (since C has only one fixed point in an orbit of type G/C); since $a_1 \in C$, we have a contradiction.

Case 3. C acts on $F(H)$ in the way the dihedral group $D_3(\approx C/H)$ acts on the unit circle in the plane, in the usual representation of D_3. This implies that the elements a_1 will map $F(H)$ into itself with exactly two fixed points, x and y, and reversing the orientation of $F(H)$.

We consider now the set $T \cdot F(H)$. Its structure is easily established; it is a 2-manifold with self-intersections: At any point z of $F(H)$ a neighborhood of the orbit $T(z)$ is homeomorphic to a fiber bundle over $T(z)$, whose fiber is the finite set of lines obtained by letting the stability group T_z operate on the tangent line to $F(H)$ at z. Except at a finite number of points z, there will be only one line in the set and the neighborhood in question is just a cylinder; i.e., product of a circle and a line. At the points x and y the element a_1 (which belongs to the stability group there) reverses the tangent line to $F(H)$, so that the neighborhood of $T(x)$ or $T(y)$ is a Möbius band with self-intersection along the middle line. One concludes easily from this description of the structure that $T \cdot F(H)$ carries a nonzero 2-cycle over Z_2, but that its second homology group over the rationals is 0. But by Alexander duality such a set cannot exist in $F(a)$ (recall that $F(a)$ is a sphere over Z_2, and therfore also over the rationals).

Case 4. C acts on $F(H)$ without fixed points (as "reflection through the center"). This implies that the stability group at any point of $F(H)$ includes the tetrahedral group S (but not C), and so equals either S or the icosahedral group I. The fixed point set $F(I)$ or I, being a manifold contained in $F(H)$, either coincides with $F(H)$ or consists of a finite number of points. The first case is impossible, since in an orbit of type G/I the group I has only one fixed point, whereas here the operation of C produces for each fixed point another one in the same orbit. In the second case we claim that there can be at most one point with stability group I on $F(H)$. Suppose x is such a point. Then in K_x the group I must operate in such a fashion that the subgroup S leaves a straight line (the tangent to $F(H)$) point wise fixed, but that I itself has no fixed line. From the list of representations of I [6, 7] one sees that I must operate by the irreducible 4-dimensional representation, obtained by letting I operate on E^5 through all even permutations of the axes and restricting to the subspace perpendicular to the main diagonal of E^5. The element α of order 5, which permutes the 5 axes cyclically, has no nonzero fixed vector. This means that the fixed points of α in S^7 in the neighborhood of the orbit $G(x)$ consist of just the fixed points in the orbit itself. which form a circle. By the Smith theorem mod 5 this must be the complete fixed point set of α in S^7, and therefore I cannot have any fixed point outside $G(x)$, i.e., different from x.

LEMMA 1. *The stability groups of points on $F(a) - F(H)$ are of type D_{2k+1} or $Z_{2(2l+1)}$.*

Proof. The only other possibility is a cyclic group of order divisible by 4 (since no subgroup isomorphic with H is allowed). The fixed point set of a_1 is a proper subset of $F(a)$, since the points of $F(H)$ are not in $F(a_1)$. By the Smith theorems mod 2 the set $F(a_1)$ must be a single circle; but in an orbit with Z_{4k} as stability group the fixed point set of Z_4 consists of two circles.

We write A for $G \cdot F(a)$, and B for $G \cdot F(H)$.

LEMMA 2. *The inclusion $F(a) \subset A$ induces a homeomorphism $F(a)/N = A/G$.*

Proof. Clearly $F(a)$ maps onto $A/G = A'$. Suppose $y = g(z)$ with with $y, z \in F(a)$. Then $a \cdot gz = gz$ or $g^{-1}ag \in G_z$. Now G_z is one of

$Z_{2(2k+1)}$, D_{2k+1}, S, I. In all cases. there exists $g' \in G_z$ with $(gg')^{-1}agg' = a$, since all elements of order two are conjugate. This implies $gg' \in N$, and $y = gz = gg'z$, q.e.d.

T operates on $F(a)$ with finite stability groups. It follows that $F(a)/T$ is a two-manifold; it is obviously orientable. Moreover, since fundamental group and first homology group map onto from $F(a)$ to $F(a)/T$, and $F(a)$ is a Z_2-sphere, $F(a)/T$ is a 2-sphere. The element b of G leaves the decomposition of $F(a)$ into T-orbits invariant, and induces an involution in $F(a)/T$; we obtain $F(a)/N$ by indentifying points under b. The image of $F(H)$ yields an S^1 in $F(a)/T$, pointwise fixed under b. It follows that $A' = F(a)/N$ is a 2-disk, whose boundary is exactly $B/G = B'$. (In other words, the N-orbits on $F(a)$ consist generally of two circles which are interchanged by b, except on $F(H)$, where the N-orbits are single circles, which are reversed in themselves by b.)

Next we compute the Z_2-cohomology of A. From the nature of the orbits in $A - B$, as described in Lemma 1, it is clear that the sheaf over $A' - B'$, formed by the Z_2-cohomology of the orbits, is constant, and that therefore (using the spectral sequence of the projection $A - B \to A' - B'$) the Z_2-cohomology (with compact supports) of $A - B$ is isomorphic to that of G/Z_2, with dimensions raised by two.

The orbits, making up the set B, have the Z_2-cohomology of the 3-sphere S^3, and the sheaf over B' formed in this fashion is constant, since it is clearly locally constant (even if there should be an orbit with I as stability group). The Z_2-cohomology of B is consequently isomorphic with that of $S^1 \times S^3$. We determine the coboundary map $d^*: H^*(B) \to H^*(A - B)$ for the pair (A, B). We have $H^1(B) \approx H^2(A - B)(\approx Z_2)$ under d^*, since these groups are obtained from the base spaces B', resp. $A' - B'$ under the projection. We claim that d^* is 0 on $H^3(B)$ and $H^4(B)$. The reason for this is that the orbits near, but not on, B (which are of type G/Z_2) are even coverings of the orbits on B (of type G/S or G/I), so that the cohomology map in dimension 3 vanishes. In more detail: We think of A' as the unit circle in the plane. Let A'_1 be a concentric circle of slightly smaller radius, with boundary B'_1; let A_1, B_1 be the inverse images under the projection. There is an obvious retraction of B_2, the closure of $A - A_1$, onto B, in fact a deformation retraction. Its restriction f to B_1 is a fiber map over the radial projection of B'_1 onto B', with the map on the fiber being the natural map from G/Z_2 to G/S, respectively to G/I. It follows from the spectral sequence that f^* is 0 in dimensions 3 and 4. The inclusions $A_1 - B_1 \subset A - B_2 \subset A - B$ clearly give isomorphisms in cohomology. It follows that d^* can be factored through f^*, which proves our claim. (Alternately one could use here the Fary spectral sequence.) The exact sequence of (A, B) shows now that the Poincaré polynomial over Z_2 of A is $1 + 2t^3 + 2t^4 + t^5$. The complement $S^7 - A$ has then Poincaré polynomial

$2t^4 + 2t^5 + t^6 + t^7$. The group H acts freely on $S^7 - A$, but now the spectral sequence of the covering leads at once to a contradiction: The direct sum of the terms of total degree 7 in E_∞ must be of rank 1. But $E_2^{3,4}$ has rank 8, whereas $E_2^{1,5}$ and $E_2^{0,6}$ (the only groups of E_2 which can contribute boundaries to $E_r^{3,4}$) are together of rank 5.

The four cases considered clearly represent all possible actions of $\mathscr{S}_3$ on the circle (up to topological equivalence), and so the proof of our theorem is finished.

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