NORMAL SUBGROUPS OF MONOMIAL GROUPS

ALLAN B. GRAY, JR.

1. Introduction. Let U be the set consisting of $x_1, x_2, x_3, \cdots x_n$. Let H be a fixed group. A *monomial substitution* of U over H is a transformation of the form,

$$y = \begin{pmatrix} x_1 \ , & x_2 \ , & x_3 \ , & \cdots, & x_n \\ h_1 x_{j_1}, & h_2 x_{i_2}, & h_3 x_{i_3}, & \cdots, & h_n x_{i_n} \end{pmatrix} \begin{matrix} x_j \in U \\ h_i \in H \end{matrix}$$

where the mapping of the x's is one-to-one. The h_j are called the factors of y. If.

$$y_1 = \begin{pmatrix} x_1 \ , & x_2 \ , & x_3 \ , & \cdots, & x_n \\ k_1 x_{j_1}, & k_2 x_{j_2}, & k_3 x_{j_3}, & \cdots, & k_n x_{j_n} \end{pmatrix}$$

then

$$y y_1 = \begin{pmatrix} x_1 \ , & x_2 \ , & x_3 \ , & \cdots, & x_n \\ h_1 k_{i_1} x_{j_{i_1}}, & h_2 k_{i_2} x_{j_{i_2}}, & h_3 k_{i_3} x_{j_{i_3}}, & \cdots, & h_n k_{i_n} x_{j_{i_n}} \end{pmatrix} .$$

By this definition of multiplication the set of all substitutions form a group $\Sigma_n(H)$. Denote by V the set of all substitutions of the form

$$y = \begin{pmatrix} x_1 \ , & x_2 \ , & x_3 \ , & \cdots, & x_n \\ h_1 x_1, & h_2 x_2, & h_3 x_3, & \cdots, & h_n x_n \end{pmatrix} = [h_1, h_2, h_3, \cdots, h_n] .$$

Then V, called the basis group, is a normal subgroup of $\Sigma_n(H)$. A permutation is an element of the form

$$\begin{pmatrix} x_1 \ , & x_2 \ , & \cdots, & x_n \\ e x_{i_1}, & e x_{i_2}, & \cdots, & e x_{i_n} \end{pmatrix} = \begin{pmatrix} 1, & 2, & \cdots, & n \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix} .$$

where e is the identity of H. Cyclic representation will also be used for elements of this type. The set S_n of all such elements is a subgroup of $\Sigma_n(H)$. Furthermore $\Sigma_n(H) = V \cup S$, $V \cap S = E$ where E is the identity of $\Sigma_n(H)$. Any element y of $\Sigma_n(H)$ can be written as $y = vs$ where $v \in V$ and $s \in S$. Ore [1] has studied this group for finite U and some of his results have been extended in [2] and [3].

The normal subgroups of $\Sigma_n(H) = \Sigma_n$ for U a finite set have been determined in [1]. The normal subgroups for $o(U) = B = \aleph_u$, $u \geqq 0$, where $o(U)$ means the number of elements of U, have been determined for rather general cases in [2] and [3]. The subset $\Sigma_{A,n}(H) = \Sigma_{A,n}$ of elements of the form $y = vs$ with s in the alternating group A_n is a

Received May 22, 1961.

subgroup of Σ_n. The normal subgroups of $\Sigma_{A,n}$ are known for all n except 3 and 4 [2]. This paper determines the normal subgroups of $\Sigma_{A,n}$ for $n = 3, 4$ that are not contained in the basis group, thus filling a gap in the theory.

2. The normal subgroups of $\Sigma_{A,3}$ not contained in the basis group V. We shall consider first the normal subgroups M that contain pure permutations.

THEOREM 1. *Let M be normal in $\Sigma_{A,3}$, $A_3 \subset M$. Then $N = M \cap V$ is a normal subgroup of $\Sigma_{A,3}$. The subgroup $M = N \cup A_3$. There exists a normal subgroup S_1 of H such that H/S_1 is Abelian and such that N consists of all elements $v = [h_1, h_2, h_3]$ for which $h_1h_2h_3 \in S_1$.*

Proof. The intersection of two normal subgroups is again normal so N is normal in $\Sigma_{A,3}$.

Clearly $M \supset (N \cup A_3)$. Let $y = vs$ be arbitrary in M. Then $ys^{-1} = v$ belongs to $M \cap V = N$ so $M \subset (N \cup A_3)$.

Let $v = [h_1, h_2, h_3]$ be arbitrary in N. Form $y = v(1, 2, 3)$, which is in M. All of the elements $y_1 = v_1yv_1^{-1}$, where v_1 is arbitrary in V are in M by M normal in $\Sigma_{A,3}$. For a proper choice of v_1, $y_1 = [h_1h_2h_3, e, e]$ $(1, 2, 3)$. Therefore N contains $[h_1h_2h_3, e, e]$. Now consider the set $N_1 \cup N$ of all elements of the form $[h, e, e]$. This is a normal subgroup of N. The elements of H that occur as the first factors of multiplications of N_1 form a normal subgroup S_1 of H. We have established that if $v \in N$ the product of the factors is in S_1. If k_1, k_2, k_3 are any elements of H satisfying $k_1k_2k_3 = k$ where k is in S_1 then $[k, e, e]$ is in N. Furthermore $[k, e, e]$ $(1, 2, 3)$ is in M and by a proper conjugation with a multiplication $[k_1, k_2, k_3]$ $(1, 2, 3)$ is in M. Hence $[k_1, k_2, k_3]$ is in N.

Since $[r_1, r_2, r_2^{-1}r_1^{-1}]$ is in N for arbitrary r_1, r_2 of H, its inverse $[r_1^{-1}, r_2^{-1}, r_1r_2]$ is also in N. Therefore $r_1^{-1}r_2^{-1}r_1r_2$ is in S_1. This shows $r_1r_2 \equiv r_2r_1 \bmod S_1$ and H/S_1 is Abelian.

THEOREM 2. *Let N be as described in the last sentence of Theorem 1. Then $N \cup A_3 = M$ is normal in $\Sigma_{A,3}$.*

Proof. Ore [1, p. 37] has shown M is normal in Σ_3 so it is normal in $\Sigma_{A,3}$.

We shall now describe those normal subgrous which do not contain a pure permutation.

THEOREM 3. *Let $S_1 \subset S_2$ be normal subgroups of H satisfying the conditions H/S_1 is Abelian and S_2/S_1 is isomorphic, by θ say, to A_3.*

Let M consist of the sets $T_i = \{vs/s = (1, 2, 3)^i\}$, $i = 0$ or 1 or 2, where the factors of substitutions of T_i run through H subject to the conditions that their product, k say, is in S_2 and the coset kS_1 maps onto $(1, 2, 3)^i$. Then M is a normal subgroup of $\Sigma_{A,3}$. Conversely if $M \notin V$ and $A_3 \notin M$, then M has the above form.

Proof. We shall establish first that M is a group. Let $y_1 = [h_1, h_2, h_3]s_1$ and $y_2 = [k_1, k_2, k_3]s_2$ be arbitrary elements in M. We know then that $h_1h_2h_3S_1\theta = s_1$ and $k_1k_2k_3S_1\theta = s_2$. Consider the product $y_1y_2 = [h_1k_{i_1}, h_2k_{i_2}, h_3k_{i_3}]s_1s_2$. Since H/S_2 is Abelian and θ is an isomorphism $h_1k_{i_1}h_2k_{i_2}h_3k_{i_3}S_1\theta = h_1h_2h_3k_1k_2k_3S_1\theta = h_1h_2h_3\theta k_1k_2k_3\theta = s_1s_2$. This shows that if y_1y_2 belongs to T_i then the coset of the product of the factors maps onto $(1, 2, 3)^i$. We show now that when y_1 as above is in M that its inverse is also in M. The inverse of y_1 is $y_1^{-1} = [h_{i_1}^{-1}, h_{i_2}^{-1}, h_{i_3}^{-1}]s_1^{-1}$. We must show $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}$ belongs to S_2 and $h_{i_1}^{-1}h_{i_2}^{-1}h_{i_3}^{-1}S_1\theta = s_1^{-1}$. The first of these follows from $h_1h_2h_3$ in S_2 and H/S_2 Abelian. The second follows from the observation that $h_3^{-1}h_2^{-1}h_1^{-1}S_1\theta = s_1^{-1}$ and H/S_1 is Abelian.

It remains to show that M is normal in $\Sigma_{A,3}$. Let $y_1 = [h_1, h_2, h_3]s_1$ and $y_3 = [g_1, g_2, g_3]s$ be arbitrary elements of M and $\Sigma_{A,3}$ respectively. We must show that the product

$$y_3y_1y_3^{-1} = [g_1h_{i_1}g_{j_1}^{-1}, g_2h_{i_2}g_{j_2}^{-1}, g_3h_{i_3}g_{j_3}^{-1}]ss_1s^{-1} = vs_1$$

is in M. The product of the factors is in S_2 since H/S_2 is Abelian and $h_1h_2h_3$ is in S_2. Finally

$$g_1h_{i_1}g_{j_1}^{-1}g_2h_{i_2}g_{j_2}^{-1}g_3h_{i_3}g_{j_3}^{-1}S_1\theta = h_1h_2h_3S_1\theta = s_1 .$$

We now give the proof of the converse. Two elements vs and v_1s_1 of M are defined to be equivalent if $s = s_1$. This is an equivalence relation and induces the partition $T_0 = \{vs/s = E\}$, $T_1 = \{vs/s = (1, 2, 3)\}$, $T_2 = \{vs/s = (1, 3, 2)\}$ on M. We note that one of the sets T_1 or T_2 is nonempty since $M \notin V$. In fact, since at least one of them is not empty, they are each nonempty.

If an arbitrary element $y = vs = [h_1, h_2, h_3]$ $(1, 2, 3)$ of T_1 is conjugated by $[h_3, h_2^{-1}, e]$ the resulting elements $[h_3h_1h_2, e, e]$ $(1, 2, 3)$ is also in T_1. Since $s_1ys_1^{-1} = s_1vs_1^{-1}s_1ss_1^{-1} = v_1s$ is in M for all s_1 of A_3 we can show that $[h_1h_2h_3, e, e](1, 2, 3)$ and $[h_2h_3h_1, e, e](1, 2, 3)$ also belong to T_1. When $y_1 = [a, e, e](1, 2, 3)$ is in T_1 then $(1, 2, 3)y_1(1, 3, 2) = [e, e, a](1, 2, 3)$ and $(1, 3, 2)y_1(1, 2, 3) = [e, a, e](1, 2, 3)$ are also in T_1.

Similarly it can be shown that T_2 contains elements of the form $[b, e, e,](1, 3, 2)$ and with every such element $[e, b, e](1, 3, 2)$, $[e, e, b](1, 3, 2)$. In particular $[h_2h_1h_3, e, e](1, 3, 2)$ is in T_2 where $[h_1, h_1, h_3](1, 3, 2)$ is arbitrary in T_2. When $[a, e, e]$ is in T_0, then $[e, a, e]$ and $[e, e, a]$ are also in T_0.

Now denote by R the set of elements of the form $[a, e, e]s$. Let S_2 be the set of elements of H that occur as first factors of elements of R. We shall show that S_2 is a normal subgroup of H. Choose arbitrary elements $m_1 = [a_1, e, e]s_1$ and $m_2 = [a_2, e, e]s_2$ of R. If $s_1 = E$ then $m_1m_2 = [a_1a_2, e, e]s_2$ is again in R and a_1a_2 belongs to S_2. If $s_1 = (1, 2, 3)$ we work with $m_3 = [e, a_2, e]s_2$ and form $m_1m_3 = [a_1a_2, e, e](1, 2, 3)s_2$. Again we have shown $a_1a_2 \in S_2$. Finally if $s_1 = (1, 3, 2)$ we let $m_4 = [e, e, a_2]s_2$ and consider $m_1m_4 = [a_1a_2, e, e](1, 3, 2)s_2$. In any case we see that S_2 is closed. When $m_1 \in R$ then m_1^{-1} which is $[a_1^{-1}, e, e]$, $[e, a_1^{-1}, e]s_1^{-1}$, or $[e, e, a_1^{-1}]s_1^{-1}$ also belongs to M. By the earlier argument we see that R must contain $[a_1^{-1}, e, e]s_1^{-1}$. This shows $a_1^{-1} \in S_2$. Let $a \in S_2$ and $h \in H$. Then, by the definition of $\Sigma_{A,3}$ and S_2, $[a, e, e]s \in M$ and $[h, h, h] \in \Sigma_{A,3}$. Now since M is normal in $\Sigma_{A,3}$, $[h, h, h][a, e, e]s[h^{-1}, h^{-1}, h^{-1}] = [hah^{-1}, e, e]s \in M$. Therefore, hah^{-1} is in S_2. We have just shown S_2 is normal in H.

Substitutions in $R \cap V = N_1$ are of the form $[a, e, e]$. The first factors form a subgroup, S_1, of H. That S_1 is normal in H follows from M normal in $\Sigma_{A,3}$. By the definition of the two groups S_1 is a subgroup of S_2.

To show that H/S_1 is Abelian we let h_1, h_2 be arbitrary elements of H and show $h_1h_2h_1^{-1}h_2^{-1}$ is in S_1. Choose an element $[b_1, b_2, b_3](1, 2, 3)$ from T_1 and conjugate it by each of the three elements $[e, h_2h_1b_1; h_1b_1b_2]$, $[e, b_1, b_1b_2]$, and $[e, h_2^{-1}h_1^{-1}b_1, h_1^{-1}b_1b_2]$. The resulting elements, which must be in M, are $y_1 = [h_1^{-1}h_2^{-1}, h_2, h_1b_1b_2b_3](1, 2, 3)$, $y_2 = [e, e, b_1b_2b_3](1, 2, 3)$, and $y_3 = [h_1h_2, h_2^{-1}, h_1^{-1}b_1b_2b_3](1, 2, 3)$. The product $y_4 = y_2y_1^{-1} = [h_2h_1, h_2^{-1}, h_1^{-1}]$ is also in M. Now form $y_5 = y_2y_3^{-1} = [h_2^{-1}h_1^{-1}, h_2, h_1]$. Finally consider $y_4y_5 = [h_2h_1h_2^{-1}h_1^{-1}, e, e]$ which is in M. Therefore, $h_2h_1h_2^{-1}h_1^{-1}$ is in S_1. In addition this also establishes that H/S_2 is Abelian. Earlier we had $[h_2h_1h_3, e, e]$ $(1, 3, 2)$ in T_2. By H/S_2 Abelian $h_1h_2h_3 \in S_2$ also.

We now define a mapping from S_2 onto A_3 as follows. For an element a of S_2 which occurs as a first factor of a substitution $y = [a, e, e]s$ we let $a\theta = s$. Certainly by this definition every element of S_2 will be mapped. If any element of S_2 is assumed to be mapped onto two different elements of A_3 a computation, using the properties already stated for R and M, will show that M contains a pure permutation contrary to the case we are currently investigating. For example, suppose $a\theta = (1, 2, 3)$ and $a\theta = (1, 3, 2)$. Then $y_1 = [a, e, e](1, 3, 2)$, $y_2 = [a, e, e](1, 2, 3)$, $y_1^{-1} = [e, e, a^{-1}](1, 2, 3)$, and $y_3 = [e, a^{-1}, e](1, 2, 3)$ all belong to M. So $[a, e, e](1, 2, 3)[e, a^{-1}e](1, 2, 3) = (1, 3, 2)$ belongs to M. This mapping also preserves multiplication. For let $a_1\theta = s_1$, $a_2\theta = s_2$. This means that R contains the elements $[a_1, e, e]s_1$, $[a_2, e, e]s_2$. But M also contains vs_2 where v has two factors of e and a_2 a factor in the position that s_1 sends x_1 into. Therefore, $[a_1a_2, e, e]s_1s_2$ belongs to R and $a_1a_2\theta = s_1s_2 = a_1\theta a_2\theta$. The definition of the mapping makes it clear that the kernel of the homomorphism is precisely S_1. Therefore, $S_2/S_1 \cong A_3$.

It has already been pointed out that if $y = vs$ is an element of T_1 or T_2 then the product of the factors $h_1h_2h_3$ of v is in S_2. If $[a_1, a_2, a_3]$ is in $M \cap V$ then since $y_5 = [h_2^{-1}h_1^{-1}, h_2, h_1]$ is also in M for arbitrary h_1, h_2 of H it follows that $[a_1, a_2, a_3][a_2a_3, a_2^{-1}, a_3^{-1}] = [a_1a_2a_3, e, e]$ is in M. This shows that the product of factors of elements in T_0 is in S_1. Now let us assume that b_1, b_2, b_3 are elements of H whose product is in S_2. Then $(b_1b_2b_3)\theta = (1, 2, 3)^i$ for $i = 0$, or 1, or 2. We will show that there is an element $y = vs$ of T_1 whose factors are b_1, b_2, and b_3. In the case where $i = 0$ we know that M contains an element $[b_1b_2b_3, e, e]$. The element $y_4 = [h_2h_1, h_2^{-1}, h_1^{-1}]$ and its inverse $y_4^{-1} = [h_1^{-1}h_2^{-1}, h_2, h_1]$ are also in M for all h_1, h_2 of H so choose $h_2 = b_2, h_1 = b_3$. Then the product $[b_1b_2b_3, e, e][b_3^{-1}b_2^{-1}, b_2, b_3] = [b_1, b_2, b_3]$ is in M. When $i = 1$ we have $[b_1b_2b_3, e, e](1, 2, 3)$ in M and by choosing $h_2 = b_3^{-1}b_2^{-1}, h_1 = b_2$ and computing $[b_1b_2b_3, e, e](1, 2, 3)[b_3, b_b^{-1}b_2^{-1}, b_2] = [b_1, b_2, b_3](1, 2, 3)$. Finally if $i = 2$ we have $[b_1b_2b_3, e, e](1, 3, 2)$ in M and by choosing $h_2 = b_3, h_1 = b_3^{-1}b_2^{-1}$ and computing we have $[b_1, b_2, b_3](1, 3, 2)$ in T_2.

3. The normal subgroups of $\Sigma_{A,4}$ not contained in the basis group V. All proofs in this section except for the proof of Lemma 1 are similar to the corresponding proofs for $\Sigma_{A,3}$ so will be omitted.

LEMMA 1. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$. Then the Klein group is contained in M.*

Proof. We will first show that M contains elements of the form $y = vs$ where $s \neq E$ is in the Klein group. Hereafter K will mean the Klein group.

There is at least one element in M of the form $y = vs$ $s \neq E, s \in A_4$. If s is not in K then s is a three cycle, and we assume without loss of generality that $s = (1, 3, 4)$. If y is conjugated by $(1, 4)(2, 3)$ the resulting element $y_1 = v_1(1, 4, 2)$ and its inverse are also in M. Therefore, $yy_1^{-1} = v_2(1, 3)(2, 4)$ is in M. We have just shown that M has an element of the form $y = vs$ where s is in K and $s \neq E$. We assume without loss of generality that $s = (1, 2)(3, 4)$ and $v = [a_1, a_2, a_3, a_4]$. Form the elements

$$\begin{aligned} y_1 &= v_1yv_1^{-1} = [e, a_2^{-1}, e, a_3][a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2 e, a_3^{-1}] \\ &= [a_1a_2, e, e, a_3a_4](1, 2)(3, 4) \text{ and } y_2 = y_3yy_3^{-1} \\ &= [e, a_2^{-1}, a_4^{-1}, e](1, 3, 4)[a_1, a_2, a_3, a_4](1, 2)(3, 4)[e, a_2, e, a_4](1, 4, 3) \\ &= [a_3a_4, e, e, a_1, a_2](1, 3)(2, 4). \end{aligned}$$

Since M is normal in $\Sigma_{A,4}$, y_1 and y_2 are in M. Therefore $y_1y_2^{-1} =$

$(1, 4)(2, 3)$ is in M. This shows $S = M \cap A_4 \neq E$. But M is normal in $\Sigma_{A,4}$ so S is normal in A_4. This means S is K or A_4.

We shall now describe the normal subgroups N which is the intersection of M and the basis group V.

THEOREM 1. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$, $A_4 \subset M$. Then $N = M \cap V$ is a normal subgroup of $\Sigma_{A,4}$, $M = N \cup A_4$. There exists a normal subgroup S_1 of H such that H/S_1 is Abelian and such that N consists of all elements $v = [h_1, h_2, h_3, h_4]$ for which $h_1h_2h_3h_4 \in S_1$.*

THEOREM 2. *Let N be as described in the last sentence of Theorem 1. Then $N \cup A_4 = M$ is normal in $\Sigma_{A,4}$.*

We shall now describe those normal subgroups which contain no elements of the form $y = vs$ where s is a three cycle.

THEOREM 3. *Let M be normal in $\Sigma_{A,4}$, $M \not\subset V$, M contains no elements of the form $y = vs$ where s is a three cycle, $M \cap V = N$. Then $M = N \cup K$. Furthermore if N_1 is as described in the last sentence of Theorem 1 then $N_1 \cup K$ is normal in $\Sigma_{A,4}$.*

We shall now describe those normal subgroups which contain elements of the form $y = vs$, where s is a three cycle, but which do not contain a pure three cycle.

THEOREM 4. *Let $S_1 \subset S_2$ be normal subgroups of H satisfying the conditions H/S_1 is Abelian and S_2/S_1 is isomorphic to A_3. Let M consist of the sets*

$$T_i = \{vs/s = (1, 2, 3)^{i} \bmod K\}\,, \qquad i = 0, 1, 2,$$

where the factors of substitutions of T_i run through H subject to the condition that their product, k say, is in S_2 and kS_1 maps onto $(1, 2, 3)^i$. Then M is a normal subgroup of $\Sigma_{A,4}$. Conversely, if M is normal subgroup of $\Sigma_{A,4}$ such that $M \not\subset V$ and $A_4 \not\subset M$, M contains elements of the form $y = vs$ where s is a three cycle, then M has the above form.

BIBLIOGRAPHY

1. R. Baer, *Die kompositionsreihe der Gruppe aller eineindeutigen abbildungen einer unendlichen Reihe auf sich*, Studia Mathematica, **5** (1934), 15-17.
2. R. Crouch, *Monomial groups*, Trans. Amer. Math. Soc., **80** (1955), 187-215.
3. R. Crouch and W. R. Scott, *Normal subgroups of monomial groups*, Proc. Amer. Math. Soc., **8** (1957), 931-936.
4. O. Ore. *Theory of monomial groups*, Trans. Amer. Math. Soc., **51** (1942), 15-64.

NEW MEXICO STATE UNIVERSITY