MEROMORPHIC FUNCTIONS AND CONFORMAL METRICS ON RIEMANN SURFACES

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1. The starting point of the present paper is the classical theory of meromorphic functions in the plane or the disk. We shall generalize fundamentals of this theory to open Riemann surfaces W_s that carry a specified conformal metric (Nos. 3, 11). The motivation is that meromorphic functions are defined by a local property and it is natural to consider them on the corresponding locally defined carrier, a 2-manifold with conformal structure.

The method we shall use largely parallels that of F. Nevanlinna [10] and L. Ahlfors [1]. We have, however, made an effort to write the presentation self-contained. The classical theory will be included as a special case.

We note in reference to earlier work generalizations given in various directions by L. Ahlfors [2], S. Chern [4, 5], G. af Hällström [6], K. Kunugui [8], L. Myrberg [9], L. Sario [14, 15], J. Tamura [19], Y. Tumura [22], and M. Tsuji [21].

2. Our principal result will be the integrated (Nevanlinna) form of the second main theorem on W_s (No. 17). No generalization of this theorem to Riemann surfaces of arbitrary genus has, to our knowledge, been given thus far. As a corollary the following extension of Picard's theorem will be established: Let P be the number of Picard values of a meromorphic function w on a Riemann surface W_p with the capacity metric (No. 21). Form the characteristic function $T(h)$ of w on the region W_h bounded by the level line $p_\beta = h$ of the capacity function p_β. Denote by $E(h)$ the integrated Euler characteristic of W_h and set

$$\eta = \overline{\lim} \frac{E(h)}{T(h)} .$$

Then

$$P \leqq 2 + \eta .$$

This bound is sharp (No. 27). Analogous extensions will be given to other classical consequences of the second main theorem (Nos. 31–36).

A generalization to arbitrary Riemann surfaces of the nonintegrated form of the second main theorem is given in [18].

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§1. Conformal metric

3. Let W be an open Riemann surface. We introduce on W a conformal metric

$$ds = \lambda(z) \mid dz \mid , \tag{1}$$

where $\lambda(z) \geqq 0$ is continuous, with no points of accumulation of its zeros, and ds is invariant under change of parameter z. Other conditions, easily met in our applications, will be imposed upon $\lambda(z)$ in the course of our reasoning. The length $l(\alpha)$ of a rectifiable arc α on W is well defined, and the distance $d(z_1, z_2)$ between two points is inf $l(\alpha)$ for arcs α from z_1 to z_2. The distance $d(E_1, E_2)$ between two subsets of W is defined as inf $d(z_1, z_2)$ for $z_1 \in E_1, z_2 \in E_2$.

Let W_0 be the interior of a compact bordered Riemann surface contained in W. We so choose the metric ds that $d(z, W_0)$ tends to a constant $\sigma_\beta \leqq \infty$ for any appoach of z to the ideal boundary β of W:

$$\sigma_\beta = \lim_{n\to\infty} d(z_n, W_0) \tag{2}$$

for any sequence $\{z_n\}$ tending to β. We consider the "level lines"

$$\beta_\sigma = \{z \mid d(z, W_0) = \sigma\} ,$$

$0 \leqq \sigma < \sigma_\beta$, and postulate that ds satisfies the condition

$$\int_{\beta_\sigma} ds = 1 . \tag{3}$$

Finally, the metric ds is chosen sufficiently regular to justify (3) and other operations to be performed on it. In particular, β_σ is assumed to be smooth at points z with $\lambda(z) > 0$.

The interesting differential geometric problem of characterizing all metrics for which these conditions are satisfied will not be discussed in the present paper. In our applications (nos. 20–37) the conditions are trivially fulfilled.

4. Schematically, the parameter σ and the arc length s along β_σ constitute a coordinate system on W. If W_σ signifies the relatively compact region bounded by β_σ, then $W_\sigma - W_0$ corresponds to a rectangle of width σ and height 1 in the (σ, s)-plane. A concrete illustration is given by $\lambda = \mid \operatorname{grad} u \mid$ for a harmonic function u on $\bar{W}_\sigma - W_0$ with $u = 0$ on β_0, $u =$ const. on β_σ such that $\int_{\beta_0} du^* = 1$. For genus

$g \geqq 0$ and connectivity $c \geqq 2$ of $W_\sigma - W_0$, $2g + 2(c-2)$ horizontal slits appear in the (σ, s)-rectangle; the edges of the slits, and the horizontal sides of the rectangle are suitably identified to form a conformal equivalent of $W_\sigma - W_0$. The slits issue from the zeros of $|\operatorname{grad} u|$. The g "handles" of $W_\sigma - W_0$ give rise to $2g$ slits in the interior of the rectangle, and the c contours cause $2(c-2)$ slits terminating at the vertical edges of the rectangle. In the general case of $ds = \lambda\,|dz|$ the end points of the slits are at the zeros of λ. The rate of growth of the number of these zeros will play a fundamental role in our approach.

§2. The first main theorem

5. Our principal aim is the second main theorem and Picard's theorem. Since they concern the behavior of a meromorphic function w on approaching β, it suffices to consider w in the boundary neighborhood $W - W_0$. The first main theorem on arbitrary open Riemann surfaces will first be needed.

The spherical distance $[w, a]$ between the points w and a is given by

$$[w, a] = \frac{|w-a|}{\sqrt{1+|w|^2}\sqrt{1+|a|^2}} .$$

We consider the proximity function

$$m(\sigma, a) = \frac{1}{2\pi}\int_{\beta_\sigma - \beta_0} \log\frac{1}{[w, a]}\,ds , \tag{4}$$

where the constant $1/2\pi$ is for convenience in later calculations. Let $n(\sigma, a)$ be the number of a-points, counted with their multiplicities, of the function w in $W_\sigma - W_0$. The counting function is defined as

$$N(\sigma, a) = \int_0^\sigma n(\sigma, a)\,d\sigma . \tag{5}$$

For $m(\sigma, \infty)$, $n(\sigma, \infty)$, $N(\sigma, \infty)$, the notations $m(\sigma, w)$, $n(\sigma, w)$, $N(\sigma, w)$ will also be used.

6. Differentiation of (4) gives for any a, b, finite or infinite, and for σ with no zeros of λ on β_σ,

$$\begin{aligned} \frac{dm(\sigma, a)}{d\sigma} - \frac{dm(\sigma, b)}{d\sigma} &= \frac{1}{2\pi}\int_{\beta_\sigma} \frac{d}{d\sigma}\log\left|\frac{w-b}{w-a}\right| ds \\ &= \frac{1}{2\pi}\int_{\beta_\sigma} d\arg\frac{w-b}{w-a} = n(\sigma, b) - n(\sigma, a) + n_0(b) - n_0(a) , \end{aligned} \tag{6}$$

where $d/d\sigma$ stands for the exterior normal derivative in the metric under consideration, and $n_0(a)$ is the number of a-points in W_0. The differentiation under the integral sign is legitimate, for $\int_{\beta\sigma}$ is an integral with respect to s from 0 to 1. For the characteristic function of w we choose

$$T(\sigma) = m(\sigma, \infty) + N(\sigma, \infty) + n_0(\infty)\sigma \ . \tag{7}$$

On integrating (6) from 0 to σ we arrive at the

FIRST MAIN THEOREM ON RIEMANN SURFACES. *Let w be a meromorphic function on an arbitrary open Riemann surface W. Then*

$$m(\sigma, a) + N(\sigma, a) + n_0(a)\sigma = T(\sigma) \tag{8}$$

for all values a.

We made no use of properties of ds outside of W_σ. In any compact subregion $\Omega \subset W$ we can choose $ds = (1/2\pi) \mid \operatorname{grad} p_\Omega \mid$, where p_Ω is the capacity function of $\Omega = W_\sigma$ (L. Ahlfors-L. Sario [3]), and let β_0 be a level line of p_Ω near its pole in Ω. Thus the first main theorem is a general property of w in any compact subregion of an arbitrary W.

7. We observe in passing that the theorem can, of course, also be written in the classical form. Let $n_\sigma = n + n_0$, $N_\sigma = N + n_0\sigma$, and $m_\sigma = (1/2\pi)\int_{\beta_\sigma} \log(1/[w, a])ds$. Then

$$m_\sigma(\sigma, a) + N_\sigma(\sigma, a) = T(\sigma) + O(1) \ . \tag{9}$$

In the case of the z-plane and for $ds = \mid dz \mid/2\pi r$, this is Nevanlinna's first main theorem.

8. As is to be expected, the Shimizu-Ahlfors interpretation of the characteristic function continues to be valid in the present case. In integrating (8) over the area elements $d\omega(a)$ of the a-sphere A the integral of $\log\,[w, a]^{-1}$ is independent of w, and the integral of $m(\sigma, a)$ vanishes. One obtains

$$T(\sigma) = \frac{1}{\pi}\iint_A N(\sigma, a)d\omega(a) + C\sigma \ , \tag{10}$$

where C is independent of σ.

For convenience we shall indicate differentiation by subindices and use the notation

$$\mid w_s \mid = \frac{\left|\dfrac{dw}{dz}\right|}{\left|\dfrac{ds}{dz}\right|} = \mid w_z \mid \lambda^{-1} \ . \tag{11}$$

Then the π^{-1}-fold spherical area of the image under w of $W_\sigma - W_0$ is

(12) $$S(\sigma) = \frac{1}{\pi}\iint_A n(\sigma, a)d\omega(a) = \frac{1}{\pi}\int_0^\sigma d\sigma\int_{\beta_\sigma}\frac{|w_s|^2}{(1+|w|^2)^2}ds$$

and we have

(13) $$T(\sigma) = \int_0^\sigma S(\sigma)d\sigma + C\sigma .$$

The derivative of the characteristic function $T(\sigma)$ is, up to an additive constant, the spherical area $S(\sigma)$.

As a corollary one concludes that $T(\sigma)$ is convex in σ.

§3. Preliminary form of the second main theorem

9. Our next task is to compare the contributions to $T(\sigma)$ of $m(\sigma, a)$ and $N(\sigma, a)$. To this end we use a mass distribution

(14) $$d\mu(a) = \rho(a)d\omega(a)$$

with density $\rho(a)$ and total mass unity on the sphere A with diameter 1 above the w-plane. Again we simply postulate that $\rho(a)$ is sufficiently regular to justify subsequent operations on it. This condition will be obviously met by the particular ρ we shall use.

In the (σ, s)-plane the density takes the form

(15) $$\delta(z) = \frac{|w_s(z)|^2}{(1+|w(z)|^2)^2}\rho(w(z)) .$$

We apply the theorem on the arithmetic and geometric mean to $\delta(z)$ on β_σ:

$$\int_{\beta_\sigma}\log\delta\, ds \leqq \log\int_{\beta_\sigma}\delta\, ds ,$$

or, equivalently,

(16) $$\int_{\beta_\sigma}\log\frac{\delta}{\rho}ds + \int_{\beta_\sigma}\log\rho\, ds \leqq \log\int_{\beta_\sigma}\delta\, ds .$$

This is the preliminary form of the second main theorem. The proof of the final form consists in evaluating the three terms in (16).

10. The first term depends only on w and λ and will be expressed in terms of $T(\sigma)$, the counting function $N_1(\sigma)$ of the multiple points of w, and the counting function $N(\sigma, \lambda^{-1})$ of the zeros of λ. The second term in (16) depends on the mass distribution $d\mu$. If ρ is chosen with suitable singularities at given points $a_1, \cdots, a_q$ of the w-plane, then

$\int \log \rho \, ds$ will be, in essence, the sum of the proximity functions $m(\sigma, a_\nu)$. In the third term of (16) the integral is the σ-derivative of the mass on $w(W_\sigma - W_0)$ and the term will appear as a remainder in the final form of (16). Thus the sum $\sum m(\sigma, a_\nu)$ will be estimated in terms of $T(\sigma)$, $N_1(\sigma)$, and $N(\sigma, \lambda^{-1})$. This is the second main theorem on open Riemann surfaces.

§4. Evaluation of $\int \log (\delta/\rho) ds$

11. We set

$$K(\sigma) = \frac{1}{4\pi}\int_{\beta_\sigma} \log \frac{\delta}{\rho} ds = \frac{1}{2\pi}\int_{\beta_\sigma} \log \frac{|w_s|}{1+|w|^2} ds \tag{17}$$

and differentiate:

$$K'(\sigma) = \frac{1}{2\pi}\frac{d}{d\sigma}\int_{\beta_\sigma} \log \frac{1}{1+|w|^2} ds + \frac{1}{2\pi}\int_{\beta_\sigma} \frac{d}{d\sigma} \log |w_z \lambda^{-1}| \, ds . \tag{18}$$

To evaluate the first integral we have from (4)

$$m(\sigma, \infty) = -\frac{1}{4\pi}\int_{\beta_\sigma - \beta_0} \log \frac{1}{1+|w|^2} ds$$

and consequently

$$\frac{1}{2\pi}\frac{d}{d\sigma}\int_{\beta_\sigma} \log \frac{1}{1+|w|^2} ds = -2\frac{dm(\sigma, \infty)}{d\sigma} . \tag{19}$$

We now impose upon λ the further condition, always met in our applications, that $\log \lambda$ is harmonic except for logarithmic poles. For the second integral in (18) the argument principle then gives

$$\frac{1}{2\pi}\int_{\beta_\sigma} \frac{d}{d\sigma} \log |w_z \lambda^{-1}| \, ds = n\left(\sigma, \frac{1}{w_z}\right) - n(\sigma, w_z) - n\left(\sigma, \frac{1}{\lambda}\right) + C , \tag{20}$$

where $n(\sigma, \lambda^{-1})$ is the number of zeros of λ in $W_\sigma - W_0$, and C is independent of σ. The number of multiple points of w in $W_\sigma - W_0$, each k-tuple point counted $k-1$ times, is

$$n_1(\sigma) = n\left(\sigma, \frac{1}{w_z}\right) - n(\sigma, w_z) + 2n(\sigma, w) , \tag{21}$$

and it follows that

$$K'(\sigma) = n_1(\sigma) - 2n(\sigma, w) - 2\frac{dm(\sigma, w)}{d\sigma} - n\left(\sigma, \frac{1}{\lambda}\right) + C ,$$

or, by (8),

(22) $$K'(\sigma) = n_1(\sigma) - 2T'(\sigma) - n\left(\sigma, \frac{1}{\lambda}\right) + C .$$

On setting

(23) $$N_1(\sigma) = \int_0^\sigma n_1(\sigma)d\sigma, \quad N\left(\sigma, \frac{1}{\lambda}\right) = \int_0^\sigma n\left(\sigma, \frac{1}{\lambda}\right)d\sigma$$

we obtain

(24) $$\frac{1}{4\pi}\int_{\beta_\sigma} \log \frac{\delta}{\rho} ds = N_1(\sigma) - 2T(\sigma) - N\left(\sigma, \frac{1}{\lambda}\right) + C\sigma .$$

§5. Estimation of $\int \log \rho \, ds$

12. Let $a_1, a_2, \cdots, a_q$ be $q \geqq 3$ points of the extended w-plane. Choose

(25) $$\frac{1}{2}\log \rho(w) = \sum_1^q \log \frac{1}{[w, a_\nu]} - \log\left(\sum_1^q \log \frac{1}{[w, a_\nu]}\right) - C .$$

As $t = [w, a_\nu] \to 0$, then $\rho(w) \to \infty$ as rapidly as $t^{-2}(\log t)^{-2}$, and the mass $\iint \rho \, d\omega$ over a t-neighborhood of a_ν is dominated by a multiple of $\int_0^t t^{-1}(\log t)^{-2}dt$. Hence the total mass is finite and C in (25) can be chosen to make it unity.

13. Integration of (25) yields

(26) $$\frac{1}{4\pi}\int_{\beta_\sigma} \log \rho(w)\, ds = \sum_1^q m(\sigma, a_\nu) - \frac{1}{2\pi}\int_{\beta_\sigma} \log \sum_1^q \log \frac{1}{[w, a_\nu]} ds - C ,$$

where

$$\int_{\beta_\sigma} \log\left(\sum_1^q \log \frac{1}{[w, a_\nu]}\right) ds \leqq \log \sum_1^q m(\sigma, a_\nu) + C .$$

On observing that, by (8), $m(\sigma, a_\nu) \leqq T(\sigma)$, we obtain

(27) $$\frac{1}{4\pi}\int_{\beta_\sigma} \log \rho(w) ds \geqq \sum_1^q m(\sigma, a_\nu) + O(\log T(\sigma)) .$$

§6. Estimation of $\log \int \delta\, ds$

14. We shall now estimate

$$\int_{\beta_\sigma} \delta\, ds = M'(\sigma)\,, \tag{28}$$

where $M(\sigma)$ is the mass distributed on the image of $W_\sigma - W_0$:

$$M(\sigma) = \int_0^\sigma \int_{\beta_\sigma} \delta\, ds = \iint_{\varDelta} n(\sigma, a)\rho(a) d\omega(a)\,. \tag{29}$$

On setting

$$Q(\sigma) = \int_0^\sigma M(\sigma) d\sigma = \iint_{\varDelta} N(\sigma, a)\rho(a) d\omega(a) \tag{30}$$

we get from (8)

$$Q(\sigma) \leqq T(\sigma)\,. \tag{31}$$

$M'(\sigma)$ will now be estimated separately in cases $\sigma_\beta = \infty$ and $\sigma_\beta < \infty$.

15. For $\sigma_\beta = \infty$ and any constant $\alpha \geqq 0$, let $\varDelta'$ be the set of values σ for which $M'(\sigma) \geqq e^{\alpha\sigma} M(\sigma)^2$. We choose an arbitrarily small fixed $\sigma_0 > 0$ and let $\sigma > \sigma_0$ in the sequel. Then

$$\int_{\varDelta'} e^{\alpha\sigma} d\sigma \leqq \int_{\varDelta'} \frac{dM}{M^2} < \frac{1}{M(\sigma_0)} < \infty\,.$$

For the set $\varDelta''$ of values σ with $M(\sigma) \geqq e^{\alpha\sigma} Q(\sigma)^2$ we obtain similarly

$$\int_{\varDelta''} e^{\alpha\sigma}\, d\sigma = \int_{\varDelta''} \frac{dQ}{Q^2} < \frac{1}{Q(\sigma_0)} < \infty\,.$$

We infer that, for $\sigma \notin \varDelta = \varDelta' \cup \varDelta''$, $M' < e^{3\alpha\sigma}\, Q(\sigma)^4$ and consequently

$$\log M'(\sigma) < 3\alpha\sigma + 4 \log Q(\sigma)\,. \tag{32}$$

From (31) it follows that for any $\alpha \geqq 0$

$$\log \int_{\beta_\sigma} \delta\, ds = O(\sigma + \log T(\sigma)) \tag{33}$$

except perhaps in a set $\varDelta$ so small that $\int_{\varDelta} e^{\alpha\sigma}\, d\sigma < \infty$.

16. In the case $\sigma_\beta < \infty$ let

$$\Delta' = \{\sigma \mid M'(\sigma) \geqq e^{\alpha/(\sigma_\beta - \sigma)} M(\sigma)^2\}$$

with $\alpha > 0$. Then

$$\int_{\Delta'} e^{\alpha/(\sigma_\beta-\sigma)}\, d\sigma \leqq \int_{\Delta'} \frac{dM}{M^2} < \infty .$$

Similarly for

$$\Delta'' = \{\sigma \mid M(\sigma) \geqq e^{\alpha/(\sigma_\beta-\sigma)} Q(\sigma)^2\}$$

we have

$$\int_{\Delta''} e^{\alpha/(\sigma_\beta-\sigma)}\, d\sigma \leqq \int_{\Delta''} \frac{dQ}{Q^2} < \infty .$$

We conclude for $\sigma \notin \Delta = \Delta' \cup \Delta''$ that

$$\log \int_{\beta_\sigma} \delta\, ds = O\Big(\frac{1}{\sigma_\beta - \sigma} + \log T(\sigma)\Big) . \tag{33$'$}$$

§7. The second main theorem

17. It remains to substitute (24), (27), and (33) or (33′) into (16). We have reached the following extension of Nevanlinna's classical theorem to Riemann surfaces W_s endowed with our conformal metric ds (Nos. 3, 11):

SECOND MAIN THEOREM ON RIEMANN SURFACES. *Let w be a meromorphic function on W_s. For any finite number $q \geqq 3$ of values $a_1, \cdots, a_q$ the sum of the proximity functions $m(\sigma, a_\nu)$ grows so slowly that, if $\sigma_\beta = \infty$,*

$$\sum_1^q m(\sigma, a_\nu) < 2T(\sigma) - N_1(\sigma) + N\Big(\sigma, \frac{1}{\lambda}\Big) + O(\sigma + \log T(\sigma) \tag{34}$$

except perhaps in a set Δ of intervals with $\int_\Delta e^{\alpha\sigma}\, d\sigma < \infty$ for $\alpha \geqq 0$.

If $\sigma_\beta < \infty$, then the term $O(\sigma + \log T(\sigma))$ in (34) is replaced by

$$O\Big(\frac{1}{\sigma_\beta - \sigma} + \log T(\sigma)\Big) \tag{34$'$}$$

and the resulting inequality holds except perhaps in a set Δ so small that $\int_\Delta e^{\alpha/(\sigma_\beta-\sigma)}\, d\sigma < \infty$ for $\alpha > 0$.

18. An equivalent formulation of (34) is readily found by substituting for $m(\sigma, a_\nu)$ from (8). For $\sigma_\beta = \infty$ we have

$$(35)\qquad (q-2)T(\sigma) < \sum_1^q N(\sigma, a_\nu) - N_1(\sigma) + N\left(\sigma, \frac{1}{\lambda}\right) + O(\sigma + \log T(\sigma)) ,$$

while for $\sigma_\beta < \infty$ the term $O(\sigma + \log T(\sigma))$ is replaced by (34′). Both inequalities are valid except perhaps in Δ.

19. The presence of exceptional intervals Δ in the second main theorem was a consequence of the nature of estimation of $M'(\sigma)$. Since we had to start from an upper bound for the integral of the integral of $M'(\sigma)$, viz., $\int M(\sigma)d\sigma \leqq T(\sigma)$, a bound cannot always be given for $M'(\sigma)$ for all σ. If, however, $T(\sigma)$ and $N(\sigma, \lambda^{-1})$ grow sufficiently slowly, we shall show that the second main theorem holds without exceptional intervals Δ.

THEOREM. *Suppose $T(\sigma)$ and $N(\sigma, \lambda^{-1})$ do not grow more rapidly than $e^{\alpha\sigma}$ for some $\alpha > 0, \sigma_\beta = \infty$. Then*

$$(36)\qquad (q-2)T(\sigma) + N_1(\sigma) < \sum_1^q N(\sigma, a_\nu) + N\left(\sigma, \frac{1}{\lambda}\right) + O(\sigma) .$$

If $\sigma_\beta < \infty$ and $T(\sigma)$ and $N(\sigma, \lambda^{-1})$ are dominated by $e^{\alpha/(\sigma_\beta - \sigma)}$ for some $\alpha > 0$, then $O(\sigma)$ in (36) is to be replaced by $O(1/(\sigma_\beta - \sigma))$.

Proof. We let $N(\sigma) = \sum_1^q N(\sigma, a_\nu)$. For $\sigma_\beta = \infty$ it follows from $T(\sigma) = O(e^{\alpha\sigma})$ that $\log T(\sigma) = O(\sigma)$, and (36) holds for $\sigma \notin \Delta$. Now let σ be an arbitrary point of an interval in Δ and denote by σ' the right end point of that interval. Then (36) is true for σ'. Since $(q-2)T(\sigma) + N_1(\sigma)$ is an increasing function, we have

$$(37)\qquad (q-2)T(\sigma) + N_1(\sigma) < N(\sigma) + N\left(\sigma, \frac{1}{\lambda}\right) + [N(\sigma') - N(\sigma)] + \left[N\left(\sigma', \frac{1}{\lambda}\right) - N\left(\sigma, \frac{1}{\lambda}\right)\right] + O(\sigma') .$$

From $N(\sigma) = O(e^{\alpha\sigma})$ and the convexity of $N(\sigma)$ it follows that $N'(\sigma) = O(e^{\gamma\sigma})$ and consequently $N(\sigma') - N(\sigma) < c\int_\sigma^{\sigma'} e^{\gamma\sigma}\, d\sigma$ for $\gamma > \alpha$. By the defining property of Δ, the integral is $O(1)$. Similarly $N(\sigma', \lambda^{-1}) - N(\sigma, \lambda^{-1}) = O(1)$. Furthermore, $\sigma' - \sigma \leqq \int_\sigma^{\sigma'} e^{\alpha\sigma} d\sigma$, hence $\sigma' - \sigma = O(1)$, and we conclude that $O(\sigma') = O(\sigma)$. Statement (36) follows.

If $\sigma_\beta < \infty$, we obtain analogously $\log T(\sigma) = O(1/(\sigma_\beta - \sigma))$ and $N'(\sigma) = O(e^{\gamma/(\sigma_\beta - \alpha)})$ for some $\gamma > \alpha$. The proof, mutatis mutandis, remains valid.

§8. Capacity metric

20. To study consequences of the second main theorem we shall now leave the above generality of λ and introduce a specific metric.

Let Ω be the interior of a compact bordered subsurface of W, with $W_0 \subset \Omega$, and choose a point $\zeta \in W_0$. Consider the capacity function p_Ω of the boundary β_Ω of Ω. By definition,

$$p_\Omega(z) = \frac{1}{2\pi} \log |z - \zeta| + h(z) \tag{38}$$

near ζ in a fixed parametric disk, $h(z)$ being harmonic with $h(\zeta) = 0$. Moreover, $p_\Omega = k_\Omega = \text{const.}$ on β_Ω. The functions p_Ω form a normal family, and any limiting function p_β is a capacity function of β on W with pole at ζ [17, 20]. The constant k_Ω increases with Ω and tends to a limit $k_\beta \leqq \infty$. The limiting function p_β is unique if $k_\beta < \infty$. The capacity of the ideal boundary β is defined as $c_\beta = e^{-k_\beta}$.

For orientation we refer here to two known [3] properties of p_β, although they will not be needed in the sequel: Among all harmonic functions p on W with the behavior (38) at ζ, $\sup_W p$ is minimized by p_β and the minimum is k_β. The surface W is parabolic if and only if $c_\beta = 0$.

21. We choose the conformal metric

$$ds = |\operatorname{grad} p_\beta| \, |dz| \, . \tag{39}$$

Set $\sigma = k$ and denote by β_k the level line $p_\beta(z) = k$ with $0 \leqq k < k_\beta$. We may assume that the parametric disk for (38) was so chosen that β_0 is an analytic Jordan curve. Then W_0 is characterized by $p_\beta(z) < 0$, and condition (2) becomes

$$\lim_{\Omega \to W} p_\beta(z) = k_\beta \tag{40}$$

with $z \notin \Omega$. Condition (3), $\int_{\beta_k} ds = 1$, is trivially fulfilled. We shall designate by W_p a Riemann surface W with property (40) and with metric (39).

22. Denote by W_k the region $p_\beta(z) < k$ and consider the Euler characteristic

$$e(k) = -n_0 + n_1 - n_2 \tag{41}$$

of $W_k - W_0$ in a triangulation with n_0 vertices, n_1 edges, and n_2 faces. Without loss of generality we may assume that β_k consists of a finite number of analytic Jordan curves. This can always be achieved by a sufficiently small decrease of k without affecting the subsequent reasoning.

Our metric (39) has the following property:

LEMMA. *The number of zeros in* $W_k - W_0$ *of grad* p_β *is the Euler characteristic* $e(k)$ *of* $W_k - W_0$.

The geometric content of the lemma is clear. In fact, the number $n(k, \lambda^{-1})$ of zeros of $\lambda = |\operatorname{grad} p_\beta|$ is the same as the number of zeros of the derivative of $\eta(z) = \exp 2\pi(p_\beta + ip_\beta^*)$ in $W_k - W_0$. If β_k consists of one analytic Jordan curve, then the image under η of $W_k - W_0$ is a circluar annulus with radii 1, $e^{2\pi k}$, and $\eta'(z)$ has no zeros. If β_k consists of two curves, some level line $p_\beta^* =$ const. issuing from β_0 branches off at a zero z_0 of $\eta'(z)$ in $W_k - W_0$ to reach the two β_k-curves. If $W_k - W_0$ is cut along this level line from z_0 to β_k, the two shores of the cut appear under $\eta(z)$ as two radial slits terminating at $|\eta| = e^{2\pi k}$. More generally, if the connectivity of $W_k - W_0$ is c, then there are $c - 2$ zeros of $\eta'(z)$ and $2(c-2)$ radial slits in the image annulus. A similar reasoning shows that, for positive genus g of $W_k - W_0$, every handle gives rise to two zeros of $\eta'(z)$ and two radial slits in the interior of the annulus. The total number of zeros of $\eta'(z)$ in $W_k - W_0$ is thus

$$n(k, \lambda^{-1}) = 2g + c - 2 \,. \tag{42}$$

But this is known to be the Euler characteristic $e(k)$ of a bordered surface of genus g and connectivity c.

23. To establish our lemma analytically we choose the following simple proof given by B. Rodin in his doctoral dissertation [13]. It shows that the lemma is an immediate consequence of the Riemann-Roch theorem.

Form the double $\hat{W}_k$ of W_k by reflecting W_k with respect to β_k ([3], p. 119), and denote by $\hat{g}$ the genus of the closed surface $\hat{W}_k$. Extend $dp_\beta + idp_\beta^*$ analytically across β_k to $\hat{W}_k$ so as to obtain a meromorphic differential with two simple poles. By the Rimann-Roch theorem (e.g. [3], p. 324) the degree of all divisors in the canonical class on $\hat{W}_k$ is $2\hat{g} - 2$. It follows that $dp_\beta + idp_\beta^*$ has $2\hat{g}$ zeros in $\hat{W}_k$. By symmetry and by our convention in No. 22, $\hat{g}$ of these zeros are in W_k; by our choice of W_0 they all are in $W_k - W_0$. But $\hat{g} = 2g + c - 2$, where g and c are the genus and the number of contours of $W. - W_0$. This completes the proof.

24. We introduce the *integrated Euler characteristic*

$$E(k) = \int_0^k e(k)\,dk \tag{43}$$

and write our result

$$n(k, |\operatorname{grad} p_\beta|^{-1}) = e(k) \tag{44}$$

in the form

$$N(k, |\operatorname{grad} p_\beta|^{-1}) = E(k)\ . \tag{45}$$

On substituting this into (35) we obtain the following form of the second main theorem:

$$(q-2)T(k) < \sum_1^q N(k, a_\nu) - N_1(k) + E(k) + O(k + \log T(k))\ , \tag{46}$$

or, equivalently,

$$\sum_1^q m(k, a_\nu) < 2T(k) - N_1(k) + E(k) + O(k + \log T(k))\ . \tag{46$'$}$$

Both inequalities hold for $k_\beta = \infty$, while for $k_\beta < \infty$ the term $O(k + \log T(k))$ is to be replaced by $O(1/(k_\beta - k) + \log T(k))$.

§9. Extension of Picard's theorem

25. We know from No. 8 that $T(k)$ is convex in k. We now exclude from our consideration the degenerate case by assuming that

$$\lim_{k\to\infty} \frac{T(k)}{k} = \infty \tag{47}$$

if $k_\beta = \infty$. By virtue of (13) this means that we only permit functions with *unbounded spherical area* $S(k)$ of the image under $w(z)$ of $W_k - W_0$.

In the case $k_\beta < \infty$ we similarly make the assumption

$$\overline{\lim_{k\to k_\beta}}\, T(k)(k_\beta - k) = \infty\ , \tag{47$'$}$$

which implies that $S(k)$ grows more rapidly than $1/(k_\beta - k)$.

An illustrative case is the extended plane punctured at a countable point set. On this region, despite its weak boundary, there trivially are meromorphic functions with infinitely many Picard values, e.g., the identity function. To exclude such functions of no interest we require that there be, in some sense, an essential singularity on the ideal boundary β. The above condition has this effect:

A meromorphic function with property (47) *or* (47′) *comes arbitrarily* *close to every value a in every boundary neighborhood $W_p - W_k$.*

To see this suppose $[w(z), a] > \varepsilon$ for all $z \in W_p - W_{k_0}$ and some a, ε, $k_0 < k_\beta$. Then $m(k, a) < (1/2\pi) \log \varepsilon^{-1}$ and $N(k, a) \leqq n(k_0, a)k$ for $k > k_0$, and we have $T(k) = O(k)$, a contradiction.

26. We set

$$\eta = \overline{\lim_{k \to k_\beta}} \frac{E(k)}{T(k)} \tag{48}$$

and denote by P the number of Picard values of $w(z)$. For nondegenerate meromorphic functions characterized by property (47) or (47′) on a Riemann surface W_p we have from (46)

$$P \leqq 2 + \eta - \underline{\lim_{k \to k_\beta}} \frac{N_1(k)}{T(k)} , \tag{49}$$

or more simply:

PICARD'S THEOREM ON RIEMANN SURFACES. *The number of Picard values of w defined on W_p exceeds at most by two the upper limit of the integrated Euler characteristic divided by the Nevanlinna characteristic:*

$$P \leqq 2 + \eta . \tag{50}$$

For Riemann surfaces W_s of No. 17 an analogous Picard theorem can be obtained by replacing k by σ in (47) and (47′), and by substituting $N(\sigma, \lambda^{-1})$ for $E(k)$ in (48).

For functions with $E(k)/T(k) \to 0$ the Picard theorem takes the simple form $P \leqq 2$. In particular, this holds for functions on a Riemann surface of finite Euler characteristic, i.e., of finite genus and a finite number of boundary components. In the special case of a plane punctured at a finite number of points this is the theorem of G. af Hällström [6]. For the nonpunctured plane we have the classical Picard theorem.

27. We claim:

THEOREM. *The bound $2 + \eta$ for P is sharp.*

Specifically, for any integer $d \geqq 2$ there is a Riemann surface W_p and a meromorphic function w on W_p such that $P = 2 + \eta = d$.

28. For an even d we can make use of the well-known function

$$w = \sqrt[n]{\frac{e^z + i}{e^z - i}} \tag{51}$$

by choosing $n = d/2$. To this end consider the covering surface W of the $z = x + iy$-plane that consists of n sheets with branch points of multiplicity n at $z = i\pi(\frac{1}{2} + h), h = 0, \pm 1, \pm 2, \cdots$. The sheets are attached to each other in the usual manner along the edges of the slits from $z = i\pi(\frac{1}{2} + 2h)$ to $i\pi(\frac{3}{2} + 2h)$. The function (51) is meromorphic on W.

To evaluate $E(k)/T(k)$ choose the capacity function $p_\beta = (1/2\pi n) \log |z|$ on W. It differs from the usual capacity function in that it has several logarithmic poles, one on each of the n sheets above $z = 0$. However, the behavior of p_β in a boundary neighborhood is unchanged and the reasoning in § 8 remains valid. The set W_0 with $p_\beta < 0$ consists of n disks $|z| < 1$, but the disconnectedness of W_0 has no bearing on our reasoning concerning $W - W_0$. The metric is $ds = |dz|/2\pi n |z|$, the set β_k lies above $|z| = e^{2\pi nk}$, and the region W_k lies above $|z| < e^{2\pi nk}$.

In evaluating $e(k)$ and $n(k, \infty)$ for η we may consider W_k instead of $W_k - W_0$, in view of $k/T(k) \to 0$ and of the fact that W_0 only contributes fixed finite quantities to the above functions for a given w.

For the Euler characteristic $e(k)$ of W_k we have

$$e(k) = n\, e_0(k) + \Sigma b_\nu , \tag{52}$$

where $e_0(k)$ is the characteristic of the disk $|z| < e^{2\pi nk}$ covered by W_k, and Σb_ν is the sum of the orders of branch points of W_k. Since $e_0(k) = -1$, and Σb_ν above $|z| < 2\pi$ is $4(n-1)$, we obtain on disregarding bounded quantities,

$$e(k) \sim 4(n-1) \cdot \frac{e^{2\pi nk}}{2\pi} . \tag{53}$$

Integration from 0 to k yields

$$E(k) \sim \frac{n-1}{\pi^2 n} e^{2\pi nk} . \tag{54}$$

The poles w are the zeros of $e^z - i$, that is, $z_j = i(\pi/2 + 2\pi j)$ with all integers j. Every pole is simple, and there is only one point of W_k above each z_j. We conclude that

$$n(k, \infty) \sim 2 \cdot \frac{e^{2\pi nk}}{2\pi} . \tag{55}$$

Consequently

$$N(k, \infty) \sim \frac{e^{2\pi nk}}{2\pi^2 n} . \tag{56}$$

From $N(k, \infty) < T(k)$ it follows that $\eta \leqq 2n - 2$. Thus theorem (50) states that the number P of Picard values cannot exceed $2n$. But this is precisely the number of values $w = e^{i\nu\pi/n}$, $\nu = 0, \cdots, 2n-1$, uncovered by $w(z)$ (cf. below), and we have proved the sharpness of bound (50) for even d.

29. A geometric description of the image under w of W with the $2n$ Picard values $e^{i\nu\pi/n}$ may be illuminating. We first take the n-sheeted horizontal strip S of W between $y = 0$ and $y = 2\pi$. It is mapped by $s = e^z$ onto an n-sheeted s-plane S_s slit along the positive real axis and with branch points of order $n - 1$ at $s = \pm i$. The linear function $t = (s + i)/(s - i)$ maps S_s onto an n-sheeted t-plane S_t slit along the upper half of $|t| = 1$, and with branch points of order $n - 1$ at $t = 0$, ∞. The function $w = \sqrt[n]{t}$ maps S_t onto a 1-sheeted region S_w^0 of the w-plane less slits L_j along $|w| = 1$ from $e^{2\mu\pi i/n}$ to $e^{(2\mu+1)\pi i/n}$, $\mu = 0, 1, \cdots, n-1$.

Each n-sheeted strip $2\pi h \leqq y \leqq 2\pi(h+1)$ of W is mapped by w onto a duplicate S_w^h of S_w^0. The image W_w of W is obtained by identifying the inner edges $|w| = 1 - 0$ of all slits L_j on S_w^h with the outer edges $|w| = 1 + 0$ of the corresponding slits L_j on S_w^{h+1}. The process creates logarithmic branch points at the end points of the slits L_j. The function w has $2n$ Picard values $e^{i\nu\pi/n}$, $\nu = 0, \cdots, 2n-1$.

Regarding the sharpness of (50) for odd integers d the reader is referred to an example constructed by B. Rodin in his doctoral dissertation [13].

30. The surface described above has no algebraic branch points, hence $N_1(k) \equiv 0$. The question now arises whether or not the bound in (49) can be reached when the surface is so strongly ramified that $\overline{\lim}\,(N_1(k)/T(k)) > 0$. We again use (51) and form the function $\hat{w} = w^m$, where m is a factor of $2n$. A computation similar to the one in Nos. 28, 29 yields $\eta \leqq (2n-2)/m$, hence by (49)

$$P \leqq \frac{2n}{m} + \left(1 - \frac{1}{m}\right). \tag{57}$$

For $m = 1$ we again have the bound $2n$. Since P and $2n/m$ are integers we conclude for $m > 1$ that P cannot exceed $2n/m$. But this is clearly the number of Picard values of $\hat{w}$. For any integer $q \geqq 2$ we can choose $n = q$ and $m = 2$, say, and obtain q Picard values. We have shown that the bound in (49) is sharp for all positive integers.

§10. Defect and ramification relations

31. We conclude by listing a number of standard consequences of

the second main theorem, extended to meromorphic functions w with (47) or (47′) on Riemann surfaces W_p.

The counterpart of the *Picard-Borel theorem* reads: There are at most $2 + \eta$ values a_ν for which $\lim (N(k, a_\nu)/T(k)) = 0$.

For the proof we only have to choose $q > 2 + \eta$ values a_ν in (46) with $N(k, a_\nu)/T(k) \to 0$ to arrive at a contradiction.

32. Consider the defect

$$\delta(a) = \varliminf_{k \to k_\beta} \frac{m(k, a)}{T(k)}$$

of w. If $\eta < \infty$, then by (46′) the number of values a with $\delta(a) > (\eta + 2)/n$ is less than n and one infers that there are only a countable number of values a with $\delta(a) > 0$. The following extension of *Nevanlinna's defect relation* results:

$$\Sigma \delta(a) \leqq 2 + \eta \,. \tag{58}$$

33. Let $n_1(k, a)$ be the number of multiple a-points of w in $W_k - W_0$, an i-tuple point being counted $i - 1$ times. Let $N_1(k, a) = \int_0^k n_1(k, a) dk$. The ramification index of a is defined as

$$\vartheta(a) = \varliminf_{k \to k_\beta} \frac{N_1(k, a)}{T(k)} \,.$$

It is clear that the set of all multiple points of a given w is countable and that

$$\Sigma \varliminf_{k \to k_\beta} \frac{N_1(k, a)}{T(k)} \leqq \varliminf_{k \to k_\beta} \frac{N_1(k)}{T(k)} \,.$$

One obtains the generalization of *Nevanlinna's ramification relation*:

$$\Sigma \vartheta(a) \leqq 2 + \eta \,. \tag{59}$$

34. Relations (58) and (59) are, of course, special cases of the following consequence of (46′):

$$\Sigma \delta(a) + \Sigma \vartheta(a) \leqq 2 + \eta \,. \tag{60}$$

For the sum $\delta(a) + \vartheta(a)$ with a given a one has the inequality

$$\delta(a) + \vartheta(a) \leqq 1 \,. \tag{61}$$

This is obtained on dividing

$$T(k) = m(k, a) + N(k, a) + O(k)$$

by $T(k)$ and on observing that $N_1(k, a) \leqq N(k, a)$.

35. The contribution to $T(k)$ by the sum $m(k, a) + N_1(k, a)$ is measured by

$$\theta(a) = \underline{\lim}_{k \to k_\beta} \frac{m(k, a) + N_1(k, a)}{T(k)} .$$

The meaning of $\theta(a)$ is clarified by considering the number $\bar{n}(k, a)$ of a-points of w, each counted only once. We set $\bar{N}(k, a) = \int_0^k \bar{n}(k, a) dk$ and note that $\bar{n}(k, a) = n(k, a) - n_1(k, a)$ and $N(k, a) = \bar{N}(k, a) + N_1(k, a)$. It follows that

$$N_1(k, a) + m(k, a) = T(k) - \bar{N}(k, a) + O(k)$$

and consequently

$$\theta(a) = 1 - \overline{\lim}_{k \to k_\beta} \frac{\bar{N}(k, a)}{T(k)} .$$

For the sum of the $\theta(a)$ we have the bound

$$\Sigma \theta(a) \leqq 2 + \eta . \tag{62}$$

In fact,

$$\Sigma \underline{\lim}_{k \to k_\beta} \frac{m(k, a) + N_1(k, a)}{T(k)} \leqq \underline{\lim}_{k \to k_\beta} \frac{\Sigma m(k, a) + N_1(k)}{T(k)} \leqq 2 + \eta .$$

36. A value a is termed totally ramified if the equation $w(z) = a$ has no simple roots. The *Nevanlinna relation for totally ramified values* also can be generalized: their number does not exceed $4 + 2\eta$. In fact, for such a, $n(k, a) \geqq 2\bar{n}(k, a)$. One concludes that

$$\theta(a) \geqq 1 - \frac{1}{2} \overline{\lim}_{k \to k_\beta} (N(k, a)/T(k)) \geqq \frac{1}{2} .$$

The statement follows from (62).

37. It is an open question whether or not there are functions on a given W with one of the following properties:

(a) $P = 2 + \eta$,
(b) $P = 0$ but $\Sigma \delta(a) = 2 + \eta$,
(c) $\Sigma \vartheta(a) = 2 + \eta$,
(d) there are $4 + 2\eta$ totally ramified points.

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