H-SPACES AND DUALITY

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1. Statement of results. It is an interesting question to determine which of the properties universally valid for Lie groups are also universally valid for H-spaces. Typical of such properties is the result, recently proved by one of us [3; Theorem 6.11], that if X is an arcwise connected H-space with finitely generated integral homology, then $\pi_2(X) = 0$, which generalizes a well known result of E. Cartan [4] asserting that the second homotopy group of a Lie group is zero.

In [3; § 7] it is also shown that in an arcwise connected H-space with finitely generated integral homology, the Poincaré duality theorem holds. Thus, such H-spaces seem to have many of the properties of group manifolds, and one can speculate as to whether they have other properties of such spaces. One might conjecture that a manifold which is an H-space is parallelizable. This is false as shown by the following example pointed out to us by J. Milnor.

The S^7 bundles over S^4 with structure group $S0(8)$ are in one-to-one correspondence with the elements of $\pi_3(S0(8))$, which is infinite cyclic [2]. If $\alpha \in \pi_3(S0(8))$, let ξ_α denote the corresponding S^7 bundle over S^4. The fiber homotopy type of ξ_α depends only on $J\alpha \in \pi_{11}(S^8)$ by [5]. Since $\pi_{11}(S^8)$ is a finite group, there is an element $\alpha_0 \in \pi_3(S0(8))$ such that ξ_{α_0} is not trivial but is fiber homotopy trivial (simply take α_0 to be 24 times a generator of $\pi_3(S0(8))$). The Pontrjagin class $p_1 \in H^4(S^4)$ of ξ_{α_0} is nonzero [6]. Let $f: S^1 \times S^3 \rightarrow S^4$ be a map of degree one, and let η be the bundle $f^*(\xi_{\alpha_0})$ induced over $S^1 \times S^3$, and let X be its total space. Since ξ_{α_0} is fiber homotopy trivial, so is η, so its total space X has the homotopy type of $S^1 \times S^3 \times S^7$, which is an H-space, and hence X itself is an H-space. On the other hand, because the Pontrjagin class f^*p_1 of η is nontrivial, it follows that the Pontrjagin class of the tangent bundle to X is nontrivial so X is not parallelizable.

The above example is also an example of a manifold which is an H-space but is not a π-manifold (i.e. its stable tangent bundle is not trivial). However, we shall show that the stable tangent bundle of such a manifold is fiber homotopy trivial. This follows from results in [8] and the fact that an H-space is self-dual in the sense defined below.

If A and B are spaces with base points $a_0 \in A$, $b_0 \in B$, then $A \vee B$ will denote the subspace $A \times b_0 \cup a_0 \times B$ of $A \times B$, and $A \# B$ will denote the space obtained from $A \times B$ by collapsing $A \vee B$ to a point. If A is a space, A^+ will denote the disjoint union of A and a base point.

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Finite polyhedra A and B will be said to be *S-dual* if for some n there is an S-map $u: A^+ \# B^+ \to S^n$ (i.e. a map of some suspension of $A^+ \# B^+$ into the corresponding suspension of S^n) such that if s^* is a generator of $H^n(S^n)$, then the slant product

$$u^*s^*/ \ : H_k(A^+) \to H^{n-k}(B^+)$$

is an isomorphism (where all the homology and cohomology groups are understood to be modulo base points). u is a duality map in the sense of [9], and the results of [9] remain valid even though A^+, B^+ are not connected (connectedness being an unnecessary restriction throughout [9]). Since $A^+ \# B^+ = (A \times B)^+$, the existence of such an S-map u would follow from the existence of a map $f: A \times B \to S^n$ such that

$$f^*s^*/ : \ H_k(A) \approx H^{n-k}(B)$$

where the homology and cohomology groups are absolute groups.

An *H-space* is a topological space X together with a continuous multiplication $\mu: X \times X \to X$ having a unit element. A *polyhedral H-space* will mean an H-space which is a finite polyhedron. Our main result is:

THEOREM 1. *If X is a connected polyhedral H-space, then X is S-dual to X.*

It follows immediately from this and [8; Theorem 1] that:

COROLLARY 1. *If X is a compact connected manifold which is an H-space, then its stable tangent bundle is fiber homotopy trivial.*

In [3; Corollary 7.2] it is shown that if X is a connected H-space with finitely generated homology then for some m, $H_m(X) = Z$ and $H_j(X) = 0$ for $j > m$. A generator of $H_m(X)$ will be called a *fundamental homology class* of X. The following is an easy consequence of Theorem 1 and [8; Theorem 1]:

COROLLARY 2. *A fundamental homology class of a connected polyhedral H-space is stably spherical* (i.e. *there is an S-map $S^m \to X$ of degree one*).

Note that a fundamental homology class of an H-space is not spherical unless the H-space is a homotopy sphere. Even the suspension of the fundamental class may not be spherical ($S0(3)$ being an example in which there is no map $S^4 \to S(S0(3))$ of degree one).

Let X be a connected polyhedral H-space. It follows from [3; Corollary 7.2] that for some m, $H^m(X) = Z$ and $H^j(X) = 0$ for $j > m$. By standard obstruction theory it follows that there is a map $g: X \to S^m$ such that g^*s^* is a generator of $H^m(X)$. We then define a map $f: X \times X \to S^m$ by $f = g \circ \mu$. Theorem 1 follows from the following special kind of Poincaré duality for H-spaces.

THEOREM 2. *Let X be a connected polyhedral H-space with $H^j(X) = 0$ for $j > m$ and v a generator of the infinite cyclic group $H^m(X)$. Then*

$$\mu^*(v)/ : \ H_k(X; G) \to H^{m-k}(X; G)$$

is an isomorphism for all k and any coefficient group G.

2. *Proof of Theorem* 2. By the same sort of considerations as in [3; § 7], to prove Theorem 2 it suffices to show that for every prime p, the map

$$\mu_*: H_k(X; Z_p) \otimes H_{m-k}(X; Z_p) \to H_m(X; Z_p)$$

is a nonsingular pairing. If μ_* were associative and commutative, this would follow (as it did in [3; § 7] for $\Delta^*: H^{m-k}(X; Z_p) \otimes H^k(X; Z_p) \to H^m(X; Z_p)$) from Borel's theorem on Hopf algebras [1]. In our case this result will follow by passing to a new Hopf algebra where the multiplication map is associative and commutative, applying Borel's theorem to obtain a nonsingular pairing there, and then showing that this gives the desired nonsingular pairing in the original Hopf algebra. This technique proves the following algebraic lemma about Hopf algebras, from which Theorem 2 follows easily.

LEMMA. *Let A be a Hopf algebra over Z_p, finitely generated over Z_p with $A_0 = Z_p$, $A_m = Z_p$ for some $m > 0$, $A_j = 0$ for $j < 0$ and $j > m$, and having an associative diagonal map $\psi: A \to A \otimes A$. Then the multiplication map*

$$\phi: A_k \otimes A_{m-k} \to A_m$$

is a nonsingular pairing.

Proof. Since A_i is finitely generated for each i, it suffices to prove that for any $a \in A_k$, $\phi(a \otimes \ ): A_{m-k} \to A_m$ is nonzero (i.e. that the map of A_k into A^*_{m-k} induced by ϕ is a monomorphism).

Let $\bar{A}$ denote the positive dimensional elements of A and let $\rho: A \to \bar{A}$ denote the projection. Define $\bar{\psi}: A \to \bar{A} \otimes \bar{A}$ by $\bar{\psi} = (\rho \otimes \rho) \circ \psi$. Define $\bar{\psi}_0 = \rho: A \to \bar{A}$, $\bar{\psi}_1 = \bar{\psi}$, and inductively, for $n > 1$,

$$\bar{\psi}_n: A \to \bar{A} \otimes \cdots \otimes \bar{A} \qquad (n + 1 \text{ times})$$

by $\bar{\psi}_n = (\bar{\psi} \otimes 1 \otimes \cdots \otimes 1) \circ \bar{\psi}_{n-1}$. Define an increasing filtration on A by $F^k = \text{kernel } \bar{\psi}_k$. Then $F^0 = A_0$, $F^1 =$ primitive elements of A, $F^i \subset F^{i+1}$ and $A = \bigcup_i F^i$ (in fact, $A_k \subset F^k$). Note that this filtration is dual to that in [7; 4.15] and hence properties of this filtration may be derived from those proved in [7]. The hypothesis that the diagonal map $\psi: A \to A \otimes A$ is associative is equivalent to the condition that the dual Hopf algebra has an associative multiplication, which was an assumption in [7; 4.15]. Since the dual filtration is compatible with the Hopf algebra structure of the dual of A, the filtration F^q is compatible with the Hopf algebra structure of A and the associated graded module $E = \Sigma E^q$, $E^q = F^q/F^{q-1}$ is a Hopf algebra (dual to the graded Hopf algebra E_0 associated to the filtration on the dual of A [7]).

It follows from [7; 4.16 and 4.17] that E_0 is primitively generated. This is equivalent to the assertion that the primitive elements of its dual E are indecomposable. It then follows from [7; Theorem 4.9] that E is a commutative and associative Hopf algebra so that Borel's theorem [1; Theorem 6.1] (see also [7; Corollary 4. 21]) applies, and so, as in [3; § 7], for any nonzero $\bar{a} \in (E)_k$, $\bar{a} \cdot (E)_{m-k} \neq 0$. To prove the lemma let $a \in F^q$ but $a \notin F^{q-1}$. Then $\bar{a} = \{a\} \neq 0$ in $E^q = F^q/F^{q-1}$, so $\bar{a} \cdot (E)_{m-k} = \{\bar{a}\bar{b} \mid \text{where } b \in A_{m-k}\} \neq 0$. Therefore, $a \cdot A_{m-k} \neq 0$, and the lemma is proved.

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