THE SPECTRUM AND THE RADICAL IN LOCALLY m-CONVEX ALGEBRAS

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Introduction. Let A be a complex algebra. A subset V of A is said to be *idempotent* if $VV \subseteqq V$; V is said to *m-convex* (multiplicatively-convex) if V is convex and idempotent; V is said to be *circled* if $cV = V$ for all scalars c such that $|c| \leqq 1$. A is a *locally convex algebra* if A is a (Hausdorff) topological algebra which has a basis for neighborhoods of the origin consisting of sets which are convex and circled; A is a *locally m-convex* (multiplicatively-convex) *algebra* if A has a basis for neighborhoods of the origin consisting of sets which are m-convex and circled. These sets can be taken to be closed. A family of closed, circled, m-convex subsets of a locally m-convex algebra A whose scalar multiples form a basis for the neighborhoods of the origin will be called an *m-base* for A.

If A is a locally m-convex algebra and $\mathscr{V}$ is an m-base for A define a nonnegative real-valued function $V(x)$ on A, V in $\mathscr{V}$, by $V(x) = \inf\{c > 0 : x \text{ in } cV\}$. Then $V(x+y) \leqq V(x) + V(y)$, $V(xy) \leqq V(x)V(y)$, and $V(cx) = |c|\,V(x)$ for all x, y in A and all scalars c. Thus with each V in $\mathscr{V}$ is associated a pseudo-norm $V(x)$. Denote the null set of $V(x)$ by N_V. Then N_V is a closed ideal and A/N_V is a normed algebra with norm $\bar{V}(x + N_V) = V(x)$. Denote A/N_V by A_V, $x + N_V$ by x_V, and the completion of A_V by B_V. Note that B_V is a Banach algebra.

1. The spectrum. An elements x in an algebra A is said to be *quasi-regular* in A if there exists an element y in A such that $x + y - xy = 0 = x + y - yx$. y is called the quasi-inverse of x. The *spectrum*, $Sp_A(x)$, of an element x in A, is given by $Sp_A(x) = \{c \neq 0\colon c \text{ complex}, c^{-1}x \text{ is not quasi-regular in } A\}$, with zero added unless A has an identity and x^{-1} exists in A. The *spectral radius*, $r_A(x)$, is defined by $r_A(x) = \sup\{|c| : c \text{ in } Sp_A(x)\}$. If A is a locally m-convex algebra and x is in A then $Sp_A(x)$ is not empty [1; 2.9].

Waelbroeck [3] has given a different definition of the spectrum of an element in a locally convex algebra with identity. Although his definition is actually given for a particular class of locally convex algebras, it can be used in any locally convex algebra with identity. However some of the properties which hold in the class of algebras which Waelbroeck considers may fail to hold in more general cases. We extend this definition to locally m-convex algebras which do not necessarily have

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an identity and show that this definition is equivalent to Waelbroeck's if the algebra does have an identity.

Let x be an element of a locally m-convex algebra A with identity e. A set S contained in C is said to be *spectral* for an element x in A, if $(x - ce)^{-1}$ is defined and bounded on $C - S$, the complement of S in the field of complex numbers, C. The spectral sets for x are a filter. Denote the Riemann sphere by C', i.e., the compact space obtained by adding a point at infinity to the complex field; and denote the set of points of C' which are limit points of the filter associated with an element x in A by $W'Sp_A(x)$. This is the spectrum defined by Waelbroeck.

Now let x be an element of an arbitrary locally m-convex algebra and let $\mathscr{S}(x) = \{S \subseteqq C : c^{-1}(c^{-1}x)^0$ is defined and bounded on $C - S\}$, where $(c^{-1}x)^0$ is the quasi-inverse of $c^{-1}x$. If S is in $\mathscr{S}(x)$ and if $S' \supseteqq S$ then S' is in $\mathscr{S}(x)$. If S and S' are in $\mathscr{S}(x)$ then $S \cap S'$ is in $\mathscr{S}(x)$. Since each S in $\mathscr{S}(x)$ contains zero the empty set is not in $\mathscr{S}(x)$. Thus $\mathscr{S}(x)$ is a filter. Let $WSp_A(x)$ be the set of limit points in C' of $\mathscr{S}(x)$, with the exception of 0 if A has an identity and 0 is not in $W'Sp_A(x)$.

THEOREM 1.1. *If A is a locally m-convex algebra with identity then $W'Sp_A(x) = WSp_A(x)$ for all x in A.*

Proof. It follows directly from the definition that 0 is in $W'Sp_A(x)$ if and only if 0 is in $WSp_A(x)$ for all x in A.

If x is an arbitrary element of A and if ∞ is not in $W'Sp_A(x)$ then there exists a bounded set S contained in C such that $(x - ce)^{-1}$ is defined and bounded on $C - S$, where e is the identity of A. Let $S' = S \cup \{c \text{ in } C : |c| \leqq 1\}$. Then $c^{-1}(c^{-1}x)^0$ is defined for c in $C - S'$ since $c^{-1}(c^{-1}x)^0 = c^{-1}e + (x - ce)^{-1}$. This equality also shows that $c^{-1}(c^{-1}x)^0$ is bounded on $C - S'$ since

$$\begin{aligned}\{c^{-1}e + (x - ce)^{-1} : c \text{ in } C - S'\} &\subseteqq \{c^{-1}e : c \text{ in } C - S'\} \\ &+ \{(x - ce)^{-1} : c \text{ in } C - S'\} \subseteqq \{c^{-1}e : |c| > 1\} \\ &+ \{(x - ce)^{-1} : c \text{ in } C - S\},\end{aligned}$$

and both of these sets are bounded. Thus $c^{-1}(c^{-1}x)^0$ is defined and bounded on the complement of a bounded set in C so that ∞ is not in $WSp_A(x)$. Conversely, if ∞ is not in $WSp_A(x)$ then there exists a bounded set S contained in C such that $c^{-1}(c^{-1}x)^0$ is defined and bounded on $C - S$. Again, let $S' = S \cup \{c \text{ in } C : |c| \leqq 1\}$; then S' is bounded and $(x - ce)^{-1}$ is defined and bounded on $C - S'$ so that ∞ is not in $W'Sp_A(x)$.

If c' is not in $W'Sp_A(x)$, $c' \neq 0, \infty$, then there exists a set S contained in C such that c' is not in the C'-closure of S and $(x - ce)^{-1}$ is defined and bounded on $C - S$. Let $S' = S \cup \{c \text{ in } C : |c| \leqq 1/2c'\}$. Then $c^{-1}(c^{-1}x)^0$

is defined and bounded on $C - S'$ and c' is not in the C'-closure of S', so that c' is not in $WSp_A(x)$. Similarly, if c' is not in $WSp_A(x)$, $c' \neq 0, \infty$, then c' is not in $W'Sp_A(x)$.

THEOREM 1.2. *If x is an element of a locally m-convex algebra A then $WSp_A(x)$ is compact in C'.*

Proof. Since C' is compact it is sufficient to show that $WSp_A(x)$ is closed in C' for each x in A, or, equivalently, that $C' - WSp_A(x)$ is open in C'.

If 0 is not in $WSp_A(x)$ then A has an identity e and there exists a set S contained in C such that $(x - ce)^{-1}$ is defined and bounded on $C - S$ and 0 is not in the C'-closure of S. Then 0 is not in $\bar{S}$, the closure of S in C, and $C - \bar{S}$ is a neighborhood of 0 in C' such that $(C - \bar{S}) \cap WSp_A(x) = \phi$, where ϕ is the empty set.

If ∞ is not in $WSp_A(x)$ then there exists a bounded set S contained in C such that $c^{-1}(c^{-1}x)^0$ is defined and bounded on $C - S$. Then $C' - \bar{S}$ is a neighborhood of ∞ such that $(C' - \bar{S}) \cap WSp_A(x) = \phi$.

If c' is not in $WSp_A(x)$, $c' \neq 0, \infty$, then there exists a set S contained in C such that c' is not in the C' closure of S and such that $C - \bar{S}$ is a neighborhood of c' in C' with $(C - \bar{S}) \cap WSp_A(x) = \phi$.

THEOREM 1.3. *If x is an element of a locally m-convex algebra then $WSp_A(x)$ is the C'-closure of $Sp_A(x)$.*

Proof. If 0 is in $Sp_A(x)$ then x is not invertible in A so that 0 is in $WSp_A(x)$. If $c' \neq 0$ and c' is in $Sp_A(x)$ then $((c')^{-1}x)^0$ does not exist so that c' is in every S in $\mathscr{S}(x)$ and hence in $WSp_A(x)$. Therefore $Sp_A(x)$ is contained in $WSp_A(x)$. Denoting the C'-closure of $Sp_A(x)$ by K, we have $K \subseteqq WSp_A(x)$.

If ∞ is not in K then there exists a positive number M in C such that $K \subseteqq \{c \text{ in } C : |c| \leqq M\}$. Let $S = \{c \text{ in } C : |c| \leqq M\}$. Then $c^{-1}(c^{-1}x)^0$ is defined for c in $C - S$. Now let $\mathscr{V}$ be an m-base for A. Since $M \geqq r_A(x) \geqq r_{B_V}(x_V)$ we have $(c^{-1}x_v)^0 = -\sum_{n=1}^{\infty} (c^{-1}x_V)^n$ for c in $C - S$. It follows that $\{c^{-1}(c^{-1}x_V)^0 : c \text{ in } C - S\}$ is bounded in B_V for each V in $\mathscr{V}$. Therefore $\{c^{-1}(c^{-1}x)^0 : c \text{ in } C - S\}$ is bounded, and S is in $\mathscr{S}(x)$. Since ∞ is not in the C'-closure of S, ∞ is not in $WSp_A(x)$.

If 0 is not in $C - K$ then A has an identity which we will denote by e. Take a compact neighborhood U of 0 contained in $C - K$. $(x - ce)^{-1}$ is then defined for c in U. Since U is compact in C it follows that $\{x - ce : c \text{ in } U\}$ is compact in A. Then $\{(x - ce)^{-1} : c \text{ in } U\}$ is compact in A since quasi-inversion and hence inversion is continuous in a locally m-convex algebra [1; 2.8]. $C - U$ is therefore a spectral set for x. Since 0 is not in the C'-closure of $C - U$, 0 is not in $WSp_A(x)$.

Now let c' be in $C - K, c' \neq 0$, and let U be a compact neighborhood of c' contained in $C - (K \cup (0))$. Then $c^{-1}(c^{-1}x)^0$ is defined for c in U and $\{c^{-1}(c^{-1}x)^0 : c \text{ in } U\}$ is bounded so that $C - U$ is in $\mathscr{S}(x)$. Since c' is not in the C'-closure of $C - U$, c' is not in $WSp_A(x)$. Therefore $C' - K \subseteqq C' - WSp_A(x)$.

COROLLARY 1.4. *In a locally m-convex algebra A*, $r_A(x) = \max\{|c| : c \text{ in } WSp_A(x)\}$, *for all x in A.*

A locally m-convex algebra A is said to be *advertibly complete* if, for every m-base $\mathscr{V}$ for A, each x in A is quasi-regular in A if and only if x_V is quasi-regular in B_V for each V in $\mathscr{V}$. See [4] for a discussion of advertible completeness and equivalent definitions. In particular note that most of the results of §§ 5–11 in [1] hold if the advertible completeness of a locally m-convex algebra is assumed in place of completeness. Thus the following two theorems are known in the case where the algebras are advertibly complete ([1] 5.3 (b) and 5.7).

THEOREM 1.5. *If A is a sequentially complete or advertibly complete locally m-convex algebra with m-base* $\mathscr{V}$, *and if x is in A, then* $r_A(x) = \sup r_{B_V}(x_V)$, V *in* $\mathscr{V}$.

Proof. The inequality $r_A(x) \geqq \sup r_{B_V}(x_V)$, V in $\mathscr{V}$, always holds, so that if $\sup r_{B_V}(x_V) = \infty$, then the equality holds. If $\sup r_{B_V}(x_V) \neq \infty$, let c be any element of C such that $|c| > \sup r_{B_V}(x_V)$, V in $\mathscr{V}$. Then $-\sum_{n=1}^{\infty} (c^{-1}x_V)^n = (c^{-1}x)^0$ for all V in $\mathscr{V}$. If A is advertibly complete this implies that $(c^{-1}x_V)^0$ exists by the definition. If A is sequentially complete then $-\sum_{n=1}^{\infty} (c^{-1}x)^n$ is a Cauchy sequence in A, hence converges, and is the quasi-inverse of $c^{-1}x$. It follows that $r_A(x) = \sup r_{B_V}(x_V)$, V in $\mathscr{V}$.

THEOREM 1.6. *Let A be a locally m-convex algebra which is either advertibly complete or sequentially complete, and let x and y be elements of A which commute. Then*

(i) $$r_A(x + y) \leqq r_A(x) + r_A(y) .$$

(ii) $$r_A(xy) \leqq r_A(x)r_A(y) .$$

(iii) $$r_A(cx) = |c| \, r_A(x) , \quad c \text{ in } C .$$

Proof. These properties are known to hold in a Banach algebra, hence the theorem follows directly from the preceding theorem.

COROLLARY 1.7. *If A is a commutative locally m-convex algebra which is either advertibly complete or sequentially complete then* $\{x \text{ in } A : r_A(x) < \infty\}$ *is a subalgebra of A.*

If A is an arbitrary locally m-convex algebra we denote $\{x$ in $A : r_A(x) < \infty\}$ by A'. Note that if A has an identity e then e is in A' since $Sp_A(e) = \{1\}$.

If A is a locally m-convex algebra we denote by A_1 the algebra which we get by adding an identity to A in the usual way. A_1 is the vector space direct sum of A and the complex numbers with the cartesian product topology, and with multiplication defined by $(x, a)(y, b) = (xy + ay + bx, ab)$. A_1 is a locally m-convex algebra with this topology [1; 2.4].

LEMMA 1.8. *If A is a locally m-convex algebra and if x is in A then $Sp_{A_1}(x) = Sp_A(x)$.*

COROLLARY 1.9. *$WSp_{A_1}(x) = WSp_A(x)$.*

COROLLARY 1.10. *$r_{A_1}(x) = r_A(x)$.*

THEOREM 1.11. *If A is a locally m-convex algebra and if x is quasi-regular in A then x^0* is *in A' if x is in A'.*

Proof. First assume that A has an identity e and let x be an element of A such that x^0 exists and $r_A(x) < \infty$. Then $(x - (1 + c^{-1})e)^{-1}$ is defined and bounded for c near zero. It follows from the identity $(x-(1+c)e)^{-1} = (x-e)^{-1} - (x-e)^{-2}((x-e)^{-1} - c^{-1}e)^{-1} = (x-e)^{-1} - (x-e)^{-2}$ $(x^0 - (1 + c^{-1})e)^{-1}$ that $(x^0 - te)^{-1}$ is defined and bounded for t near infinity, and hence that ∞ is not in $WSp_A(x^0)$. The theorem then follows from Corollary 1.9.

COROLLARY 1.12. *If A is a locally m-convex algebra and if x is in A' then $WSp_{A'}(x) = WSp_A(x)$.*

COROLLARY 1.13. *If A is a locally m-convex algebra and if x is in A' then $r_A(x) = r_{A'}(x)$.*

2. The radical, the spectrum and $M(A)$.

Denote the set of continuous homomorphisms of a locally m-convex algebra A into the complex numbers by $M(A)$. If A is a commutative advertibly complete locally m-convex algebra then $Sp_A(x) \cup (0) = \{f(x) : f \text{ in } M(A)\}$, for all x in A [4; § 3] and [1; 5.5]. We now look at this equality.

THEOREM 2.1. *If A is a locally m-convex algebra and if $Sp_A(x) \cup (0) = \{f(x) : f \text{ in } M(A)\}$, all x in A, then A* is *advertibly complete.*

Proof. If $Sp_A(x) \cup (0) = \{f(x) : f \text{ in } M(A)\}$ then $f(x) \neq 1$ for all f

in $M(A)$ implies that 1 is not in $Sp_A(x)$ so that x is quasi-regular. This condition implies that A is advertibly complete [4; cor. of Thm. 5].

COROLLARY 2.2. *If A is a commutative locally m-convex algebra then A is advertibly complete if and only if $Sp_A(x) \cup (0) = \{f(x) : f$ in $M(A)\}$ for all x in A.*

The multiplicative radical of a locally m-convex algebra A is denoted by $MR(A)$ and defined by $MR(A) = \{x \text{ in } A : f(x) = 0, \text{ all } f \text{ in } M(A)\}$. We denote the Jacobson redical of an algebra A by $R(A)$. The following theorem and the first corollary are given in [4; § 3].

THEOREM 2.3. *Let A be a locally m-convex algebra. Then $MR(A) = R(A)$ if and only if for all x in A, if x is not quasi-regular then there exists an f in $M(A)$ such that $f(x) \neq 0$.*

COROLLARY 2.4. *If A is a commutative advertibly complete locally m-convex algebra then $R(A) = MR(A)$, and, in particular, $R(A)$ is closed.*

COROLLARY 2.5. *Let A be a locally m-convex algebra such that $Sp_A(x) \cup (0) = \{f(x) : f$ in $M(A)\}$ for all x in A. Then*
(i) *$R(A) = MR(A)$.*
(ii) *$R(A)$ is closed.*
(iii) *A is commutative modulo $R(A)$.*
(iv) *A is commutative if A is semi-simple.*

Proof. We only need look at (iii). This follows since $f(xy - yx) = f(xy) - f(yx) = f(x)f(y) - f(y)f(x) = 0$ for all f in $M(A)$.

COROLLARY 2.6. *If A is a semi-simple advertibly complete locally m-convex algebra then A is commutative if and only if $Sp_A(x) \cup (0) = \{f(x) : f$ in $M(A)\}$, all x in A.*

COROLLARY 2.7. *If A is a semi-simple Banach algebra then A is commutative if and only if $Sp_A(x) \cup (0) = \{f(x) : f$ in $M(A)\}$ for all x in A.*

3. The radical and quasi-nilpotent elements and topological divisors of zero. We follow Michael in defining topological divisors of zero. Let A be a Banach algebra, let x be an element of A, and let c be a scalar. $x + c$ is called a *left* (*right*) *strong topological divisor of zero* if there exists a sequence y_n in A such that $\| y_n \| = 1$, $n = 1, 2, \cdots$, and $(x + c)y_n \to 0(y_n(x + c) \to 0)$. $x + c$ is said to be a *strong topological divisor of zero* if $x + c$ is either a left or a right strong topological divisor of zero. If $x + c$ is both a left and a right strong topological divisor of zero then it is called a *two-sided strong topological divisor of zero.*

If A is a locally m-convex algebra and x is an element of A, and c is a scalar, then $x + c$ is called a (*left, right, two-sided*) *topological divisor of zero*, if, whenever $\mathscr{V}$ is an m-base for A, there exists a V in $\mathscr{V}$ such that $x_V + c$ is a (left, right, two-sided) strong topological divisor of zero in B_V. See [1; § 6] for a discussion of topological divisors of zero. Note that if x is an element of a Banach algebra then $x + c$ is a strong topological divisor of zero if and only if it is a topological divisor of zero.

We call an element x in a locally m-convex algebra *quasi-nilpotent* if $r_A(x) = 0$ and denote the set of quasi-nilpotent elements by N_A. N_A is called the *quasi-radical* of A. An ideal I in A is said to be a *quasi-nil ideal* if every element in I is quasi-nilpotent.

LEMMA 3.1. *If A is a Banach algebra then every element of N_A is a two-sided topological divisor of zero.*

Proof. This is proved in [5; 2.26]. One could also use the proof of Rickart that each element of $R(A)$ is a topological divisor of zero [2; 2.3.5 (iii)], since the spectrum of each element of N_A also consists only of zero, and N_A can contain no nonzero idempotents.

COROLLARY 3.2. *If A is a locally m-convex algebra then every element of N_A is a two-sided topological divisor of zero.*

THEOREM 3.3. *If A is a locally m-convex algebra then $R(A)$ is a quasi-nil ideal.*

Proof. Recall that $R(A)$ consists of all elements x in A such that $(c + y)x$ is quasi-regular for every y in A and c in C. Now let x be in $R(A)$. Then cx is quasi-regular for all c in C, hence $Sp_A(x) = (0)$ and $r_A(x) = 0$.

COROLLARY 3.4. *If A is a locally m-convex algebra then every element of $R(A)$ is a two-sided topological divisor of zero.*

THEOREM 3.5. *N_A is an ideal in A if A is a commutative locally m-convex algebra which is either advertibly complete or sequentially complete.*

Proof. This follows from Theorem 1.6.

THEOREM 3.6. *If A is a locally m-convex algebra then $R(A)$ is a quasi-nil ideal equal to the sum of all quasi-nil left (or right) ideals.*

Proof. $R(A)$ is a quasi-nil ideal by Theorem 3.3. Let I be any quasi-nil left (or right) ideal in A. Then $Sp_A(x) = (0)$ for each x in I so that I is a quasi-regular ideal and hence contained in $R(A)$.

COROLLARY 3.7. *If A is a locally m-convex algebra and if N_A is a left (or right) ideal then $R(A) = N_A$. In particular if A is commutative and is advertibly or sequentially complete then $R(A) = N_A$.*

COROLLARY 3.8. *If A is a commutative semi-simple locally m-convex algebra which is either advertibly complete or sequentially complete then any subalgebra is semi-simple.*

Proof. Let B be a subalgebra of A and let x be in $R(B)$. Then cx is quasi-regular in B for all c in C so that cx is quasi-regular in A for all c in C. Therefore $Sp_A(x) = (0)$ and hence x is in $N_A = R(A)$ so that $x = 0$.

REFERENCES

1. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs of the American Mathematical Society, no. 11, 1952.
2. C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand Co. (1960).
3. L. Waelbroeck, *Locally convex algebras: spectral theory*, Seminar on Complex Analysis, Institute of Advanced Studies 1957-58.
4. S. Warner, *Polynomial completeness in locally multiplicatively convex algebras*, Duke Math. J., **23** (1956), 1-12.,
5. B. Yood, *Seminar on Banach Algebras*, University of California (1956-57).

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