REMARKS ON AFFINE SEMIGROUPS

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A problem of fundamental importance in the study of measure semigroups is the following: if S is a compact topological semigroup and $\tilde{S}$ is the convolution semigroup (with the weak-$*$ topology) of nonnegative normalized regular Borel measures on S, what relationship exists between a measure μ in $\tilde{S}$ and its carrier? In the paper numbered [9, Lemma 5], Wendel proved that when S is a group and μ an idempotent in $\tilde{S}$, then carrier μ is a group and μ is Haar measure on carrier μ. He proved further that the mapping $\mu \to$ carrier μ is a one-to-one mapping from the set of idempotents of $\tilde{S}$ onto the set of closed subgroups of S. Glicksberg in [6] extended these results to the case when S is an abelian semigroup, In addition he showed (when S is a group or an abelian semigroup) the structure of the closed subgroups of $\tilde{S}$ to be quite simple: each closed subgroup of $\tilde{S}$ consists of the G-translates of Haar measure on some closed normal subgroup of a suitably chosen closed group G of S.

It is our purpose in this paper to prove in § 2 that these properties are equivalent in general, each being equivalent to several other properties of some interest (see Theorem 2). One of these conditions is the geometric requirement that $\tilde{S}$ can contain no 'parallelogram' whose vertices are μ, ν, $\mu\nu$, and $\nu\mu$, with all four of these measures idempotent and μ and ν distinct. A crucial lemma of independent interest is that found in Theorem 1 of § 1, where it is shown that a line segment of an affine semigroup (see [3] for definitions) which contains three distinct idempotents consists entirely of idempotents. Several corollaries are drawn from this theorem, among them the result that a compact affine semigroup consists of idempotents (i.e., is a *band* in the sense of [2]) if and only if it is rectangular, and that this occurs if and only if it is *simple* (i.e., contains no proper ideals).

References for terminology and notation used here may be found in [3, 6, 8, 9].

1. General affine semigroups. This section is devoted primarily to several results about general affine semigroups. However, we list first without proof two lemmas ([3, Theorem 3] and [4, Theorem 2]) needed in the sequel.

LEMMA 1. *Let T be a compact affine topological semigroup and*

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let K be its kernel (=minimal ideal). Then

(a) *Each minimal left or right ideal is convex.*

(b) $x \in K$ *if and only if* $xTx = \{x\}$; *in particular, each point of K is idempotent.*

LEMMA 2. *Let S be a compact topological semigroup and μ be an idempotent in $\tilde{S}$. Then $H =$ carrier μ is a compact simple semigroup, and for each continuous complex function f on S the mapping $y \to \int f(xy)d\mu(x)$ is constant on each minimal left ideal of H.*

THEOREM 1. *Suppose T is an affine semigroup and L is a line segment in T. If there exist three distinct idempotents on L, then L consists entirely of idempotents, and $xLx = \{x\}$ for all $x \in L$.*

Proof. Let e, f, and g be distinct idempotents on L, with e between f and g. Then there exists $0 < a < 1$ such that $e = af + (1-a)g$, so $af + (1-a)g = e = e^2 = a^2f + a(1-a)fg + a(1-a)gf + (1-a)^2g$. Multiplication on the left by f yields $af + (1-a)fg = a^2f + a(1-a)fg + a(1-a)fgf + (1-a)^2fg$, or $af = a^2f + a(1-a)fgf$. Rewriting this as $a(1-a)f = a(1-a)fgf$ and using the fact that a is neither zero nor one, we obtain $f = fgf$. By similar arguments one can show $gfg = g$, and it follows that both fg and gf are idempotents. Again using the fact that e is an idempotent, $af + (1-a)g = e = e^2 = a^2f + a(1-a)gf + a(1-a)fg + (1-a)^2g$. This can be rewritten as $a(1-a)f + a(1-a)g = a(1-a)fg + a(1-a)gf$, so $f + g = fg + gf$. If now x is any point on L, say $x = bf + (1-b)g$, then $x^2 = b^2f + b(1-b)fg + b(1-b)gf + (1-b)^2g = b^2f + b(1-b)[f + g - gf] + b(1-b)gf + (1-b)^2g = bf + (1-b)g = x$, so each $x \in L$ is an idempotent. By direct computation it then follows readily that $xLx = \{x\}$, all $x \in L$.

COROLLARY 1. *Every element of an affine semigroup T is idempotent if and only if $xTx = \{x\}$, all $x \in T$. In addition, if T is a compact affine topological semigroup, the requirement that T be simple (i.e., T is its own kernel) is equivalent to each of the above conditions.*

Proof. If $xTx = \{x\}$ for all x in T, then $x^3 = x$, so $x^2 = x^3x = xx^2x = x$. Conversely, if T consists of idempotents, fix x in T. Then y in T implies the line segment L joining x and y contains more than two idempotents, so by the theorem $xyx \in xLx = \{x\}$; i.e. $xTx = \{x\}$.

When T is compact, Lemma 1 shows that T is simple if and only if $xTx = \{x\}$, all $x \in T$.

COROLLARY 2. *When T is the convolution semigroup $\tilde{S}$ of measures* *on a compact semigroup S, then each of the conditions of the preceding corollary is equivalent to each of* (1) *the multiplication in S is either left or right singular; i.e., $xy = x$ for all $x, y \in S$ or $xy = y$ for all $x, y \in S$,* (2) *the multiplication in $\tilde{S}$ is either left or right singular.*

Proof. It was shown in [5, Corollary 3] that $\tilde{S}$ is simple if and only if (1) holds. Now it is clear that here (2) implies (1) since S is a semigroup of $\tilde{S}$. To show the converse, let C be the convex hull of S in $\tilde{S}$. It is known [2, Lemmas 3.1 and 3.2] that C is dense in $\tilde{S}$. From this fact and the requirements on the multiplication in $\tilde{S}$ it follows readily that (1) implies (2).

COROLLARY 3. *If T is an affine semigroup, the following are mutually equivalent*:

(1) *there exist three distinct collinear idempotents in T.*

(2) *there exist distinct idempotents f and g in T such that fg and gf are also idempotents and $f + g = fg + gf$.*

(3) *T contains an affine semigroup affinely equivalent to either the closed unit square of the Euclidean plane under the multiplication $(x, y)(a, b) = (x, b)$ or the closed unit interval of reals under left or right singular multiplication.*

Proof. It was seen in the proof of Theorem 1 that (1) implies (2,) To prove (2) implies (3), let f and g be distinct idempotents, with fg and gf idempotent and $f + g = fg + gf$. Denote by M the manifold generated by $\{f, g, gf\}$ (i.e., M is composed of all sums of the form $af + bg + cgf$, with $a + b + c = 1$). Since $fg = f + g - gf$, it follows that $M \cap T$ contains the convex hull C of $\{f, g, fg, gf\}$. If gf is on the line through f and g, say $gf = af + (1 - a)g$, then $gf = gff = af + (1 - a)gf$, so $af = agf$. If $a = 0$, then $gf = g$ and $fg = f + g - gf = f + g - g = f$. It is then easy to see that the closed line segment L from f to g is a semigroup, with left singular multiplication. If $a \neq 0$, then $gf = f$ and $fg = f + g - gf = f + g - f = g$. In this case L is a semigroup, with right singular multiplication.

In the alternate case, gf is not on the line through f and g. We use here the identities $gfg = g$ and $fgf = f$ (easily deducible from the equation $fg + gf = f + g$) to show that if x and y are any points of C (say $x = af + bg + [1 - (a + b)]gf$ and $y = cf + dg + [1 - (c + d)]gf$, where $a, b, c, d \geqq 0$), then $xy = af + dg + [1 - (a + d)]gf$. The mapping $x \rightarrow (a, b)$ can now be easily verified to be an affine equivalence between C and the unit square, where the latter is given the multiplication $(a, b)(c, d) = (a, d)$. Thus (3) holds.

The final implication (3) implies (1) is obvious, for each of the three affine semigroup mentioned in (3) clearly contain entire line segments of idempotents.

COROLLARY 4. *The kernel K of a compact affine topological semigroup T is non-convex if and only if there exist distinct points x and y of K such that the open line segment between x and y misses K.*

Proof. If such a pair of points exists it is obvious that K is non-convex. Conversely, if K is non-convex one can find distinct points x and y of K and a point of T outside K on the open line segment L joining x to y. It is then clear (since by Lemma 1 every point of K is an idempotent) that L misses K, for if L and K meet Theorem 1 implies $\{z\} = zLz = zxz \in zKz \subset K$, for all $z \in L$. This concludes the proof.

The preceding corollary shows that the examples of nonconvex kernels given in [3, pp. 111–112] were the only possible kind, for in both of these the non-convexity was shown by exhibiting points x and y such that L missed K. It seems likely that the only way in which a kernel can fail to be convex is for there to be in T a usual real interval semigroup whose two idempotents are in K.

2. Measure semigroups. Preliminary to our main Theorem 2, several lemmas will be stated and proved. Throughout this section $\tilde{S}$ will be (as before) the convolution semigroup of measures on a compact semigroup S. Recall that the carrier of a measure μ in $\tilde{S}$ is the complement of the largest open set of S whose μ measure is zero. A result needed repeatedly is the fact that the carrier of a product of two measures is the product of the carriers [6, Lemma 2.1]. We say, following Wallace, that a semigroup of S is *simple* if it contains no proper (two-sided) ideals. The proof of the following lemma is obvious, and is omitted. In Lemma 4, the *carrier* of a subset Γ of $\tilde{S}$ is the closure of the set $\cup\,\{\text{carrier } \mu : \mu \in \Gamma\}$.

LEMMA 3. *Let H be a compact semigroup of S, let $\tilde{H}$ denote the semigroup of measures on H, and let H' be the set of measures $\mu \in \tilde{S}$ such that carrier $\mu \subset H$. Then $\tilde{H}$ and H' (the latter with the multiplication and topology inherited from $\tilde{S}$) are affinely equivalent (both topologically and algebraically).*

LEMMA 4. *Let Γ be a compact group in $\tilde{S}$ with η its identity element. Let H be the carrier of η, and denote by G the carrier of Γ. Then both H and G are compact simple semigroups of S and G and have the same idempotents. In particular, G is a group if and only if H is; in this case, H is a normal subgroup of G and η is Haar measure on H.*

Proof. If $\mu \in \Gamma$, then $\mu = \eta\mu$, so carrier $\mu = H \cdot$ carrier μ. But then $S_0 = \bigcup \{\text{carrier } \mu : \mu \in \Gamma\} = \bigcup \{H \cdot \text{carrier } \mu : \mu \in \Gamma\} = HS_0$. Similarly $S_0H = S_0$; by compactness and the definition of G, it follows that $G = \bar{S}_0 = \bar{H}\bar{S}_0 = \bar{H} \cdot \bar{S}_0 = HG$ and $G = GH$, where $\bar{A}$ denotes the topological closure of A. We show now that the kernel K of G (G is known to be a semigroup [6, p. 55]) contains H. Let $x \in S_0$. There exists $\mu \in \Gamma$ such that $x \in$ carrier μ, so x carrier μ^{-1}(μ^{-1} denotes the inverse in Γ) $\subset$ carrier $\mu \cdot$ carrier $\mu^{-1} = H$. Thus each set xG meets H, where $x \in S_0$, and by similar arguments Gx meets H for any x in S_0. It is then easily seen that the same is true for $x \in G$. In particular if $x \in K$, there exist $y \in H \cap xG$, $z \in H \cap Gx$, and then $yz \in xG \cdot Gx \subset xGx \subset KGK \subset K$. Thus H and K intersect, so fix $p \in H \cap K$. Since H is simple (Lemma 2), $H = HpH \subset HKH \subset GKG \subset K$. But then $G = GH \subset GK \subset K$, so $G = K$ and G is simple.

To prove G and H have the same idempotents, it suffices (since $H \subset G$) to show $e^2 = e \in G$ implies $e \in H$. By [8, Theorem 4.1], eGe is a maximal group of G, and the argument used above shows H meets eGe. Since H is also simple, there exists $f^2 = f \in H$ such that eGe meets the maximal group fHf of H. However, if two groups meet their identity elements are the same: $e = f$. Thus $e \in H$. Now it is known [8, Theorem 4.3] that a compact simple semigroup is a group if and only if it contains exactly one idempotent; thus it is clear that H is a group if and only if G is.

To conclude the proof, suppose H(hence G) is a group, and let $x \in S_0 \cap$ carrier μ, where $\mu \in \Gamma$. Then x carrier $\mu^{-1} \subset H$, so if $y \in$ carrier μ^{-1}, $z = xy \in H$ implies $x^{-1} = yz^{-1} \in$ carrier μ^{-1}. $H =$ carrier μ^{-1}. Thus $x^{-1}Hx \subset$ carrier $\mu^{-1} \cdot H \cdot$ carrier $\mu = H$ (here all inverses are taken in G). Since this is true for x in the dense subset S_0 of G, it is true also for $x \in G$; i.e., H is normal in G. Finally, it is clear by Lemma 2 that η is Haar measure on H. This completes the proof.

It should be remarked that the above proof of our Lemma 4 owes much to Glicksberg's proof of Theorem 2.3 of [6].

LEMMA 5. *Let H be a compact semigroup such that $\tilde{H}$ contains at most two distinct collinear idempotents. Then the kernel of H is a group.*

Proof. Let μ be in the kernel of $\tilde{H}$. By Lemma 1, μ is idempotent; and $\mu\tilde{H}$ and $\tilde{H}\mu$ are convex. Since here $\tilde{H}$ has at most two collinear idempotents, it is clear that $\mu\tilde{H} = \{\mu\} = \tilde{H}\mu$; i.e., μ is the zero of $\tilde{H}$. But then (since μ is both right and left invariant) Rosen's result [7, Corollary 1] implies the kernel of H is a group.

THEOREM 2. *The following conditions are mutually equivalent:*

(1) *The carrier of each idempotent measure in* $\tilde{S}$ *is a group.*

(2) *No three idempotents of* $\tilde{S}$ *are collinear.*

(3) $\tilde{S}$ *contains no affine image of any of the three semigroups mentioned in Corollary 3,*

(4) *Every compact simple semigroup of* S *is a group.*

(5) *The mapping* $\mu \to \text{carrier}\,\mu$ *is one-to-one onto between the set* $\tilde{E}$ *of idempotents of* $\tilde{S}$ *and the set of compact simple semigroups of* S.

(6) *The mapping* $\mu \to \text{carrier}\,\mu$ *is one-to-one on* $\tilde{E}$.

(7) *Each compact group of* $\tilde{S}$ *consists of the G-translates of Haar measure on a compact normal subgroup of some compact group* G *of* S.

Proof. (1) implies (2). Let $\mu, \nu \in \tilde{E}$, $0 < a$, $0 < b$, and $a + b = 1$ be such that $\phi = a\mu + b\nu \in \tilde{E}$. Let $A = \text{carrier}\,\mu$ and $B = \text{carrier}\,\nu$. By (1), A, B and $A \cup B = \text{carrier}\,\phi$ are groups. It follows then from Lemma 2 that μ, ν, and ϕ are Haar measure on A, B, and $A \cup B$ respectively. Let e, f, and g be the identities of A, B, and $A \cup B$ respectively. It is then clear (since A, B are subgroups of the group $A \cup B$) that $e = f = g$. Suppose there is t in B/A and let $x \in A$. Then $xt \in AB \subset (A \cup B)(A \cup B) \subset A \cup B$, so $xt \in A$ or $xt \in B$. If $xt \in A$, then (inverse of x in A)$\cdot xt \in A$. This implies $t = ft = et \in A$, a contradiction. Thus $xt \in B$, so $x = xe = xf = xt\cdot$(inverse of t in B)$\in BB \subset B$; thus $A \subset B$. But then $A \cup B = B$, so $\text{carrier}\,\phi = B = \text{carrier}\,\nu$. Since normalized Haar measure on the compact group B is unique, it follows that $\phi = \nu$, so (2) is proved.

The equivalence of (2) and (3) follows immediately from Corollary 3 of § 1.

(2) implies (4). Let H be a compact simple semigroup of S. It is clear (assuming (2)) that the H' of Lemma 3 cannot contain three distinct collinear idempotents, so the same is true (by Lemma 3) of $\tilde{H}$. Lemma 5 then implies that H (being its own kernel) is a group.

(4) implies (5). If $H = \text{carrier}\,\mu = \text{carrier}\,\nu$, with $\mu, \nu \in \tilde{E}$, then by (4) and Lemma 2, μ and ν are both normalized Haar measure on the group H. Thus $\mu = \nu$ and the mapping $\mu \to \text{carrier}\,\mu$ is one-to-one on $\tilde{E}$. To complete the proof of (4) implies (5), let H be a compact simple semigroup of S. By (4), H is a group, and then Haar measure μ on H (extended to S, of course) is idempotent and carrier $\mu = H$; i.e., the mapping is onto.

(5) implies (6) is clear. To show (6) implies (2), suppose there exist three distinct collinear idempotents in $\tilde{S}$. There is then by Theorem 1 a nondegenerate line segment L of idempotent measures. In particular then, there exist distinct measures μ and ν on L such that carrier $\mu =$ carrier ν, contradicting (6).

(4) implies (7). Let Γ be a compact group in $\tilde{S}$ with identity element η, let G be the carrier of Γ, and let $H = \text{carrier}\,\eta$. By (4) and

Lemmas 4 and 2, G and H are groups with H normal in G and η is Haar measure on H. Then the proof given by Glicksberg (starting on page 57 of [6] with the phrase "Now suppose S is a (non-abelian) compact group—") applies to our situation to prove (7) holds, for an examination of his proof reveals that all he needs there is that H be a normal subgroup of the group G, with η being Haar measure on H (or one could apply Glicksberg's result to $\tilde{G}$).

To conclude, we show (7) implies (1). Let $\mu^2 = \mu \in \tilde{S}$ and let Γ be the maximal group cantaining μ [8, Theorem 2.1]. Then Γ is a compact group of $\tilde{S}$ so by (7) there are compact groups G and H of S, with H a normal subgroup of G, such that $\Gamma = \eta G$, where η is Haar measure on H. The measure η is then invariant on H ($\eta x = x\eta = \eta$, all $x \in H$), so $\{\eta\} = \eta H \subset \eta G = \Gamma$ implies (Γ being a group)$\eta = \mu$. Thus carrier $\mu =$ carrier $\eta = H$, a group. This completes the theorem.

It has already been remarked that condition (1) of Theorem 2 holds in case S is either a group or an abelian semigroup. More generally, this is true if the idempotents of S commute. In fact, if H is a compact simple semigroup of S and e and f are idempotents of H, then $ef \in Hf \cap eH$ and $fe \in He \cap fH$. Since here $fe = ef$, this says that the maximal groups $eHe = eH \cap He$ and $fHf = fH \cap Hf$ of H meet. However, two maximal groups which meet coincide [8, Theorem 2.1], so $eHe = fHf$ and $e = f$. But then H has exactly one idempotent and so is a group [8, Theorem 4.3].

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