DECOMPOSITION AND HOMOGENEITY OF CONTINUA ON A 2-MANIFOLD

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1. Introduction. Many partial results have been obtained in attempting to characterize homogeneous plane continua; a history of this problem can be found in [4]. The question arises; which of these results hold for homogeneous proper subcontinua of a 2-manifold, and indeed do there exist such continua which cannot be embedded in the plane? The main purpose of this paper is to extend some results for plane homogeneous continua to corresponding results for continua on a 2-manifold, with a long range aim of investigating the embedding problem.

Let X be a nondegenerate homogeneous plane continuum. F. B. Jones [10] has shown that X is a simple closed curve if it is aposyndetic or if it contains a noncutpoint, H. J. Cohen [7] has shown that X is a simple closed curve if it either contains a simple closed curve or is arcwise connected, and R. H. Bing [3] has shown that X is a simple closed curve if it contains an arc. In § 4 the above results of Cohen's and Jones' are generalized to homogeneous continua on 2-manifolds. Section 3 contains results on collections of continua which arise rather naturally in considering the generalizations of Cohen's work.

Jones [12] has shown that if X is decomposable and is not a simple closed curve, at least it becomes one under a natural aposyndetic decomposition. In § 5 this result is extended to homogeneous continua on a 2-manifold as well as to homogeneous continua with a multicoherence restriction.

In extending plane results to results on arbitrary 2-manifolds, we will use as a generalization of the Jordan curve theorem the fact that for any 2-manifold M there exists a positive integer k such that M is separated by the sum of any k disjoint simple closed curves on M.

2. Definitions. Only separable metric spaces will be considered here. A connected compact metric space is called a *continuum*. A *2-manifold* is a continuum such that each of its points lies in an open set topologically equivalent to Euclidean 2-space. A *2-manifold with boundary* is a continuum such that each of its points lies in an open set whose closure is topologically equivalent to a closed 2-cell.

A point set X is said to be *n-homogeneous* if for any n points $x_1, x_2, \cdots, x_n$ of X and any n points $y_1, y_2, \cdots, y_n$ of X there is a home-

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omorphism of X onto itself that carries $x_1 + x_2 + \cdots + x_n$ onto $y_1 + y_2 + \cdots + y_n$. For $n = 1$, the term *homogeneous* is used. A set X is said to be *nearly homogeneous* if for any point x of X and open set D of X there exists a homeomorphism of X onto itself carrying x into D. A set X is *locally homogeneous* if for each two points x and y of X there exists a homeomorphism between two open subsets of X containing x and y respectively such that x is mapped onto y.

A continuum X is said to be *aposyndetic at the point* p of X if for any point q of $X - p$ there is a subcontinuum Y of X and an open subset U of X such that $X - q \supset Y \supset U \supset p$. The continuum X is said to be *aposyndetic* if it is aposyndetic at each of its points.

A continuum X is said to be *semi-locally connected at a point* p if for each positive number ε there exists a positive number δ such that $X - V_\varepsilon(p)$ is contained in a finite number of components of $X - V_\delta(p)$, (Note: In general, $V_r(X)$ is the r-neighborhood of the set X; i.e., the set of all points x such that the distance, $\rho(x, X)$, from x to X is less than r.) If X is semi-locally connected at each of its points, X is said to be *semi-locally connected.*

A simple triod is the sum of three arcs each having a point p as an end point such that p is the common part of each two of these three arcs.

If G is an upper semi-continuous collection filling a continuum X, the decomposition space relative to G will be denoted by X'. The projection map of X onto X' relative to G will be denoted by f throughout this paper.

3. Collections of continua which fill a continuum. We will state a theorem and a corollary, from G. T. Whyburn [17, pp. 43-44], which are needed in the proofs of some of the theorems of this section.

THEOREM W. *If G is any uncountable collection of disjoint cuttings of a connected set M, then some element X of G separates in M a pair of points belonging to $G^* - X$.*[1]

COROLLARY W. *No continuum of convergence K of a connected set M contains an uncountable collection of disjoint cuttings of M. Indeed, if a and b are points of K, no subset of K separates a and b in M.*

THEOREM 1. *If G is a nondegenerate collection of disjoint continua filling a continuum X on a 2-manifold M, G_0 is a countable subcollection of G, k is an integer such that M is separated by the sum of*

[1] For any collection G, G^* denotes the sum of the sets of G.

every k elements of $G - G_0$, and D is a complementary domain of X, then the boundary of D is the sum of a finite number of continua $B_1, \cdots, B_m$ each lying in some element of G.

Proof. It follows from work of J. H. Roberts' and N. E. Steenrod's [14, Lemma 1] that the boundary of D is the sum of a finite number of continua $B_1, \cdots, B_m$. Suppose B_1 intersects each continuum of an uncountable subcollection G_1 of G. There exists an uncountable collection Z such that each element of Z is the sum of k continua of $G_1 - G_0$ and no two distinct elements of Z intersect. It follows from Theorem W that there is an element Q of Z such that $M - Q$ is the sum of two mutually separated sets H_1 and H_2 containing two continua g_1 and g_2, respectively, of $G_1 - G_0$. This involves a contradiction, since D does not intersect Q, and each of g_1 and g_2 contains a boundary point of D. Thus, since a continuum cannot be the sum of a countable number (greater than one) of disjoint closed sets, B_1 must be contained in some element of G.

COROLLARY 1.1. *Under the hypothesis of Theorem 1, let G_2 be the set of elements of $G - G_0$ not intersecting the boundary of any complementary domain of X; then any k elements of G_2 separate X, G_2 is uncountable, and $f(G_2^*)$ is dense in X'.*

Proof. Suppose C is the sum of k elements of G_2, $M - C$ is the sum of two mutually separated sets H and K, and $X - C \subset H$. All of the complementary domains of X must then lie in H, contradicting the existence of K. Thus C must separate X. From Theorem 1, since X has at most a countable number of complementary domains, at most a countable number of elements of G intersect the boundary of a complementary domain of X; therefore G_2 is uncountable and $f(G_2^*)$ is dense in X'.

THEOREM 2. *If G is a collection of disjoint continua filling a continuum X in a connected space M, G_0 is a countable subcollection of G, and k is an integer such that M is separated by every k elements of $G - G_0$, then G is upper semi-continuous and X' is locally connected.*

Proof. Suppose the sequence of points $p_1, p_2, \cdots$ converges to p_0, where p_i $(i = 0, 1, \cdots)$ is a point in a continum g_i of the collection G. It will be shown that $g_0 \supset \limsup \{g_i\}$. If lim sup $\{g_i\}$ intersects each continuum of an uncountable subcollection G_1 of G, then, as in Theorem 1, obtain an uncountable collection Z of cuttings of M, each being the sum of k continua of $G_1 - G_0$, and no one containing g_0. By Theorem

W, there is an element Q of Z such that $M - Q$ is the sum of two mutually separated sets H_1 and H_2 containing two continua g and g', respectively, of $G_1 - G_0$ and such that $H_1 \supset g_0$. But there exists an integer n such that $H_1 \supset g_i$ for all $i > n$; thus g' cannot intersect lim sup $\{g_i\}$. This contradiction implies that $g_0 \supset \limsup \{g_i\}$ and G is upper semi-continuous.

If X' were not locally connected there would exist a sequence of disjoint nondegenerate continua $X_1, X_2, \cdots$ in X' converging to a nondegenerate continuum X_0 in X'. Let F be the collection consisting of G and the individual points of $M - G^*$. The collection F is upper semi-continuous, and M' is connected. The nondegenerate continuum $f^{-1}(X_0)$ contains an uncountable collection of mutually exclusive cuttings of M, each consisting of k elements of $G - G_0$; thus X_0 contains an uncountable collection of mutually exclusive cuttings of M', each consisting of k points. This contradicts Corollary W; hence X' is locally connected.

THEOREM 3. (a) *If G is a nondegenerate collection of disjoint continua filling a continuum X on a 2-manifold, and G_0 is a countable subcollection of G such that every continuum of $G - G_0$ separates the manifold, then G is upper semi-continuous and X' is a dendron.*

(b) *If each element of G separates the manifold into two complementary domains, then X' is an arc.*

Proof of (a). From Theorem 2, G is upper semicontinuous, and X' is locally connected. From the proof of Corollary 1.1, all but a countable number of elements of G separate X; thus X' has at most countably many nonseparating points. Every nondegenerate subcontinuum of X' then contains uncountably many separating points of X' so that X' is a dendron [17, (1.1), p. 88].

Proof of (b). For each point x in X', let $g_x = f^{-1}(x)$. Suppose that some complementary domain D of X has a boundary which intersects two elements g_a and g_b of G. Since X' is a dendron by (a), there exists an arc $[ab]$ in X'. Let D_1 be the complementary domain of $f^{-1}([ab])$ which contains D. By Corollary 1.1, there is a point c of (ab) such that g_c does not intersect the boundary of any complementary domain of $f^{-1}([ab])$. Let the two complementary domains of g_c be H and K, with D_1 lying in H. Since D lies in H, g_a and g_b together with $f^{-1}([ac))$ and $f^{-1}((cb])$ must lie in H. All complementary domains of $f^{-1}([ab])$ must then lie in H, and K is empty. From this contradiction we conclude that the boundary of any complementary domain of X must lie in one element of G.

If g is an element of G which contains the boundary of some complementary domain of X, then as in a proof of Cohen's [7, Lemma 2.3], it can be shown that g does not separate X. As in further proofs of Cohen's [7, Lemma 4.2 and Lemma 4.3], if $[cd]$ is an arc of X' and p a point of the open arc (cd), then g_p must separate g_c from g_d in M; thus, X' cannot contain a simple triod. From part (a) above, X' is a dendron; therefore, X' is an arc.

THEOREM 4. *If G is a nondegenerate collection of disjoint continua filling a plane continuum X such that each element of G separates the plane into two complementary domains, then there exist two elements g_0 and g_1 of G such that $X - (g_1 + g_0)$ is an open annulus.*

Proof. By Theorem 3, X' is an arc. From the proof of Theorem 3, no element of G containing the boundary of a complementary domain of X can separate X. Using Theorem 1 and proceeding as in the case where G is a collection of simple closed curves [7, Theorem 4], we may show that the boundary of the unbounded complementary domain of X must be contained in an element g_1 of G corresponding to an end point of X', the element g_0 of G corresponding to the other end point of X' must lie in the interior complementary domanin of g_1, and every point common to the interior domain of g_1 and the exterior domain of g_0 must be in X.

THEOREM 5. *If G is a collection of disjoint continua filling a plane continuum such that each element of G separates the plane into two complementary domains and is irreducible with respect to separating the plane, then G is a continuous collection.*

Proof. Let X be the plane continuum filled by G. As in Theorem 3 and 4, G is upper semi-continuous, X' is an arc $[ab]$, and the interior of X is an open annulus. For each x in $[ab]$ let g_x be the element of G such that $f(g_x) = x$. It will be sufficient to show that no sequence of elements of G converges to a proper subset of an element of G.

Suppose there is a sequence $g_{x_1}, g_{x_2}, \cdots$ of elements of G converging to a proper subset h of an element g_{x_0} of G. Suppose without loss that $x_0 \neq a$. Since g_{x_0} is irreducible with respect to separating the plane, there exists an open disk D containing h but not containing all of g_{x_0} and such that $D \cdot g_a = \phi$ if $x_0 = b$ and $D \cdot (g_a + g_b) = \phi$ if $x_0 \neq b$. For some x in $[ab]$, g_x lies entirely in D. Since x is neither a nor b, there exists a sub-arc $[cd]$ of (ab) such that $f^{-1}([cd])$ is contained in D and x is in (cd). From the proof of Theorem 4, $f^{-1}((cd))$ is an open annulus with inner boundary contained in g_c (or g_d). Then $f^{-1}([ac])$ (or $f^{-1}([db])$) lies in D, contradicting the choice of D.

Note. In the following theorem, we are justified in referring to

X', since Theorem 2 and Theorem 3 assure us that G is upper semi-continuous.

THEOREM 6. *If G is a collection of disjoint simple closed curves filling a continuum X on a 2-manifold M such that X' is an arc, then X is an annulus, a Möbius strip, or a Klein bottle.*

LEMMA 6.1. *If G is a collection of disjoint simple closed curves filling a continuum X on a 2-manifold M such that X' is an arc or a simple closed curve, then G is a continuous collection.*

Proof of Lemma 6.1.

Case 1. Suppose X' is an arc. The proof proceeds in the same fashion as the proof of Theorem 5, since a proper subset h of an element of G is an arc or a point, and each open set containing h contains an open disk containing h.

Case 2. Suppose X' is a simple closed curve, and $g_1, g_2, \cdots$ is a sequence of elements of G converging to a proper subset h of an element g' of G. We may break X' into two arcs A_1 and A_2 with $f(g')$ interior to A_1. We may then choose a subsequence of $g_1, g_2, \cdots$ whose elements correspond to points in A_1 and, considering this subsequence and the arc A_1, proceed as in Case 1.

LEMMA 6.2. *Under the hypotheses of Theorem* 6, *X is a 2-manifold with boundary.*

Proof of Lemma 6.2. Let X' be the arc $[ab]$, and for each x in $[ab]$ let g_x be the element of G such that $f(g_x) = x$. By covering g_x with a circular chain of open disks, an open set of M containing g_x may be obtained which is homeomorphic to an open annulus or an open Möbius strip.

Case 1. Suppose g_x lies in an open annulus R, and x is in (ab); then there is an arc A of X' such that $f^{-1}(A)$ is in R and x is an interior point of A. Using a theorem of Cohen's [7, Theorem 4], $f^{-1}(A)$ is a closed annulus, and each point of g_x is contained in an open disk in X.

Case 2. Suppose g_x lies in an open annulus R, and x is an endpoint (say b) of X'; then, as in Case 1, there exists a closed annulus R_1 contained in R such that $R_1 = f^{-1}([cb])$, where $[cb]$ is a subarc of X'. Let

p be a point of g_x, and D be a Euclidean neighborhood of p in R such that the diameter of D is less than the distance from g_x to $f^{-1}(X' - (cb])$. Then $D \cdot X$ is contained in R_1, and considering X as space, p has a neighborhood in $D \cdot X$ whose closure is homeomorphic to a closed disk.

Case 3. Suppose g_x lies in an open Möbius strip R, but not in the interior of any annulus. Suppose without loss that $x \neq a$. If g_x separates R then $R - g_x = H + K$ where H is an open annulus and K an open Möbius strip, or H is an open disk and K an open Möbius strip with a closed disk removed. In either case K contains a simple closed curve J which fails to separate K, and thus fails to separate R; then g_x is contained in the open annulus $R - J$. Thus g_x does not separate R, and $R - g_x$ is an open annulus. Let $[cx]$ be a subarc of $[ab]$ such that $f^{-1}([cx))$ is in $R - g_x$; then $f^{-1}([cx))$ is a half open annulus R_1 from [7, Theorem 4]. By Lemma 6.1, G is continuous, and the boundary of R_1 must be the sum of g_c and g_x. If $x \neq b$ there is another half open annulus $R_2 = f^{-1}((xd])$ in $R - g_x$ whose boundary is the sum of g_x and g_d. But then g_x would lie interior to the annulus $f^{-1}([cd])$. This contradicts the choice of g_x; thus x must be equal to b. Let p be a point of g_b. Choose a disk R_p which contains p, has a simple closed curve C for a boundary, does not intersect $f^{-1}([ac])$, and is such that the sum of $g_b \cdot cl(R_p)$ and C is a theta-curve. Let R_1 and R_2 be the two complementary domains of the theta-curve which lie in R_p. Since p is on the boundary of $f^{-1}([cb))$, R_1 (say) must intersect $f^{-1}([cb))$. If $R_1 - X$ is not empty, R_1 must intersect the bonndary of $f^{-1}([cb))$ since R_1 is connected. But neither g_c nor g_b intersect R_1, and thus R_1 is contained in X. Similarly, either R_2 does not intersect X, or R_2 lies entirely in X; in either case, considering X as space, p has a neighborhood in $X \cdot R_p$ whose closure is homeomorphic to a closed disk.

Thus, in any case, X is a 2-manifold with boundary.

Proof of Theorem 6. It follows from results of J. M. Slye's [15, Theorem 1 and Corollary 10] that if G is an upper semi-continuous collection of simple closed curves filling a continuum X which is a 2-manifold with boundary, and X' is an arc, then X must be an annulus, a Möbius strip or a Klein bottle. Thus, Theorem 6 follows directly from Lemma 6.2 and Slye's results.

REMARK. The following is a brief outline of a direct proof of Theorem 6, which does not use Slye's results.

In Case 3 of the proof of Lemma 6.2, cover g_b with a set of open disks $R_1, R_2, \cdots, R_n$ in $M - f^{-1}([ac])$ with boundaries consisting of simple closed curves $C_1, C_2, \cdots, C_n$ such that $R_1 + R_2 + \cdots + R_n$ is an open

Möbius strip and such that, for $i = 1, 2, \cdots, n$, the sum of C_i and $g_b \cdot cl(R_i)$ is a theta-curve with interior complementary domain H_i and K_i. Suppose that the complementary domains have been numbered so that each domain in the sequence $H_1, H_2, \cdots, H_n, K_1, K_2, \cdots, K_n, H_1$ intersects the next domain in the sequence. As before, any domain intersecting X must lie in X; thus by an inductive process the whole Möbius strip $R_1 + R_2 + \cdots + R_n$ lies in X. If c is in (ab), it can be shown that, depending on whether g_a and g_b fall under Case 2 or Case 3, each of $f^{-1}([ac])$ and $f^{-1}([cb])$ is an annulus or a Möbius strip. Thus X itself must be an annulus, a Möbius strip, or a Klein bottle.

COROLLARY 6.1. *If G is a collection of disjoint simple closed curves filling a continuum X on a 2-manifold M such that X' is a simple closed curve, then X must fill M and be a torus or a Klein bottle.*

THEOREM 7. *If G is a nondegenerate collection of disjoint simple closed curves filling a proper subcontinuum X of a 2-manifold M, then X must be an annulus or a Möbius strip.*

Proof. By Theorem 2 and Theorem 13, G must be upper semicontinuous and X' locally connected. Thus X' must be a dendron, for if X' contains a simple closed curve, then, by Corollary 6.1, X would fill M. Since M is a 2-manifold, Theorem 6 implies that X' contains no simple triod. Thus X' must be an arc, and Theorem 7 follows from Theorem 6.

REMARK. If the restriction that X be a proper subcontinuum of M were removed in Theorem 7, X could also be a torus of a Klein bottle.

COROLLARY 7.1. *If G is a collection of disjoint simple closed curves filling a continuum X on a 2-manifold M, then G is continuous and X' is an arc or a simple closed curve.*

Proof. As in the proof of Theorem 7, either the decomposition space X' is an arc or it contains a simple closed curve. If X' contains a simple closed curve, it must be one by Corollary 6.1. The continuity of G follows from Lemma 6.1.

4. Conditions under which a homogeneous continuum on a 2-manifold is a simple closed curve.

THEOREM 8. *If X is a homogeneous proper subcontinuum of a 2* *manifold M, and X contains a simple closed curve, then X must be a simple closed curve.*

Proof. Suppose X is not a simple closed curve.

(1) X is one-dimensional, for otherwise X would contain an open set in M and thus would contain M. This contradicts the hypothesis of Theorem 8.

(2) X is not locally connected, for using (1) and a result of Anderson's [1, Theorem 13], X must be either a simple closed curve or the universal one-dimensional curve. This is a contradiction in either case, since the universal curve contains no open set which can be embedded in the plane.

(3) Because X is not locally connected, there is a disk on M containing an open set of X which has uncountably many components [7, Lemma 2.1 and Corollary 2.11].

(4) Suppose X contains a simple triod. By (3), some open set D of X is contained in a disk of M and has uncountably many components; thus the homogeneity of X implies that D contains uncountably many disjoint simple triods. This contradicts a theorem of R. L. Moore's [13, Theorem 75, p. 254]; thus X contains no simple triod.

(5) No two simple closed curves in X intersect, for if some two did intersect then X would contain a simple triod, contrary to (4).

(6) Let G be the collection of all simple closed curves in X; then G fills X and the elements of G are disjoint, since by homogeneity each point of X lies on a simple closed curve and by (5) no two simple closed curves intersect.

By Theorem 7, X must be an annulus or a Möbius strip; this contradicts (1), and Theorem 8 follows.

COROLLARY 8.1. *If a nondegenerate proper subcontinuum of a 2-manifold is locally connected and homogeneous, then it must be a simple closed curve.*

Proof. This corollary follows from (1) and (2) in the proof of Theorem 8.

COROLLARY 8.2. *No locally homogeneous proper subcontinuum of a 2-manifold contains a simple triod.*

Proof. This corollary is obtained as in (3) and (4) of the proof of Theorem 8, where the homogeneity condition may be replaced by local homogeneity.

REMARK. Cohen [7, Theorem 3] has shown that a homogeneous, arcwise connected, plane continuum must be a simple closed curve. In a similar fashion, we have the following result.

THEOREM 9. *If the nondegenerate continuum X is arcwise connected, contains no simple triod, and is either nearly homogeneous or locally homogeneous; then X is a simple closed curve.*

COROLLARY 9.1. *If a nondegenerate proper subcontinuum of a 2-manifold is locally homogeneous and arcwise connected then it must be a simple closed curve.*

This corollary follows from Corollary 8.2 and Theorem 9.

THEOREM 10. *If a nondegenerate proper subcontinuum of a 2-manifold is aposyndetic and homogeneous, then it must be a simple closed curve.*

LEMMA 10.1. *Suppose A is an arc, and G is a countably infinite collection of disjoint arcs such that if $A_i = [x_i y_i]$ is an arc of G for $i = 1, 2, \cdots$, then $A_i \cdot A = x_i + y_i$; then for any positive integer k, there exist k disjoint simple closed curves contained in $A + G^*$.*

Proof of Lemma 10.1. For convenience, let A be the unit interval $[01]$. Without loss of generality, suppose that $x_1, x_2, \cdots$ is a monotone sequence converging to a point x of A, and $y_1, y_2, \cdots$ is a monotone sequence converging to a point y of A.

Case 1. If $x \neq y$, let I_x and I_y be disjoint open intervals (or half open intervals if x or y is an endpoint of A) of A, containing x and y respectively. Suppose, without loss, that each point of the sequence $x_1, x_2, \cdots$ lies in I_x and each point of the sequence $y_1, y_2, \cdots$ lies in I_y. Let $[pq]_A$ denote a subinterval of A with the points p and q as endpoints. Then $J_{1,2} = [x_1 x_2]_A + [y_1 y_2]_A + A_1 + A_2$ is a simple closed curve in $A + G^*$. Indeed, $\{J_{(2n-1),2n}\}$ for $n = 1, 2, \cdots, k$ is a set of k disjoint simple closed curves in $A + G^*$, where $J_{2n-1,2n} = [x_{2n-1} x_{2n}]_A + [y_{2n-1} y_{2n}]_A + A_{2n-1} + A_{2n}$.

Case 2. Suppose $x = y$. If the two sequences $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ converge to x from opposite sides, then the construction in Case 1 will give the desired set of simple closed curves. Suppose for convenience, that both sequences converge to x from the left. There exists an increasing sequence of positive numbers $r_1, r_2, \cdots$ such that for each i, both $x_{r_{i+1}}$ and $y_{r_{i+1}}$ lie to the right of both x_{r_i} and y_{r_i} on A. Then $\{J_n\}$, for $n = 1, 2, \cdots, k$, is a set of k disjoint simple closed curves in $A + G^*$, where $J_n = [x_{r_n} y_{r_n}]_A + A_{r_n}$.

LEMMA 10.2. *If X is a continuum satisfying the hypothesis of Theorem 10, then the boundary of each complementary domain of X is locally connected.*[2]

Proof of Lemma 10.2. Let D be a complementary domain of X. The boundary of D must consist of a finite number of continua $B_1, B_2, \cdots, B_m$ by a lemma of Roberts' and Steenrod's [**14**, Lemma 1]. Suppose B_1 fails to be locally connected at a point q. Let R be a disk containing q such that $cl(R)$ intersects no B_i for $i = 2,3, \cdots, m$. By a standard construction, there are two open sets R_1 and R_2 with closures in R, a continuum X_0 in R, and a sequence of disjoint continua $X_1, X_2, \cdots$ in R with the following properties:

(1) $cl(R_1)$ does not intersect $cl(R_2)$,

(2) R_1 and R_2 have simple closed curves C_1 and C_2, respectively, for boundaries,

(3) for each i, X_i contains both a point of C_1 and a point of C_2 and is a component of the common part of B_1 and $R - (R_1 + R_2)$, and

(4) the sequence $X_1, X_2, \cdots$ converges to X_0.

Let p be a point of $X_0 - (cl(R_1) + cl(R_2))$, ε be a positive number less than the distance from p to $cl(R_1) + cl(R_2)$, and $V_\varepsilon(p)$ be a circular neighborhood of p with a circle C_ε as boundary. Then there exists a circular neighborhood $V_\delta(p)$ with a circle C_δ as boundary such that $\delta < \varepsilon$ and all of $X - V_\varepsilon(p)$ lies in one component N of $X - V_\delta(p)$. That such a $V_\delta(p)$ exists follows as in a theorem of Whyburn's [**16**, (6.22)], since X, being compact and aposyndetic, must be semi-locally connected, and p must not be a cut point of X because X is homogeneous. Without loss of generality, suppose that the X_i $(i = 1, 2, \cdots)$ have been chosen so that each intersects $V_\delta(p)$ and such that $(X_1 \cdot C_j)$, $(X_2 \cdot C_j)$, $\cdots$ are ordered, as named, along the simple closed curve C_j $(j = 1, 2)$. Without change of notation, consider X_i $(i = 1, 2, \cdots)$ to be irreducible from C_1 to C_2.

An open set O_1 in R is bounded by $X_1 + X_2 + A_{11} + A_{12}$, where A_{11} is an arc in C_1 irreducible from X_1 to X_2 and intersecting no X_j with $j > 2$, and A_{12} is a similar arc in C_2. In the same way, for $i = 1, 2, \cdots$ obtain a "corridor" O_i between the continua X_i and X_{i+1} bounded by X_i, X_{i+1} and arcs A_{i1} and A_{i2} in C_1 and C_2 respectively. Now, since each X_i is on the boundary of D, for $i = 2, 3, 4, \cdots$ we may choose a point z_i in $X_i \cdot V_\delta(p)$ and a neighborhood U_i of z_i in $V_\delta(p)$ such that U_i contains a point p_i of D and intersects no X_j for $j \neq i$. The point

[2] As noted by the referee for this paper, Lemma 10.2 is closely related to results (mainly for the plane or n-sphere) of Jones' [**9**], Whyburn's [16], and Wilder's [**18** and **19**]. Indeed, the proof given here was motivated by the proof of Theorem 14 in [**16**].

p_i must lie in O_i or O_{i-1}. Possibly discarding some of the p_i ($i = 1, 2, \cdots$) and X_i ($i = 2, 3, \cdots$), and re-numbering the remaining points, continua, and corresponding corridors (retaining the same order as before); we arrive at a set of points $\{p_i\}$ of D with p_i in $O_i \cdot V_\delta(p)$ ($i = 1, 2, \cdots$).

Run an arc $[p_1p_2]$ in D from p_1 to p_2. Let x_1 be the last point of $C_\delta \cdot O_1$ on $[p_1p_2]$ in the order p_1 to p_2. Let y_1 be the first point of C_δ in the order x_1 to p_2 along $(x_1p_2]$, a subarc of $[p_1p_2]$. Then y_1 is in some O_{j_1} with $j_1 \neq 1$, and (x_1y_1) lies in $D - cl(V_\delta(p))$, where (x_1y_1) is an open subarc of $[p_1p_2]$. Choose n_2 such that $n_2 > 1$ and $n_2 > j_1$. Now run an arc $[p_{n_2}p_{n_2+1}]$ in $D - [x_1x_2]$ from p_{n_2} to p_{n_2+1}. As before let x_2 be the last point of $C_\delta \cdot O_{n_2}$ on $[p_{n_2}p_{n_2+1}]$ in the order p_{n_2} to p_{n_2+1}, and y_2 the first point of C_δ in the order x_2 to p_{n_2+1} along $(x_2p_{n_2+1}]$; then y_2 is in some O_{j_2} with $j_2 \neq n_2$, and (x_2y_2) lies in D-$cl(V_\delta(p))$. Continue constructing disjoint arcs $[x_iy_i]$ in this manner such that for $i = 1, 2, \cdots$:

(1) x_i is in $C_\delta \cdot O_{n_i}$,

(2) y_i is in $C_\delta \cdot O_{j_i}$ with $j_i \neq n_i$,

(3) $n_k > n_i$ for $k > i$ and $n_k > j_i$ for $k > i$,

(4) $[x_iy_i]$ lies in $D - \sum_{k=1}^{i-1}[x_ky_k]$, and

(5) (x_iy_i) lies in $D - cl(V_\delta(p))$.

The set $\sum_{i=1}^{\infty}(x_i + y_i)$ is a subset of an arc A in C_δ. Lemma 10.1 may now be applied to the arc A and the set of arcs $\{A_i\}$ where, for $i = 1, 2, \cdots, A_i = [x_iy_i]$. Using Theorem 13 and the construction in Lemma 10.1, obtain m disjoint simple closed curves $J_1', J_2', \cdots, J_m'$ whose sum separates M and such that each one is the sum of one or two arcs from the set $\{A_i\}$ and one or two arcs of C_δ.

Case 1. Suppose the simple closed curves are of the form given in Case 1 and the first part of Case 2 of Lemma 10.1. We can then re-number the arcs and corridors so that $J_n' = [x_{2n-1}x_{2n}]_\delta + [y_{2n-1}y_{2n}]_\delta + A_{2n-1} + A_{2n}$ ($n = 1, 2, \cdots, m$), and $x_1, x_2, \cdots, x_{2m}$ forms an ordered set along C_δ, where $[pq]_\delta$ denotes a subarc of A with endpoints p and q. For each n ($n = 1, 2, \cdots, m$), replace the arcs $[x_{2n-1}x_{2n}]_\delta$ and $[y_{2n-1}y_{2n}]_\delta$ with arcs $[x_{2n-1}x_{2n}]_V$ and $[y_{2n-1}y_{2n}]_V$, respectively, such that the $2m$ arcs of $([x_{2n-1}x_{2n}]_V, [y_{2n-1}y_{2n}]_V\}$ are disjoint, and $(x_{2n-1}x_{2n})_V$ and $(y_{2n-1}y_{2n})_V$ are open arcs lying in $V_\delta(p)$. Let $J_1, \cdots, J_m$ be the new simple closed curves obtained from $J_1', \cdots, J_m'$ by the replacement of arcs as described above.

We will now show that if z_1 and z_2 are points of C_δ which lie in different corridors O_1 and O_2, respectively, and each of z_1 and z_2 is an end point of some arc in D like the $[x_iy_i]$ described above; then the arc $[z_1z_2]_\delta$ of C_δ must intersect N. Let $[z_1z_3]$ be an arc in D such that z_3 is in $C_\delta \cdot O_i$ where $i \neq 1$ and (z_1z_3) lies in $D - cl(V_\delta(p))$. Let $[z_2z_4]$ be an arc in D such that z_4 is in $C_\delta \cdot O_j$, where $j \neq 2$ and (z_2z_4) lies in

$D - cl(V_\delta(p))$. Let B be the subarc $[z_1z_1']$ of $[z_1z_3]$, such that z_1' is the first point of C_ε on $[z_1z_3]$ in the order z_1 to z_3. Let E be the subarc $[z_2z_2']$ of $[z_2z_4]$, such that z_2' is the first point of C_ε on $[z_2z_4]$ in the order z_2 to z_4. Let $C = [z_1'z_2']_\varepsilon$ be an arc of C_ε in the same direction as the arc $F = [z_1z_2]_\delta$ on C_δ. Let J be the simple closed curve $B + C + E + F$ whose interior lies between C_δ and C_ε. Then F must intersect X since z_1 and z_2 lie in different corridors, and B and E are in D. Suppose N does not intersect F, then no component of $X - V_\delta(p)$ can intersect both F and C, for such a component would be contained in N. However, every component of $X - V_\delta(p)$ must intersect C_δ, and thus each component of $X - V_\delta(p)$ which intersects the interior of J must intersect either F or C. Let H be the set of components of $X - V_\delta(p)$ intersecting F, and let K be the set of components intersecting C. Then H^* and K^* are disjoint closed sets. Then by a theorem proved by Moore [13, Theorem 12, p. 189], there exists an arc from B to E, lying interior to J except for end points, which does not intersect X and thus must lie in D. But then z_1 can be connected to z_2 by an arc in $D \cdot cl(V_\varepsilon(p))$; this contradicts the choice of ε and the fact that z_1 and z_2 lie in different corridors O_1 and O_2. Therefore N must intersect the interior of $F = [z_1z_2]_\delta$.

By the construction of Lemma 10.1, all of the $[x_{2n-1}x_{2n}]_\delta$, for $n = 1, 2, \cdots, m$, must lie in an arc of C_δ containing none of the y_i ($i = 1, 2, \cdots, 2m$). Then from the discussion above, since x_1, x_2, and x_3 each lie in different corridors, the interior of each of the arcs $[x_1x_2]_\delta$ and $[x_2x_3]_\delta$ must contain a point of N. We can then choose an arc $[u_1x_2u_2]_\delta$ of C_δ such that

(1) $u_1 + u_2 \subset N$,
(2) $(u_1x_2u_2)_\delta$ does not intersect N,
(3) u_1 is a point of $(x_1x_2)_\delta$,
(4) u_2 is a point of $(x_2x_3)_\delta$, and
(5) $[u_1x_2u_2]_\delta - x_2$ intersects no J_n ($n = 1, 2, \cdots, m$).

Case 2. Suppose the simple closed curves $J_1', \cdots, J_m'$ are of the form given in the second part of Case 2 in the proof of Lemma 10.1. We can then re-number the arcs and corridors so that $J_n' = [x_ny_n]_\delta + A_n$ ($n = 1, 2, \cdots, m$), and $x_1, y_1, x_2, y_2, \cdots, x_m, y_m$ forms an ordered set on A. As before, for $n = 1, 2, \cdots, m$ replace $[x_ny_n]_\delta$ by the arc $[x_ny_n]_V$ such that $(x_ny_n)_V$ lies in $V_\delta(p)$; thus obtain the set of disjoint simple closed curves $\{J_n\}$, where $J_n = [x_ny_n]_V + A_n$.

Since x_1 and y_1 lie in different corridors as do y_1 and x_2, we can obtain an arc $[u_1y_1u_2]_\delta$ of C_δ such that

(1) $(u_1 + u_2) \subset N$,
(2) $(u_1y_1u_2)_\delta$ does not intersect N,

(3) u_1 is a point of $(x_1y_1)_\delta$,
(4) u_2 is a point of $(y_1x_2)_\delta$, and
(5) $[u_1y_1u_2]_\delta - y_1$ intersects no J_n $(n = 1, 2, \cdots, m)$.

Notice that, in each of the above cases, N does not intersect $J_1 + J_2 + \cdots + J_m$.

For purposes of the remainder of the proof, Case 1 and Case 2 are identical, and we will use the notation of Case 2. Assume, without loss, that $J_1 + J_2 + \cdots + J_m$ separates M, but $J_2 + \cdots + J_m$ does not. There exist points c and d separated from each other on M by $J_1 + J_2 + \cdots + J_m$ and an arc $[cd]$ of M which does not intersect $J_2 + \cdots + J_m$ and thus must intersect J_1. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min[\rho(u_1, J_1), \rho(u_2, J_1), \rho(c, J_1), \rho(d, J_1), \rho(J_1, J_k), k = 2, 3, \cdots, m]$. Let U be an annulus or Möbius strip contained in an ε_1 cover of J_1 such that J_1 is interior to U. Let c_1 be the first point of J_1 on $[cd]$ in the order c to d and c_0 a point of $U \cdot [cd]$ preceding c_1 on $[cd]$ in the order c to d. Then in $U - J_1$ there is an arc B_1 from c_0 to a point a_1 of $(u_1y_1)_\delta$ (consider u_1 and u_2 re-numbered if necessary). Similarly construct an arc B_2 in $U - J_1$ from d_0 to a point b_1 of $(y_1u_2)_\delta$ or $(u_1y_1)_\delta$, where d_0 is a point of $U \cdot [cd]$ preceding the first point d_1 of J_1 on $[cd]$ in the order d to c. If b_1 is in $(y_1u_2)_\delta$ then $[cc_0] + B_1 + [u_1a_1]_\delta + N + [b_1u_2]_\delta + B_2 + [d_0d]$ is a continuum in $M - \sum_{i=1}^{m} J_i$ containing both c and d, and if b_1 is in $(u_1y_1)_\delta$ then $[cc_0] + B_1 + [u_1a_1]_\delta + N + [u_1b_1]_\delta + B_2 + [d_0d]$ is a continuum in $M - \sum_{i=1}^{m} J_i$ containing both c and d. This contradiction establishes Lemma 10.2.

Proof of Theorem 10. Suppose X is an aposyndetic homogeneous nondegenerate subcontinuum of a 2-manifold, and X is not a simple closed curve. By Theorem 8, X contains no simple closed curve. It follows from Corollary 8.2 and Lemma 10.2 that a component B_1 of the boundary of a complementary domain D of X must be an arc. Cover B_1 with an open disk R that does not intersect any other component of the boundary of D and does not contain X. Since X is connected, R must intersect $X - B_1$. Since B_1 is part of the boundary of D, $R - B_1$ is a connected set intersecting both X and D, and thus intersecting the bounday of D; this contradicts our choice of R. Therefore X is a simple closed curve.

COROLLARY 10.1. *Every 2-homogeneous nondegenerate proper subcontinuum of a 2-manifold is a simple closed curve.*

Proof. C. E. Burgess has shown that any 2-homogeneous continuum is aposyndetic [6, Theorem 7]. Corollary 10.1 then follows directly from Theorem 10 and the fact that each 2-homogeneous continuum is homogeneous [5, Theorem 1].

COROLLARY 10.2. *Every homogeneous nondegenerate continuum, which contains a non-cutpoint and lies on a 2-manifold, is a simple closed curve.*

Proof. This follows directly from Theorem 10 and the proof of a theorem of Jones' [10, Theorem 2].

5. Decomposition of decomposable homogeneous continua.

THEOREM 11. *If a proper subcontinuum X of a 2-manifold M is decomposable and homogeneous, then there exists a continuous collection G of disjoint continua filling X such that X' is a simple closed curve, and the elements of G are mutually homeomorphic, homogeneous tree-like continua.*

Proof. A theorem of Jones' [12, Theorem 1] gives a nondegenerate continuous collection G of mutually exclusive continua filling X such that

(1) X' is a homogeneous aposyndetic continuum,

(2) the elements of G are mutually homeomorphic, homogeneous continua, and

(3) if g is a continuum of the collection G and K a subcontinuum of X containing both a point of g and a point of $X - g$, then g is a subset of K.

Case 1. Suppose that each element g of G is treelike. A theorem of Roberts' and Steenrod's [14, Theorem 1] implies that the collection consisting of the elements of G together with the individual points of $M - X$ forms an upper semi-continuous decomposition of M such that M' is homeomorphic to M. Then since X' is an aposyndetic homogeneous continuum on a 2-manifold, it must be a simple closed curve by Theorem 10.

Case 2. Suppose that each element of G is nontreelike. Since X' is homogeneous, it can have no separating point; thus, for any element g of G, $X - g$ is connected and lies in a complementary domain D of $M - g$. By a result due to Roberts and Steenrod [14, Lemma 1], D must contain a continuum K such that $D - K = H_1 + \cdots + H_s$ where the H_i are disjoint open cylinders. By the continuity of G, there is some subcollection G_1 of G filling a continuum A in one of these open cylinders. Think of this cylinder as embedded in the plane. Each element of G_1 must separate the plane [2, Theorem 6]; indeed, each element of G_1 must have two complementary domains, since the elements of G are homeomorphic and the plane does not contain uncountably many disjoint continua each having three or more complementary domains. By Theorem 4, A is two-dimensional; this is a contradiction, since X is one-dimensional. Case 2 is thus vacuous, and Theorem 11 is established.

REMARK. In the proof of Theorem 11, each element of G_1 fails to separate the plane, and thus each element of G is indecomposable by a theorem of Jones' [11, Theorem 2]. The indecomposability of the elements of G also follows, as in the proof of Theorem 12, from a theorem proved by E. Dyer [8, p. 591].

THEOREM 12. *If X is a decomposable continuum which is homogeneous and hereditarily finitely multicoherent*[3], *then there is a nondegenerate continuous collection G of disjoint continua filling X such that X' is a simple closed curve and the elements of G are mutually homeomorphic, homogeneous, indecomposable continua.*

LEMMA 12.1. *An aposyndetic hereditarily finitely multicoherent continuum must be locally connected.*

Proof of Lemma 12.1. Let X be an aposyndetic hereditarily finitely multicoherent continuum. As in the proof Burgess [6, Theorem 8] has given for the case where X is hereditarily unicoherent, for any subcontinuum K of X and point p of $X - K$, there exists a positive integer m and continua $X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_m$ such that $K \subset [(X - X_1) + (X - X_2) + \cdots + (X - X_m)]$ and, for each i $(i \leqq m)$, $X_i + Y_i = X$ and $p \subset X - Y_i$. Since X is hereditarily finitely multicoherent, the common part of the continua $X_1, X_2, \cdots, X_m$ is the sum of a finite number of continua $Z_1, Z_2, \cdots Z_n$ not intersecting K. Then $X = Z_1 + Z_2 + \cdots + Z_n + Y_1 + Y_2 + \cdots + Y_m$, and X is locally connected by a theorem of Moore's [13, Theorem 51, p. 134].

LEMMA 12.2 *A locally connected, hereditarily finitely multicoherent continuum X must be hereditarily locally connected.*

Proof of Lemma 12.2. Suppose Y is a subcontinuum of X which fails to be locally connected. Then there exists a sequence of disjoint nondegenerate continua $N_1, N_2, \cdots$ in Y converging to a nondegenerate continuum N. Let p_1 and p_2 be points of N and R_1 and R_2 be open sets containing, respectively, p_1 and p_2 such that $cl(R_1) \cdot cl(R_2) = O$. Since X is locally connected, there exist connected open sets U_1 and

[3] A continuum X is said to be finitely multicoherent if for any two subcontinua X_1 and X_2 of X such that $X = X_1 + X_2$, the common part of X_1 and X_2 is the sum of a finite number of continua. If every subcontinuum of X is finitely multicoherent, then X is said to be hereditarily finitely multicoherent.

U_2 containing p_1 and p_2, respectively, such that U_1 and U_2 are contained in R_1 and R_2, respectively. We may choose a sequence of disjoint continua $M_1, M_2, \cdots$ such that each M_i is a subcontinuum of some N_j, each M_i is irreducible from $cl(U_1)$ to $cl(U_2)$, and the sequence $M_1, M_2, \cdots$ converges to a subcontinuum M of N. Then $cl(U_1) + \sum_{i=1}^{\infty} M_i + M$ and $cl(U_2) + \sum_{i=1}^{\infty} M_i + M$ are two subcontinua of X whose intersection is the sum of an infinite number of disjoint continua. This is a contradiction; hence, X must be hereditarily locally connected.

Proof of Theorem 12. There exists a nondegenerate continuous collection G of disjoint continua filling X having the properties given at the beginning of the proof of Theorem 11. If A is a subcontinuum of X', a result of Whyburn's [17, p. 154] for monotone decompositions implies that A is finitely multicoherent if $f^{-1}(A)$ is finitely multicoherent; thus X' must be hereditarily finitely multicoherent. But X' is aposyndetic and thus, by Lemma 12.1, must be locally connected. By Lemma 12.2, X' is hereditarily locally connected, and Burgess [6, Theorem 14] has shown that a nondegenerate continuum which is homogeneous and hereditarily locally connected must be a simple closed curve.

To show the indecomposability of the elements of G, let us choose a continuous subcollection G_1 of G which is an arc with respect to its elements. Let g_0 and g_1 be the end elements of this arc, and x_0 and x_1 be points in g_0 and g_1 respectively. A continuum K in G_1^* which contains x_0 and x_1 must also contain each element of G_1 which it intersects. But K must then contain both g_0 and g_1, and, since K is connected, K must fill G_1^*; i.e. G_1^* is irreducible from x_0 to x_1. Dyer [8, p. 591] has shown that such a collection must contain an indecomposable continuum; thus, each continuum of G is indecomposable.

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