COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

ELLIOT CARL WEINBERG

Loosely speaking, a lattice is called distributive if the order of performing the operations of finite suprema and infima may be interchanged. A lattice is called completely distributive if the order of performing the operations of infinite suprema and infima may be interchanged. The purpose of this note is to relate the property of complete distributivity in l-groups to other l-group properties. We shall prove, for example, that an l-group G which has an atomistic lattice of polars is completely distributive. The most interesting results are obtained for Archimedean l-groups. In this case the two above-mentioned properties are equivalent to each other and to the existence of certain 'nice' representations of G as subdirect unions of simply ordered groups. In the last section examples are given to distinguish these properties for non-Archimedean groups.

1. Preliminaries. We shall follow the notation and terminology of Chapter XIV of [1], to which the reader is referred for general background concerning l-groups.

1.1. DEFINITION. Let $\mathscr{C} = \{C_i: i \in I\}$ be a family of simply ordered groups. The *complete direct union* of $\mathscr{C}$ is the l-group of all functions $a: I \rightarrow \cup \mathscr{C}$ such that $a_i = a(i) \in C_i$ for all $i \in I$ with the operations defined by $(a \vee b)_i = a_i \vee b_i$ and $(a + b)_i = a_i + b_i$ for all $i \in I$. The *discrete direct union* of $\mathscr{C}$ is the l-subgroup of the complete direct union which consists of those functions which are zero at all but a finite number of points of I. The l-group H is a *subdirect union* of $\mathscr{C}$ if it is an l-subgroup of the complete direct union for which the projection map P_i of H into the factor group C_i maps H onto C_i for each $i \in I$. The subdirect union H of the family $\mathscr{C}$ is called *regular* if the projection map P_i is a *complete lattice homomorphism* for each $i \in I$; i.e., $P_i(\bigvee_{j \in J} g_j) = \bigvee_{j \in J} P_i(g_j)$ for each $i \in I$. (More generally, a sublattice L of a lattice M is called a regular sublattice if the injection map of L into M preserves infinite suprema and infima.) A *complete subdirect union* of $\mathscr{C}$ is an l-subgroup of the complete direct union which contains the discrete direct union. An l-group is called *representable* if it is isomorphic to a subdirect union of simply ordered groups.

Recall that a *polar* of an l-group G is a subset of G which consists of the elements disjoint from each element of some subset of G.

Received October 18, 1961.

Partially ordered by inclusion the set of polars of an l-group forms a complete Boolean algebra. The role of polars in the study of l-groups is indicated by the following proposition, a proof of which may be found in [3].

1.2. PROPOSITION. Let G be an l-group.

(a) If J is an l-ideal of G, then G/J is simply ordered if and only if J contains each element of some prime ideal in the Boolean algebra of all polars of G.

(b) Each polar of G is an l-ideal if and only if G is representable.

(c) G is isomorphic to a complete subdirect union of simply ordered groups if and only if each polar of G contains a minimal direct factor of G.

1.3. PROPOSITION. Let J be an l-ideal of the l-group G. In order that the canonical homomorphism $\varphi: G \to G/J$ be a complete lattice homomorphism it is necessary and sufficient that J be closed (i.e., if $\{j_i: i \in I\} \subset J$ and $\bigvee_{i\in I} j_i$ exists in G, then $\bigvee_{i\in I} j_i \in J$).

Proof. Assume that J is a closed l-ideal. Let $g = \bigvee_{i\in I} g_i (g, g_i \in G)$. Then $\bigwedge_{i\in I}(g - g_i) = 0$. Suppose that $h \in G^+$ satisfies $0 \leqq \varphi(h) \leqq \varphi(g - g_i)$ for each $i \in I$. Then, for each $i \in I$, there exists $h_i \in J \cap G^+$ such that $h \leqq g - g_i + h_i$, from which we can calculate $0 \bigvee (h - h_i) \leqq g_i - g$ and $0 = \bigwedge_{i\in I}(g - g_i) = \bigwedge_{i\in I}[(h - h_i) \bigvee 0] = h + \bigwedge_{i\in I}(-h_i \bigvee - h)$, so $h = \bigvee_{i\in I}(h \bigwedge h_i)$. Since J is convex, $h \bigwedge h_i \in J$; since J is closed, $h \in J$. Hence $\varphi(h) = 0$ and $\bigwedge_{i\in I} \varphi(g - g_i) = \varphi(g) - \bigvee_{i\in I} \varphi(g_i) = 0$. The converse is obvious.

1.4. COROLLARY. *If a polar J of an l-group G is an l-ideal, then the homorphism $\varphi: G \to G/J$ is complete.*

Proof. Let J be the set of all elements of G disjoint from each element of the set H. Suppose that $\{j_i: i \in I\} \subset J$ and $\bigvee_{i\in I} j_i$ exists in G. It suffices to assume that $j_i \geqq 0$ for each $i \in I$. Let $h \in H$. Since $j_i \bigwedge |h| = 0$ for each i, $(\bigvee_{i\in I} j_i) \bigwedge |h| = \bigvee_{i\in I}(j \bigwedge |h|) = 0$. Thus $\bigvee_{i\in I} j_i \in J$.

1.5. DEFINITION. An l-group G is *completely distributive* if $\bigwedge_{i\in I} \bigvee_{j\in J} g_{ij} = \bigvee_{\varphi\in J^I} \bigwedge_{i\in I} g_{i\varphi(i)}$ whenever $\{g_{ij}: i \in I, j \in J\}$ is a subset of G for which all of the indicated suprema and infima exist.

1.6. PROPOSITION. Let G be an l-group. The following are equivalent.

(a) G is completely distributive.

(b) If $\{g_{ij}: i \in I, j \in J\} \subset G^+$ and $0 < g = \bigvee_{j \in J} g_{ij}$ for each $i \in I$, then there exists $\varphi \in J^I$ such that $\bigwedge_{i \in I} g_{i\varphi(i)} = 0$ is false.

(c) If $0 < g \in G$, then there exists $g^* > 0$ such that $g = \bigwedge_{i \in I} g_i$ implies $g^* < g_i$ for some $i \in I$.

Proof. The equivalence of (a) with (b) is discussed in [5], while the equivalence of (b) with (c) is clear.

1.7. PROPOSITION. Let G be an l-subgroup of H such that each positive element of H is the supremum of some family of elements of G. If G is completely distributive, then H is completely distributive.

Proof. Suppose that $0 < h \in H$. There exists $g \in G$ such that $0 < g \leqq h$. Let g^* be the element whose existence is guaranteed by 1.6(c). Suppose that $h = \bigvee_{i \in I} h_i$ where $\{h_i: i \in I\} \subset H^+$. For each $i \in I$ there exists a family $\{g_{ij}: j \in J\} \subset G^+$ such that $h_i = \bigvee_{i \in J} g_{ij}$. Then

$$g = g \wedge h = g \wedge (\bigvee_{i \in I} h_i) = g \wedge (\bigvee_{i \in I,\, j \in J} g_{ij}) = \bigvee_{i,j} (g \wedge g_{ij}) .$$

There exists a pair $i \in I \in I, j \in J$ such that $g^* \leqq g_{ij}$. Hence $g^* \leqq h_i$, so H is completely distributive.

2. The main result. Let G be an l-group. Consider the following properties.

(A) G is isomorphic to a complete subdirect union of simply ordered groups.

(B) The Boolean algebra $P(G)$ of polars of G is atomistic.

(C) G is isomorphic to a regular subdirect union of simply ordered groups.

(D) G is completely distributive.

2.2. THEOREM. *If G is representable, then* (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D). *If G is Archimedean, then the four properties are equivalent. For arbitritary l-groups,* (B) $\Rightarrow$ (D).

Proof.

(A) implies (B). This is an immediate consequence of 1.2(c).

(B) implies (C). Observe that (C) is equivalent to the requirement that there exist a family of l-ideals L_i such that each G/L_i is simply ordered, each homomorphism $G \to G/L_i$ is complete, and $\bigcap_i L_i = \{0\}$. Let a be a strictly positive element of G. There exists a maximal polar L_a containing the set of all elements disjoint from a but not containing a. Since G is representable, each L_a is an l-ideal. By 1.3 and 1.2(a), the family $\{L_a: 0 < a \in G\}$ has the requisite properties.

(C) implies (D). Simply ordered groups, the complete direct union of a family of simply ordered groups, and regular l-subgroups of such groups are completely distributive. The l-group G belongs to the last category.

If G is Archimedean, then (D) implies (A). Let G_∞ denote the completion of G [1, p. 229]; i.e., G_∞ is a complete l-group which contains G as an l-subgroup in such a way that each element of G_∞ is the supremum of the set of all elements of G which it contains. By 1.7, G_∞ is completely distributive. We shall first show that $P(G_\infty)$ is atomistic. If P is a nonempty polar of G_∞, then in contains a strictly positive element f. Consider the family $\{(f_{i0}, f_{i1}): i \in I\}$ of all pairs of elements of G_∞^+ such that $f = f_{i0} \vee f_{i1}$ for each $i \in I$. Since G_∞ is completely distributive and complete, there exist $\varphi \in 2^I$ and $h \in G_\infty$ such that $0 < h = \bigwedge_{i \in I} f_{i\varphi(i)}$. We claim that $\bar{h}$, the smallest polar of G_∞ which contains h, is an atom which is contained in P. Indeed, since $h \leqq f$, $\bar{h} \subset P$. Now, recalling from [1, p. 233] that every closed l-deal of a complete l-group is a direct factor, we see that if K and K' are complementary polars of G_∞, then G_∞ is the direct union of K and K'. Thus there exist $k \in K$ and $k' \in K'$ such that $f = k + k' = k \vee k'$, from which it follows that $h \leqq k$ or $h \leqq k'$. Hence $\bar{h} \subset K$ or $\bar{h} \subset K'$. Since $\bar{h}$ is contained in one element of each complementary pair of $P(G_\infty)$, it is an atom.

Now observe that the map $P \to P \cap G$ is an isomorphism of $P(G_\infty)$ onto $P(G)$. Thus to complete the proof it suffices to show that, for each atom A of $P(G_\infty)$, $A \cap G$ is a direct factor of G. Let A' denote the complement in $P(G_\infty)$ of A. Let $f \in G^+$. There exist unique elements $a \in A$ and $a' \in A'$ such that $f = a + a'$; moreover, a and a' belong to G_∞^+. If $a' = 0$, then $f = a \in A \cap G$, so suppose $a' > 0$. There exists $x \in G$ such that $0 < x \leqq a'$. Since G is Archimedean, there exists a positive integer n such the $nx \nleqq a'$. Since A is an atom of $P(G_\infty)$, it is simply ordered, so $a' \leqq nx$; moreover, $x \leqq a'$ implies that $x \in A'$. We can calculate $f \wedge nx = (a \vee a') \wedge nx = (a \wedge nx) \vee (a' \wedge nx) = 0 \vee a' = a'$, so $a' \in G$ and $a = f - a' \in G$. This completes the proof.

(B) implies (D). Let $\bar{f}$ be the smallest polar of G containing the strictly positive element f. If a is an element of an atom A of $P(G)$ contained in $\bar{f}$ such that $0 < a$, then $0 < a \wedge f \in A$. Since A is a simply ordered l-subgroup of G, it is completely distributive, so there exists $a^* \in A$ such that $a \wedge f = \bigvee_{i \in I} a_i \in A^+$, implies $0 < a^* \leqq a_i$ for some $i \in I$. Suppose that $f = \bigvee_{i \in I} f_i, f_i \in G^+$. Than $a \wedge f = \bigvee_{i \in I}(a \wedge f_i)$, and $a \wedge f_i \in A$ for each $i \in I$, so there exists $i \in I$ such that $0 < a^* \leqq a \wedge f_i \leqq f_i$. In other words, G is completely distributive.

2.2. REMARKS. The observant reader will have noticed that the proof that (D) implies (A) may be modified to prove that a completely distributive l-group G has an atomistic lattice of polars provided that G may be embedded as an l-subgroup of a group H in which each polar is a direct factor and each positive element is the supremum of a set of elements of G. There are l-groups for which such embeddings cannot be found (see. 3.4.)

Some of the implications of 2.1 are suggested by earlier lattice-theoretic propositions. That (B) and (D) are related sounds like Tarski's theorem [4] that a Boolean algebra is completely distributive if and only if it is atomistic. That (C) and (D) are related sounds like a theorem of Raney [2] which says that a complete lattice is completely distributive if and only if it is isomorphic to a regular sublattice of a complete direct union of chains.

3. EXAMPLES. We will exhibit the following examples.

3.1. A representable l-group which satisfies (B) but not (A).

3.2. A representable l-group which satisfies (C) but not (B).

3.3. A nonrepresentable l-group which satisfies (D) but not (B).

3.4. A completely distributive l-group whose completion by cuts is not completely distributive.

Unfortunately, we do not know if there exists a representable l-group which satisfies (D) but not (C).

3.1. Consider the lexicographic product $J \oplus J_2$ of the l-group J of integers and the direct union J_2 of two copies of J. This l-group has precisely two proper nonzero polars, but it has no direct factors. The details have been discussed in [3].

3.2. Let $L = \bigoplus_{i \in N} J_i$ be the lexicographic sum of a countable family of copies of the l-group J of integers indexed by the set N of positive integers. Let Q denote the set of rational numbers of the form $p/2^k$ for $p = 0, 1, \cdots, 2^k - 1$ and $k \in N$.

Let

$$H_{p,k} = \{x \in Q : p/2^k \leqq x < (p+1)/2^k\} .$$

Denote by G the set of all elements g of the l-group L^Q of all functions from Q into L such that there exists $k = k(g) \in N$ satisfying the two conditions:

(i) if $k < m$, then $g(x)(m) = 0$ for all $x \in Q$, and

(ii) if $m \leqq k$, then $g(x)(m) = g(y)(m)$ for all $x, y \in H_{p,m}$. We shall show that G is an l-subgroup of L^Q which has the desired properties.

It is clear that $g, h \in G$, implies $g\text{-}h \in G$, so G is a partially ordered group. Let $g \in G$, and denote by $g \vee 0$ the supremum of g and 0 in L^Q. Since L is simply ordered, for each $x \in Q$, either $(g \vee 0)(x) = 0$ or $(g \vee 0)(x) = g(x)$. Certainly, $(g \vee 0)(x)(j) = 0$ for all $j > k(g)$ and for all $x \in Q$. Now let $x, y \in H_{p,j}$ for some $j \leqq k(g)$. Observe that $x \in H_{q,m}$ for $m \leqq j$ if and only if $y \in H_{q,m}$. If $(g \vee 0)(x)(j) \neq (g \vee 0)(y)(j)$, then we may assume, since $g(x)(j) = g(y)(j)$, that $(g \vee 0)(x) = 0$ while $0 < g(y) = (g \vee 0)(y)$. If m is the first integer such that $g(y)(m) \neq 0$, then $m \leqq j$. Let q be an integer such that $y \in H_{q,m}$. Then $x \in H_{q,m}$, $0 < g(x)(m) = g(y)(m)$, and $g(x)(m') = 0$ for all $m' < m$. Hence $g(x) > 0$. With this contradiction we have completed the proof that $g \vee 0 \in G$. Hence G is an l-subgroup of L^Q.

For each $x \in Q$, let $M_x = \{g \in G: g(x) = 0\}$. The reader can easily verify that each M_x is an l-ideal of G, that $\bigcap_{x \in Q} M_x = \{0\}$, and that G/M_x, being isomorphic to L, is simply ordered. We shall prove that each M_x is closed. Suppose, on the contrary, that there exists a set $\{g_i: i \in I\} \subset M_x$ such that $\bigvee_{i \in I} g_i = g$ while $g(x) = a > 0$. Let $\bar{a}$ denote the function in L^Q with constant value a. Let $h_i = (\bar{a} - g_i) \vee 0$. Then $\bigwedge_{i \in I} h_i = (\bar{a} - \bigvee_{i \in I} g_i) \vee 0 \in M_x$, while $h_i(x) = a$ for each $i \in I$. We can show, however that $\{h_i: i \in I\}$ has a positive lower bound not in M_x, a contradiction which will complete the argument that G satisfies property (C). Indeed, let j be the first integer such that $a(j) > 0$. (Suppose that $x = p/2^k$.) Then $h_i(p/2^k)(j) = a(j) > 0$ for each $i \in I$. If $k \leqq j$, then $h_i(y)(j) = a(j)$ for all $y \in H_{p2^{j-k},j} \subset H_{p,k}$. In this case each h_i is bounded below by the function f defined by

(i) for $y \in H_{p2^{j-k},j}$, $f(y)(j) = a(j)$, $f(y)(j+1) = a(j+1) - 1$, and $f(y)(m) = 0$ otherwise; and

(ii) for $y \notin H_{p2^{j-k},j}$, $f(y) = 0$.

If $j < k$, then let $b \in L$ be given by $b(k) = 1$ and $b(m) = 0$ for $m \neq k$. Then $b < a$. Letting $f_i = h_i \wedge \bar{b}$, we see that $0 < f_i$ and $f_i(p/2^k) = b$ for each $i \in I$. Now the previous case applies to the new set of functions $\{f_i: i \in I\}$.

It is easy to verify that $P(G)$ is not atomistic. For $g \in G$, let $Z(g) = \{x \in Q: g(x) = 0\}$. Since $|g| \wedge |h| = 0$ if and only if $Z(g) \cup Z(h) = Q$, while, for each $g \in G$, we can find $h \in G$ such that $|g| \wedge |h| = 0$ and $Z(g) = Q - Z(h)$, we conclude that the smallest polar containing an element g consists of all elements h such that $Z(g) = Z(h)$. Hence the family $\{H_{p,k}: k \in N;\ p = 0, 1, \cdots, 2^k - 1\}$, partially ordered by inclusion, is isomorphic to a coinitial subset of the partially ordered set of nonzero polars. Since the first family has no minimal elements, $P(G)$ has no atoms.

3.3. Consider the l-group C of one-to-one order preserving maps of the closed unit interval onto itself ($(f+g)(x) = f(g(x))$ and $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all x). That C is completely distributive follows from the fact that C is a regular sublattice of the lattice of all functions from the unit interval into itself. That $P(C)$ is not atomistic follows from the fact that each strictly positive element of C contains two disjoint strictly positive elements.

3.4. We again call upon the l-group of example 3.1. Let $\bar{J}_2$ denote the lattice obtained by adjoining a largest element (∞, ∞) to J_2. It is easily verified that the completion of $J \oplus J_2$ is $J \oplus \bar{J}_2$.

The conditionally complete lattice $J \oplus \bar{J}_2$ is not completely distributive. For each pair of integers (a, b), let $f_{ab} = (1, (a, b))$. Then

$$\bigwedge_{a \in J} \bigvee_{b \in J} f_{ao} = (1, (\infty, \infty)), \text{ while}$$
$$\bigvee_{\varphi \in J^J} \bigwedge_{a \in J} f_{a\varphi(a)} = (0, (\infty, \infty)) .$$

This example is interesting for another reason. The l-group $J \oplus J_2$ is maximal in the sense that it cannot be embedded as a proper l-subgroup of any l-group H in such a way that each element of H is the supremum of a set of elements of $J \oplus J_2$.

References

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, rev. ed., **25** (1948).
2. G. N. Raney, *A subdirect union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc., **4** (1953), 518-522.
3. F. Sik, *Über Summen einfach geordneter Gruppen*, Czech. Math. J., **8** (1958), 22-53.
4. E. C. Smith Jr. and A. Tarski, *Higher degrees of completeness and distributivity in Boolean algebreas*, Trans. Amer. Math. Soc., **84** (1957), 230-257.
5. E. C. Weinberg, *Higher degrees of distributivity in lattices of continuous functions*, Trans. Amer. Math. Soc., **104** (1962), 334-346.

UIVERSITY OF ILLINOIS