THE ABSTRACT THEOREM OF CAUCHY-WEIL

ANDREW M. GLEASON

1. Let M be a separable, complex-analytic manifold. It is well-known that, if f is a bounded analytic function on M and $p \in M$, then $f(p)$ can be expressed as an integral of the "boundary values" of f. In general the boundary on which the integration is carried out and the boundary values to be integrated are abstract but in special cases a concrete description can be given. Suppose that M is an open subset having compact closure in some larger manifold M' and we consider only analytic functions f on M which have continuous extensions to $\bar{M}$. Then the boundary B is a subset of the topological boundary of M, the boundary values are given by the continuous extension and we may write

$$f(p) = \int_B f(b) d\mu_p(b) \tag{1.1}$$

where μ_p is a measure on B which is independent of f. When M is a region in the plane with rectifiable Γ we have the familiar Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)dt}{t - z} .$$

Here the integral is expressed with a fixed measure and a kernel function which is analytic in z. In the abstract proof of (1.1) each of the measures μ_p is derived by a separate application of the Hahn-Banach theorem so we cannot directly replace (1.1) with a formula involving an integral kernel depending analytically on p. It is the object of this paper to prove that this is possible.

An explicit formula involving an integral kernel for domains in C^n having sufficiently smooth boundary has been given by Weil [7]; a new proof of this formula in slightly more general circumstances will appear in [3]. Proofs under various circumstances have also been given by Arens [1], Hervé [4] and others. In the present paper we require no regularity on the boundary, but the integral kernel is not given explicitly. Perhaps the most remarkable fact about our result is that in the ordinary complex plane it is new or at least not standard.

2. This section is devoted to incorporating into the literature a proof of the well-known fact that several, apparently different, definitions of an analytic mapping of a complex-analytic manifold into a

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Banach space coincide. For the one-dimensional case these results are in [5; p. 92ff].

2.1 DEFINITION. Let M be a complex-analytic manifold of dimension n. Let φ be a map of M into a Banach space A. We shall say that φ is analytic at $p \in M$ if and only if φ can be represented in the neighborhood of p by a norm-convergent multiple power series in a local coordinate system centered at p. This means

$$\varphi(q) = \sum_{j_1, j_2, j_n} a_{j_1, j_2, \cdots, j_n} z_1(q)^{j_1} z_2(q)^{j_2} \cdots z_n(q)^{j_n} \tag{2.2}$$

for all q sufficiently near p, where $\langle z_1, \cdots, z_n \rangle$ is a local coordinate system centered at $p(z_1(p) = z_2(p) = \cdots = z_n(p) = 0)$.

We shall say that φ is analytic if it is analytic at each point p of M.

2.3 NOTATION. It is convenient to introduce the abbreviations (j) for $\langle j_1, j_2, \cdots, j_n \rangle$, $z^{(j)}$ for $z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$ and $|j|$ for $j_1 + j_2 + \cdots + j_n$.

By paraphrasing the standard proofs for the corresponding facts concerning complex-valued analytic functions we can prove

2.4. A necessary and sufficient condition that the power series

$$\sum_{(j)} a_{(j)} \lambda^{(j)}$$

should converge in the polycylinder $\{\lambda: |\lambda_i| < \rho,\ i = 1, \cdots, n\}$ is that for every positive ε there exist a constant C such that

$$\| a_{(j)} \| \leqq C(\rho - \varepsilon)^{-|j|}$$

for all (j).

2.5. If φ can be represented on an open neighborhood N of p by a power series in terms of a local coordinate system z centered at p, then, for any $q \in N$ and any local coordinate system w centered at q, φ can be represented in a neighborhood of q by a power series in w. Hence the notion analytic is independent of local coordinate systems.

2.6. If T is a bounded linear operator from one Banach space A to another B and φ is an analytic map of M into A, then $T \circ \varphi$ is an analytic map of M into B. In particular, if a^* is a bounded linear functional on A, then $a^* \circ \varphi$ is an analytic function in the usual sense.

We shall prove a converse of this result below (2.11).

2.7. If a^* is a linear functional on A and φ has the local representation (2.2) near p, then $a^* \circ \varphi$ has the local representation

$$\sum a^*(a_{(j)})z^{(j)}$$

and we can conclude that $a^* \circ \varphi$ vanishes locally at p if and only if $a^*(a_{(j)}) = 0$ for all (j). Then it follows from the Hahn-Banach theorem that the coefficients $a_{(j)}$ span the same closed linear manifold in A as does the φ-image of any connected open set containing p.

2.8 THEOREM. *Suppose φ is a map of a complex-analytic manifold M into the conjugate A^* of a Banach space A. Suppose that, for each $a \in A$, the map $q \to \varphi(q)(a)$ is analytic. Then φ is analytic.*

Proof. Choose a point p of M and a local coordinate system z centered at p. For each $a \in A$ we can expand

$$\varphi(q)(a) = \sum f_{(j)}(a)z(q)^{(j)} . \tag{2.9}$$

The coefficients $f_{(j)}(a)$ depend linearly on a because the power series representation of an analytic function is unique. We must show that the functionals $f_{(j)}$ are bounded.

For sufficiently small ρ the equations $|z_1| = |z_2| = \cdots = |z_n| = \rho$ determine a compact set T in the domain of the coordinate system z. For every $a \in A$, the analytic function $q \to \varphi(q)(a)$ carries T into a bounded subset of the complex plane, hence by the Banach-Steinhaus theorem there is a constant K such that $\|\varphi(q)\| \leqq K$ for all $q \in T$.

We can express the coefficients in (2.9) by the iterated Cauchy integral

$$f_{(j)}(a) = \frac{1}{(2\pi i)^n} \int_T \frac{\varphi(q)(a)}{z_1^{j_1+1} z_2^{j_2+1} z_n^{j_n+1}} \, dz_1 dz_2 \cdots dz_n$$

where q represents the points having coordinates $z_1, z_2, \cdots, z_n$. The usual estimate gives

$$|f_{(j)}(a)| \leqq K \|a\| \rho^{-|i|}$$

whence $f(j)$ is bounded; in fact, $\|f_{(j)}\| \leqq K\rho^{-|j|}$. Therefore the power series

$$\sum f_{(j)}z^{(j)} \tag{2.10}$$

converges in the norm in A^* for $|z_i| < \rho$, $i = 1, \cdots, n$ and determines an analytic map ψ of this polycylindrical neighborhood of p into A^*. Now $\psi(q)(a) = \varphi(q)(a)$ for all $a \in A$ and all q near p. Hence φ has the expansion (2.10) in a neighborhood of p. Since p is arbitrary φ is analytic.

2.11 THEOREM. *Suppose φ is a map of a complex-analytic mani-*

fold M into a Banach space A. If $a^*\circ\varphi$ is an analytic function on M for each $a^* \in A^*$, then φ is analytic.

Proof. Let U be the natural enbedding of A in A^{**} given by $(Ua)(a^*) = a^*(a)$. Now the map $U\circ\varphi$ satisfies the hypothesis of Theorem 2.8 so we conclude that $U\circ\varphi$ is an analytic map of into A^{**}.

Say that $U\circ\varphi$ has the local representation (2.10) near a point $p \in M$, where now $f_{(j)} \in A^{**}$. We must prove that $f_{(j)} \in UA$. The latter is a closed linear manifold in A^{**} which contains the range of $U\circ\varphi$, hence it contains also the coefficients as proved in 2.7. We may write therefore $f_{(j)} = Ua_{(j)}$ and, because U is an isometry, it follows that φ has the local representation $\sum a_{(j)}z^{(j)}$. Therefore φ is analytic.

2.12 COROLLARY. *Suppose φ is a map of a complex-analytic manifold M into a Banach space A. Suppose B is a closed linear subspace of A^* such that, for some constant K and all $a \in A$*

$$\|a\| \leqq K \sup\{|b(a)| : \|b\| = 1,\ b \in B\}\ .$$

If $b\circ\varphi$ is an analytic function for each $b \in B$, then φ is analytic.

Proof. Map A into B^* by defining $T(a)(b) = b(a)$. We have then $\|Ta\| \leqq \|a\|$ and our condition on B gives $\|a\| \leqq K\|Ta\|$. It follows that T is an isomorphism of A onto a closed linear manifold of B^* and the previous proof is valid.

2.13 THEOREM. *Let φ be a map of a complex-analytic manifold M into a Banach space A such that, for every compact subset K of M, $\varphi(K)$ is bounded in the norm. Suppose that $a^*\circ\varphi$ is analytic for every a^* in some total subset of A^*. Then φ is analytic.*

Proof. Let B be the set of all bounded linear functionals a^* on A for which $a^*\circ\varphi$ is analytic. Clearly B is a linear subspace of A^*.

Let $\{b_\theta\}$ be a directed system in B such that $b_\theta \to f$ weak* in A^* and $\{\|b_\theta\|\}$ is bounded. By virtue of our hypothesis on φ, the directed system of functions $\{b_\theta\circ\varphi\}$ is uniformly bounded on every compact subset of M. Since it converges pointwise to $f\circ\varphi$ the latter is analytic. Thus $f \in B$. This proves that B is closed with respect to norm-bounded weak* convergence. By a well-known theorem of Banach [5, p. 39], B is weak* closed in A^*. But by hypothesis B is total, so $B = A^*$ and Theorem 2.11 applies.

There are a number of cases in which the boundedness required by Theorem 2.13 can be deduced from the Banach-Steinhaus theorem. We shall give only the case of linear operators from one Banach space to another, but it is clear that similar theorems apply to a great range of spaces of multilinear functionals.

2.14 THEOREM. *Let A and B be Banach spaces. Let L be the Banach space of all bounded linear operators from A to B with the operator norm. Suppose φ is a map of a complex-analytic manifold M into L such that, for all $a \in A$ and all $b^* \in B^*$, the function $p \to b^*(\varphi(p)(a))$ is analytic. Then φ is analytic.*

Proof. If K is a compact subset of M, then two applications of the Banach-Steinhaus theorem show that $\varphi(K)$ is norm bounded in L. Thus 2.13 applies.

2.15. We conclude this section by noting that analyticity of maps into Banach spaces can equally well be defined in terms of differentiability. Suppose for example that φ maps a complex-analytic manifold M into a Banach space A and that φ has first partial derivatives in terms of some local coordinate system either in the norm or weakly. Then Hartogs' theorem implies that $a^* \circ \varphi$ is an analytic function on the coordinate domain for every $a^* \in A^*$ and therefore φ is analytic on that domain. The same considerations apply to weak* differentiability for functions into conjugate spaces or spaces of linear operators.

3. We shall now study the analytic maps of a complex-analytic manifold into some of the classical function spaces. For the sake of future applications we give the results in a more general form that we shall need.

3.1. Starting with a measure space $\langle X, \mathscr{E}, \mu \rangle$ we shall be concerned with the associated Banach spaces L^p, $1 \leqq p \leqq \infty$ of classes of pth power integrable (or bounded) measurable functions. Our work requires that we distinguish between a function and the member of L^p which it represents.

3.2 DEFINITION. Let $\langle X, \mathscr{E}, \mu \rangle$ be a measure space and let M be a topological space. A function defined on $M \times X$ will be called measurable with respect to μ if and only if it is measurable with respect to the least σ-field containing all sets of the form $G \times E$ where $E \in \mathscr{E}$ and G is a Baire open set in M. (That is, a set of the form $\{m \colon f(m) > 0\}$ for some continuous real function f defined on M. All open sets have this form if M is metrizable.)

3.3 LEMMA. *Let $\langle X, \mathscr{E}, \mu \rangle$ be a measure space and let M be a complex-analytic manifold. Let P be a bounded measurable complex-*

valued function on $M \times X$ *such that*

(a) $(\forall x \in X) m \to P(m, x)$ *is analytic*

and

(b) $(\forall m \in M) x \to P(m, x)$ *is integrable.*

Then, for any set E *in* $\mathscr{E}$ *of finite measure,* $\int_E P(m, x) d\mu(x)$ *is an analytic function of* m.

Proof. Let f be the function defined by the integral. Choose any point m_0 on M and a system z of local coordinates centered at m_0. Say that the range of z contains the closed polycylinder $\{\lambda: |\lambda_j| \leqq \rho\}$ and let $T = \{\lambda: |\lambda_1| = \rho\}$. For each $x \in X$ and all m near m_0, the iterated Cauchy formula gives

$$P(m, x) = \frac{1}{(2\pi i)^n} \int_T \frac{P(q, x) d\lambda_1 \cdots d\lambda_n}{(\lambda_1 - z_1(m)) \cdots (\lambda_n - z_n(m))}$$

where q represents the point whose coordinates are $\lambda_1, \cdots, \lambda_n$. Under our hypotheses Fubini's theorem applies when we integrate over the set E. Interchanging the order of integration gives

$$f(m) = \frac{1}{(2\pi i)^n} \int_T \frac{f(q) d\lambda_1 \cdots d\lambda_n}{(\lambda_1 - z_1(m)) \cdots (\lambda_n - z_n(m))}$$

which shows that f is analytic near m_0 since f is bounded and the integral expands in a power series by the usual arguments. Since m_0 was arbitrary, f is an analytic function.

3.4 LEMMA. *Let* $\{\bar{f}_{(j)}\}$ *be a multiple sequence of points in* L^p, $1 \leqq p \leqq \infty$, *and let* $\{f_{(j)}\}$ *be functions representing these points respectively. Then, for almost all* $x \in X$,

$$\limsup_{|j| \to \infty} (|f_{(j)}(x)| / \|\bar{f}_{(j)}\|)^{1/|j|} \leqq 1 . \tag{3.5}$$

(If $\|\bar{f}\| = 0$, we interpret $0/0 = 1$).

Proof. The result is trivial for $p = \infty$, so we assume $p < \infty$. Suppose that $t > 1$ and that g is a pth power integrable function. Consider the set $E = \{x: |g(x)| > t \|\bar{g}\|_p\}$. Evidently $\mu(E) t^p \|\bar{g}\|_p^p \leqq \int |g|^p = \|\bar{g}\|_p^p$ and therefore $\mu(E) \leqq 1/t^p \leqq 1/t$. (If $\|\bar{g}\|_p = 0$, then $\mu(E) = 0$, surely.)

Suppose $0 \leqq \rho < 1$ and let $E_{(j)} = \{x: |f_{(j)}(x)| > \rho^{-|j|} \|\bar{f}_{(j)}\|_p\}$. Then $\mu(E_{(j)}) \leqq \rho^{|j|}$. Therefore

$$\mu(\textstyle\bigcup_{|j|>k} E_{(j)}) \leqq \sum_{|j|>k} \rho^{|j|} = \sum_{h>k} \binom{h+n-1}{h} \rho^h .$$

Since the latter tends to zero as k increases, the set $D_\rho = \bigcap_k \bigcup_{|j|>k} E_{(j)}$ has measure zero. For points $x \notin D_\rho$, the inequality $|f_{(j)}(x)| \leqq \rho^{-|j|} \| \bar{f}_{(j)} \|_p$ is violated for at most a finite number of (j). Hence (3.5) holds except on $\bigcup_m D_{1-1/m}$, which is a null set.

3.6 THEOREM. *Let $\langle X, \mathscr{E}, \mu \rangle$ be a measure space and let L^p be the associated Lebesgue space where $1 \leqq p \leqq \infty$. Let ψ be a map of a separable complex-analytic manifold M into L^p which takes compact sets in M into bounded sets in L^p. Then ψ is analytic if and only if there exists a measurable function Q on $M \times X$ such that*

(a) $(\forall x \in X) m \to Q(m, x)$ *is analytic*

and

(b) $(\forall m \in M) x \to Q(m, x)$ *represents* $\psi(m)$.

Proof. Sufficiency. Let K be a compact subset of M and let $m_1, m_2, \cdots$ be a dense sequence of points in K. Then

$$\sup_{m \in K} |Q(m, x)| = \sup_i |Q(m_i, x)|$$

The right hand expression is clearly a measurable function of x, hence

$$E_k = \{x : \sup_{m \in K} |Q(m, x)| \leqq k\}$$

is a measurable set in X for each integer k.

Let q be the index conjugate to p. Let g be the characteristic function of some set of finite measure contained in E_k for some k. Now g represents a member $\bar{g}$ of L^q and the usual pairing of L^p and L^q gives

$$\langle \psi(m), \bar{g} \rangle = \int Q(m, x) g(x) d\mu(x)$$

which is an analytic function of m on Int K by Lemma 3.3. Since $\bigcup_k E_k = X$, the collection of all such functions g determines a total system of linear functionals on L^p. Then Theorem 2.13 asserts that ψ is analytic on Int K. Thus ψ is analytic in a neighborhood of any point of M and is therefore analytic.

Necessity. Suppose ψ is an analytic map of M into L^p.

Choose any point m_0 of M and let z be a local coordinate system centered at m_0. Let ψ be expanded in the series

$$\sum \bar{f}_{(j)} z(m)^{(j)} \tag{3.7}$$

which converges, say, in the region N_0 determined by the inequalities $|z_i(m)| < \rho$, $i = 1, 2, \cdots, n$. Choose representative functions $f_{(j)}$ for $\bar{f}_{(j)}$.

Applying 2.4 and 3.4 we see that

$$\sum f_{(j)}(x)z(m)^{(j)} \tag{3.8}$$

converges for $m \in N_0$ unless x is in a certain set Y_0 of measure zero. We define Q_0 by (3.8) on $N_0 \times (X - Y_0)$ and let Q_0 be 0 on $N_0 \times Y_0$.

For a fixed value of $m \in N_0$, the partial sums of (3.8) converge almost everywhere to $Q_0(m, x)$. These partial sums represent the partial sums of (3.7) which converge to $\psi(m)$ in the norm of L^p. It follows that $x \to Q_0(m, x)$ represents $\psi(m)$. Moreover, the individual terms and hence the partial sums of (3.8) are measurable on $M \times X$, so Q_0 is measurable. This solves the problem locally on M.

To obtain a global function Q we select a sequence of functions $\{Q_i\}$, with Q_i defined on $N_i \times X$ where N_i is an open set in M, so that $\bigcup N_i = M$ and conditions (a) and (b) hold mutatis mutandis. If $N_i \cap N_j \neq \Lambda$, then choose a dense sequence $\{m_s\}$ in $N_i \cap N_j$. Since both sides represent $\psi(m_s)$, $Q_i(m_s, x) = Q_j(m_s, x)$ unless x is in the certain null set A_s. Since Q_i and Q_j are continuous in the first argument, $Q_i(m, x) = Q_j(m, x)$ for all $m \in N_i \cap N_j$ unless $x \in \bigcup_s A_s = B_{i,j}$. Hence, if $x \notin \bigcup_{i,j} B_{i,j}$ we can define $Q(m, x) = Q_i(m, x)$ for any i such that $m \in N_i$. For other values of x take $Q(m, x) = 0$. Now (a) and (b) hold and Q is measurable. This concludes the proof.

3.9. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of complex-valued continuous functions of X with the usual norm. If μ is a positive Radon measure on X, then the space $L^1(\mu)$ is isometrically embedded in $C(X)^*$ by the map ξ defined by

$$(\xi\bar{f})(g) = \int_X f(x)g(x)d\mu(x)$$

where $g \in C(X)$, $\bar{f} \in L^1(\mu)$, and f is any representative of $\bar{f}$. We shall refer to the ξ-image of $L^1(\mu)$ as an L^1 subspace of $C(X)^*$. The Riesz representation theorem and the Radon-Nikodyn theorem together characterize the image as consisting of those measures which are absolutely continuous with respect to μ.

3.10 LEMMA. *Every subset of $C(X)^*$ which is separable in the norm topology is contained in an L^1 subspace.*

Proof. Let E be a separable subset of $C(X)^*$ and let $e_1, e_2, \cdots$ be a sequence in E which is dense in the norm topology. Each of the functional e_i is determined by a Radon measure π_i on X, and π_i is absolutely continuous with respect to some positive Radon measure μ_i. Now put $\mu = \sum \mu_i/\mu_i(X)2^i$. Evidently each of the measures π_i is absolutely continuous with respect to μ. Since the map ξ given in 3.9 is an isometry, its range is a closed linear manifold in $C(X)^*$ which contains $e_1, e_2, \cdots$, and therefore all of E.

3.11 THEOREM. *Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of complex-valued continuous functions on X. Let ψ be an analytic map of a separable complex-analytic manifold M into $C(X)^*$. There exists a positive Radon measure μ on X and a complex-valued function Q on $M \times X$ measurable with respect to μ such that*

(a) $(\forall x \in X) m \rightarrow Q(m, x)$ *is analytic on M*

(b) $(\forall m \in M) x \rightarrow Q(m, x)$ *is μ-integrable*

(c) $m \rightarrow \int_X |Q(m, x)|\, d\mu(x)$ *is continuous on M*

and

(d) $$(\forall f \in C(X))\ \psi(m)(f) = \int_X Q(m, x) f(x) d\mu(x)\ .$$

Proof. The range of ψ is a separable subset of $C(X)^*$ because ψ is continuous in the norm. By Lemma 3.10 there is a Radon measure μ such that the image of $L^1(\mu)$ in $C(X)^*$ contains $\psi(M)$. Hence we may regard ψ as the composition of the map ξ of 3.9 and an analytic map ψ_0 of M into $L^1(\mu)$. Now the theorem follows from Theorem 3.6 applied to the map ψ_0.

4. We shall now turn our attention to the lifting problem

$$\begin{tikzcd} & A \arrow[d, "\eta"] \\ M \arrow[ur, dashed, "?"] \arrow[r, "\varphi"'] & B \end{tikzcd}$$

where η is a continuous linear projection of A onto B and φ is an analytic map of M into B. The required map is to be an analytic map ψ of M into A such that $\eta \circ \psi = \varphi$.

It is known [5, p. 42] that there is a constant K such that for any $b \in B$ we can choose $a \in A$ with $\eta(a) = b$ and $||a|| \leqq K||b||$. If we look at the local representation of φ as a power series, say $\varphi = \sum b_{(j)} z^{(i)}$, we can lift the coefficients one at a time choosing $a_{(j)}$ so that $\eta(a_{(j)}) = b_{(j)}$ and $||a_{(j)}|| \leqq K||b_{(j)}||$. The new series $\sum a_{(j)} z^{(j)}$ has the same region of convergence as the old and defines a function ψ mapping an open set in M analytically into A satisfying $\eta \circ \psi = \varphi$. Thus the lifting problem is trivial locally. The global problem can therefore be treated as a problem in homology using coefficients in the kernel of the projection η. We shall not pursue this attack, however. It is easier to construct a continuous lifting of φ and then smooth it to an analytic map using a kernel function.

In this section it is not important to distinguish between a function which is square integrable and its equivalence class which is a member of L^2.

4.1 DEFINITION. Let M be a (real) manifold of class C'. A measure μ on M will be called proper if and only if it is a Borel measure and, for any local coordinate system x on M having domain D, the Lebesgue measure ν on D determined by x is absolutely continuous with respect to μ, and its Radon-Nikodym derivative with respect to μ is bounded on compact subsets of D.

4.2 LEMMA. *Let M be a manifold of class C' satisfying the second axiom of countability. Let f be any continuous complex-valued function on M. There exists a proper measure μ on M such that $f \in L^2(\mu)$.*

Proof. Since M is separable and paracompact we can find a countable, locally finite partition $\{g_i\}$ of unity such that each function g_i is supported on a compact subset of the domain of a single coordinate system $\langle x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)}\rangle$. Let μ_i be the measure on M defined by $g_i \,|\, dx_1^{(i)} dx_2^{(i)} \cdots dx_n^{(i)} \,|$ and put $\mu_0 = \sum \mu_i / 2^i \mu_i(M)$. Then μ_0 is a finite measure on M and it is clear that on the domain of any local coordinate system x it is a locally finite sum of nonnegative continuous functions multiplied by the Lebesgue measue ν determined by x with at least one of the functions positive at each point. The Radon-Nikodym derivative of ν with respect to μ_0 is therefore an everywhere positive continuous function. Therefore μ_0 is a proper measure. If we divide μ_0 by the positive continuous function $1 + |f|^2$ we obtain a new proper measure μ for which $f \in L^2(\mu)$.

4.3 LEMMA. *Let N be a neighborhood of the origin in C^n. There is a neighborhood P of the origin in C^n and a bounded Borel measurable function ψ on N having compact support in $N - P$ such that, for every complex analytic function f defined on N and every $\lambda \in P$*

$$f(\lambda) = \int_N \frac{f(\sigma)\psi(\sigma)}{(\sigma_1 - \lambda_1) \cdots (\sigma_n - \lambda_n)} \, d\nu(\sigma) \tag{4.4}$$

where ν stands for Lebesgue measure.

Proof. There is no loss of generality in assuming that N is the polycylinder $\{\lambda: |\lambda_j| < a, j = 1, 2, \cdots, n\}$. Choose numbers b and c so that $0 < b < c < a$ and take P as the polycylinder $\{\lambda: |\lambda_j| < b\}$.

Consider to begin with the case $n = 1$. From the ordinary Cauchy integral formula we see that, for every g analytic on N and every $\lambda \in P$,

$$g(\lambda) = \frac{1}{2\pi(c-b)} \int_b^c \left[\int_0^{2\pi} \frac{g(\rho e^{i\theta})\rho e^{i\theta}}{\rho e^{i\theta} - \lambda}\, d\theta\right] d\rho$$
$$= \frac{1}{2\pi(c-b)} \int_D \frac{g(\sigma)\sigma}{\sigma - \lambda}\, \frac{d\nu_1(\sigma)}{|\sigma|}$$

where ν_1 is the Lebesgue measure in the plane and D is the annulus $\{\lambda: b \leqq |\lambda| \leqq c\}$. If ψ_0 is the function $z(2\pi(c-b)|z|)^{-1}$ on D and 0 elsewhere, then $g(\lambda) = \int_N g(\sigma)\psi_0(\sigma)(\sigma - \lambda)^{-1} d\nu_1(\sigma)$.

In the general case put $\psi(\sigma) = \psi(\sigma_1, \sigma_2, \cdots, \sigma_n) = \psi_0(\sigma_1)\psi_0(\sigma_2)\cdots\psi_0(\sigma_n)$ and we obtain (4.4) by breaking the $2n$-dimensional integral into an n times iterated 2-dimensional integral.

4.5 THEOREM. *Let M be a complex-analytic manifold. Let μ be a proper measure on M. Let $\boldsymbol{H}$ be the set of all complex-analytic functions on M which are in $L^2(\mu)$. Then $\boldsymbol{H}$ is a closed linear manifold in $L^2(\mu)$. For any point $p \in M$ the valuation functional $\chi_p: f \to f(p)$ is continuous in the L^2 norm on $\boldsymbol{H}$. The map $p \to \chi_p$ of M into $\boldsymbol{H}^*$ is analytic. There exists a Hermitian symmetric continuous function K on $M \times M$ such that, for each $q \in M$, $p \to K(q, p)$ is in $\boldsymbol{H}$ and, for every function g in $L^2(\mu)$, the orthogonal projection f of g on $\boldsymbol{H}$ is given by*

$$f(p) = \int_M K(p, q) g(q) d\mu(q) \; . \tag{4.6}$$

Proof. Let q be any point of M and let z be a local coordinate system centered at q with domain N. By 4.3 there is a neighborhood P of q and a bounded Borel measurable function ψ with compact support in $N - P$ such that, for every complex-analytic function f defined on N and every, $p \in P$,

$$f(p) = \int_N \frac{f(r)\psi(r)}{(z_1(r) - z_1(p))\cdots(z_n(r) - z_n(p))}\, d\nu(r) \tag{4.7}$$

since μ is a proper measure we can write $d\nu = \zeta d\mu$ where ζ is a Borel measurable function bounded on the support of ψ. Evidently the formulas

$$\theta_p(r) = \psi(r)\zeta(r)(z_1(r) - z_i(p))^{-1}\cdots(z_r^n - z_n(p))^{-1} \text{ for } r \in N$$
$$\theta_p(r) = 0, \text{ for } r \notin N$$

define a square integrable function and (4.7) becomes $f(p) = \langle f, \bar{\theta}_p \rangle$ for

$f \in \boldsymbol{H}$ where $\langle , \rangle$ represents the inner product and the bar stands for complex conjugation. This shows that the valuation functional χ_p is bounded in the L^2 norm.

Moreover, $\|\theta_p\|$ is bounded for p in any compact subset of P. Hence, if $\{f_n\}$ is a sequence in $\boldsymbol{H}$ which converges in the L^2 norm to g, then $f_n \to g$ pointwise and uniformly on compact subsets of P. Therefore g is analytic on a neighborhood of q. Since q is arbitrary g is analytic on M. This proves that $\boldsymbol{H}$ is closed in $L^2(\mu)$.

The map $p \to \chi_p$ is analytic by virtue of Theorem 2.8 and a fortiori continuous. The map which identifies linear functionals on $\boldsymbol{H}$ with vectors of $\boldsymbol{H}$ is a conjugate linear isometry, hence if we choose, for each p, $h_p \in \boldsymbol{H}$ so that $f(p) = \langle f, h_p \rangle$, the map $p \to h_p$ is continuous. Now set $K(p, q) = \langle h_q, h_p \rangle = h_q(p)$. It is obvious that K is Hermitian symmetric and continuous on $M \times M$. If g is any member of $L^2(\mu)$ and f is its orthogonal projection on $\boldsymbol{H}$, then, for any $p \in M$, we have $\langle g, h_p \rangle = \langle f, h_p \rangle + \langle g - f, h_p \rangle = \langle f, h_p \rangle = f(p)$. Writing the inner product as an integral this becomes (4.6).

4.8 THEOREM. *Let A and B be Banach spaces and let η be a linear map of A onto B. Let M be a complex-analytic manifold and let φ be an analytic map of M into B. There exists an analytic map ψ of M into A such that $\varphi = \eta \circ \psi$.*

Proof. It has been shown by Bartle and Graves [2] that there is a continuous map ξ of B into A such that $\eta \circ \xi$ is the identity. Hence, if $\psi_0 = \xi \circ \varphi$, then ψ_0 is a continuous, but not necessarily analytic, solution of the lifting problem.

Each component of a complex-analytic manifold is separable and we may deal with the components of M one at a time in constructing ψ, so we may assume that M is separable. According to Lemma 4.2 we can choose a proper measure μ on M such that the function $p \to \|\psi_0(p)\|$ is in $L^2(\mu)$. The norm inequality shows that $a^* \circ \psi_0$ is in $L^2(\mu)$ for every $a^* \in A^*$; moreover, the function $b^* \circ \varphi$ is analytic and in $L^2(\mu)$ for every $b^* \in B^*$. Let K be the kernel function associated with μ as in Theorem 4.5 and put

$$\psi(p) = \int_M K(p, q)\psi_0(q)d\mu(q) .$$

Since the integrand is a continuous function and the norm of the integrand is itself integrable, this integral exists in the sense of norm convergence of appropriate approximating sums.

Now, for any $a^* \in A^*$, $a^*\psi(p) = \int K(p, q)a^*\psi_0(q)dq(\mu)$ which is analytic function of p by Theorem 4.5. This proves that ψ is an analytic map of M into A, using Theorem 2.11. Finally, for any $b^* \in B^*$, $b^*\eta\psi(p) = \int K(p, q)b^*\eta\psi_0(q) = \int K(p, b)b^*\varphi(q)d\mu(q) = b^*\varphi(p)$ according to 4.5. This proves that $\eta\circ\psi = \varphi$ and concludes the proof.

5. We can now prove our abstract version of the Cauchy-Weil integral formula after a brief discussion of the Shilov boundary.

5.1. Let $\boldsymbol{A}$ be a Banach algebra with unit. The unitary homomorphisms of $\boldsymbol{A}$ into the complex numbers are a subset H of the dual space $\boldsymbol{A}^*$ which is compact in the weak * topology. In what follows H is endowed with this topology.

With every $a \in \boldsymbol{A}$, we can associate a continuous function $\hat{a}$ on H by defining $\hat{a}(h) = h(a)$. The mapping $a \to \hat{a}$ is then a continuous homomorphism of $\boldsymbol{A}$ into the algebra $C(H)$. It is known that $\|\hat{a}\| \leqq \|a\|$, but inequality is possible, in fact the map need not even be one-to-one.

From now on suppose that $\boldsymbol{A}$ is a unitary subalgebra of the bounded continuous functions on some space X. We define a map φ of X into H by $\varphi(x)(a) = a(x)$ for all $a \in A$. It is easily seen that φ is continuous and that φ is one-to-one if and only if $\boldsymbol{A}$ separates the points of X. Evidently $\hat{a}\circ\varphi = a$ for any $a \in \boldsymbol{A}$, and therefore $\|\hat{a}\| \geqq \|a\|$. Hence the map $a \to \hat{a}$ is an isometry in this case.

For any set $J \subseteqq H$ we can form $\|\hat{a}\|_J = \sup\{|\hat{a}(h)|: h \in J\}$. It turns out that among the compact subsets of H there is at least one, B, for which $\|\ \|_B = \|\ \|_H$. This set is called the Shilov boundary of $\boldsymbol{A}$.

As we showed above $\|\hat{a}\|_{\varphi(X)} = \|a\|_H$ for all $a \in A$ and therefore $B \subseteqq \overline{\varphi(X)}$. If X is compact and φ is one-to-one, then $\varphi(X) = \overline{\varphi(X)}$ and φ is a homeomorphism. Therefore we can regard B as a subset of X. If X is compact but φ is not one-to-one, then every function in $\boldsymbol{A}$ achieves its norm on the set $\varphi^{-1}(B)$, but this set need not be minimal. In general there will be many minimal compact subsets of X which carry the norms of the function in $\boldsymbol{A}$.

For a more complete discussion of the Shilov boundary see [6].

5.2 THEOREM. *Let M be a separable complex-analytic manifold ane let $\boldsymbol{A}$ be the algebra of bounded complex-analytic functions on M. Let B be the Shilov boundary of the ideal space of $\boldsymbol{A}$. There exists a positive Radon measure μ on B and a μ-measurable function Q on $M \times B$ such that*

(a) $(\forall b \in B)$ $m \to Q(m, b)$ *is analytic*

(b) $(\forall m \in M)$ $b \to Q(m, b)$ *is μ-integrable*

(c) $m \to \int_B |Q(m, b)|\, d\mu(b)$ *is continuous, and*

(d) *for any function f in $\boldsymbol{A}$ and any m in M,*

$$f(m) = \int_B Q(m, b)\hat{f}(b)d\mu(b) ,$$

where $\hat{f}$ represents the function induced by f on B.

Proof. The map $f \to \hat{f}$ embeds $\boldsymbol{A}$ isometrically in $C(B)$. Denote the transpose of this map by η; it is a continuous linear map of $C(B)^*$ onto $\boldsymbol{A}^*$.

Define the map φ of M into $\boldsymbol{A}^*$ by $\varphi(m)(f) = f(m)$ for all f in $\boldsymbol{A}$ and all m in M. According to Theorem 2.8, φ is an analytic mapping. By Theorem 4.8 we can find an analytic map ψ of M into $C(B)^*$ such that $\eta\circ\psi = \varphi$. Finally Theorem 3.11 tells us that ψ can be represented by a kernel. Conditions (a), (b), and (c) of 3.11 become (a), (b), and (c) above while (d) above comes from 3.11 (d) and $f(m) = \varphi(m)(f) = \eta\psi(m)(f) = \psi(m)(\hat{f})$.

5.3 REMARK. It follows from the minimality properties of the Shilov boundary that, for any nonempty open set V of B, $\mu(V) > 0$.

5.4 COROLLARY *Let G be an open subset of a complex-analytic manifold M having compact closure. Let $\boldsymbol{A}$ be the algebra of continuous functions on $\bar{G}$ which are analytic on G. There is a subset B of $\bar{G} - G$, a positive Radon measure μ on B, and a μ-measurable function Q on $G \times B$ such that* (a), (b), *and* (c) *of* 5.2 *hold (with M replaced by G) and*

(d) *for any function f in $\boldsymbol{A}$ and any m in G*

$$f(m) = \int_B Q(m, b)f(b)d\mu(b) .$$

Proof. The argument of 5.2 holds provided only that the restriction map which sends $\boldsymbol{A}$ into $C(B)$ is an isometry. The maximum modulus principle shows that we can take $B = \bar{G} - G$. There may be smaller sets which will work equally well. If A separates points in $\bar{G}$ then there will be a least B which will do; it is the inverse image of the Shilov boundary under the natural embedding φ of $\bar{G}$ into $\boldsymbol{A}^*$. If A does not separate points there may be more than one minimal set B.

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HARVARD UNIVERSITY