THREE SPECTRAL THEOREMS FOR A PAIR OF SINGULAR FIRST-ORDER DIFFERENTIAL EQUATIONS

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1. Preliminaries. In the regular case the classical method of obtaining eigenvalues and eigenfunctions of the equation

$$y''(x) + [\lambda - q(x)]y = 0 \quad \left[' \equiv \frac{d}{dx}\right] \tag{1}$$

under Sturmian boundary conditions involves the use of asymptotic expansions. For the singular cases of (1) when the range of x is infinite or semi-infinite instead of finite, Titchmarsh [6] has shown that such asymptotic solutions are also necessary in obtaining spectral and expansion theorems by the method of complex variables. The objective of this paper is to generalize for a particular case these types of results to the following pair of equations

$$\begin{aligned} u'(x) - [\lambda a(x) + b(x)]v(x) &= 0\ , \\ v'(x) + [\lambda c(x) + d(x)]u(x) &= 0\ . \end{aligned} \tag{2}$$

Interest in this system arises from a consideration of the Dirac relativistic wave equations for a particle in a central field. The equations (2) correspond in this case to the radial wave equations. Conte and Sangren [2] and the authors [3] have shown that most of the results of Titchmarsh can be generalized for (2) over the interval $(0 \leqq x < \infty)$, under the restriction $a(x) = c(x) = 1$. Also, the spectral properties of (2) for $a(x) = c(x) = 1$ over the infinite interval $(-\infty, \infty)$ have been investigated [4]. In this paper a discussion of the system (2) for $a(x) = x^{2k}$, $c(x) = x^{-2k}$ over the interval $(0, \infty)$ is presented. It is assumed throughout, that k is a nonzero integer.

Let $\phi(x, \lambda) = [\phi_1(x, \lambda), \phi_2(x, \lambda)]$ and $\theta(x, \lambda) = [\theta_1(x, \lambda), \theta_2(x, \lambda)]$ be two solutions of system (2) over the interval $a \leqq x \leqq b$, where $a > 0$ and $b < \infty$, such that $\phi_1(l, \lambda) = 1$, $\phi_2(l, \lambda) = 0$, $\theta_1(l, \lambda) = 0$, $\theta_2(l, \lambda) = 1$, where $a \leqq l \leqq b$. It can be shown that the Wronskian $W_x[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1$ is independent of x so that since $W_l(\phi, \theta] = 1$, $\phi(x, \lambda)$ and $\theta(x, \lambda)$ are linearly independent. For the singular case it can be shown that for complex values of λ the system (2) has a solution $\psi(x, \lambda) = [\psi_1, \psi_2] = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$. A limit circle case is determined separately at each of the end points, 0 and ∞, by the conditions that all functions $c|\psi|^2 + a|\psi_2|^2$ are integrable, that is, belong to the class $L(0, l)$ or $L(l, \infty)$. In the limit point case, at either end, there exist only one $m(\lambda)$ and

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$\psi(x, \lambda)$ where $c|\psi_1|^2 + a|\psi_2|^2$ is integrable. The existence at each end of at least one such integrable function is guaranteed by the direct extension of Weyl's limit-point, limit-circle theorem [6, 1, 5]. Furthermore, the $m(\lambda)$ at either end is such that $\overline{m(\lambda)} = m(\bar{\lambda})$ and $m(\lambda)$ is analytic in either the upper or lower half plane. The spectral properties can be obtained from these functions, $m_0(\lambda)$ and $m_\infty(\lambda)$.

2. The nature of the spectrum; interval $(1, \infty)$. To obtain the spectral properties of the system (2) over the interval $(1, \infty)$ the order properties as $x \to \infty$ of the solution vectors ϕ and θ have to be investigated. This is most easily done by using the corresponding integral equations which may be obtained by the method of variation of coefficients. It is easily verified that if $\phi_1(1) = 1$, $\phi_2(1) = 0$, $\theta_1(1) = 0$ and $\theta_2(1) = 1$ the appropriate integral equations are given by

$$\begin{aligned}(4)\qquad \phi_1(x, \lambda) &= G_1(\lambda x) + \int_1^x \{b\phi_2[H_2G_1(\lambda x) - G_2H_1(\lambda x)] \\ &\quad + d\phi_1[H_1G_1(\lambda x) - G_1H_1(\lambda x)]\}\, ds \\ &= G_1(\lambda x) + L(x, \phi_1, \phi_2)\end{aligned}$$

$$\begin{aligned}(5)\qquad \phi_2(x, \lambda) &= G_2(\lambda x) + \int_1^x \{b\phi_2[H_2G_2(\lambda x) - G_2H_2(\lambda x)] \\ &\quad + d\phi_1[H_1G_2(\lambda x) - G_1H_2(\lambda x)]\}\, ds\,, \\ &= G_2(\lambda x) + K(x, \phi_1, \phi_2) \\ \theta_1(x, \lambda) &= H_1(\lambda x) + L(x, \theta_1, \theta_2) \\ \theta_2(x, \lambda) &= H_2(\lambda x) + K(x, \theta_1, \theta_2)\end{aligned}$$

where

$$G_1(\lambda x) = +\frac{x^p}{\Delta}\,[J_{-(p-1)}(\lambda)J_p(\lambda x) + J_{p-1}(\lambda)J_{-p}(\lambda x)]\,,$$

$$G_2(\lambda x) = +\frac{x^{-(p-1)}}{\Delta}\,[J_{-(p-1)}(\lambda)J_{p-1}(\lambda x) - J_{p-1}(\lambda)J_{-(p-1)}(\lambda x)]\,,$$

$$H_1(\lambda x) = +\frac{x^p}{\Delta}\,[J_{-p}(\lambda)J_p(\lambda x) - J_p(\lambda)J_{-p}(\lambda x)]\,,$$

$$H_2(\lambda x) = +\frac{x^{-(p-1)}}{\Delta}\,[J_{-p}(\lambda)J_{p-1}(\lambda x) + J_p(\lambda)J_{-(p-1)}(\lambda x)]\,,$$

$$\Delta = J_p(\lambda)J_{-(p-1)}(\lambda) + J_{-p}(\lambda)J_{p-1}(\lambda) = \frac{2\cos k\pi}{\pi\lambda}\,,$$

and

$$p = k + \frac{1}{2}\,.$$

Let $\lambda = \sigma + it$ where $t > 0$, and let

$$h_1(x, \lambda) = x^{-k}e^{-tx}\phi_1(x, \lambda) \quad \text{and} \quad h_2(x, \lambda) = x^k e^{-tx}\phi_2(x, \lambda) .$$

The well-known asymptotic formula $J_p(z) = (2/\pi z)^{1/2} \cos(z - 1/2\, p\pi - \pi/4) + 0\{/e^{-iz}//z/^{-3/2}\}$ for $0 \leqq \arg z \leqq \pi/2$ and $z \gg 1$ is used in the following. The substitution of h_1 and h_2 in (4) and (5) and the taking of absolute values yields

$$|h_1(x, \lambda)|; |h_2(x, \lambda)| \leqq M + \int_1^x \{|g_1|\,|h_2| + |g_2|\,|h_1|\}\, ds$$

where

$$M = \max_{x\to\infty} \left\{ \frac{x^{1/2}}{|\Delta|} [J_p(\lambda x)J_{-p+1}(\lambda) + J_{-p}(\lambda x)J_{p-1}(\lambda)]; \right.$$
$$\left. \frac{x^{1/2}}{|\Delta|} [-J_{p-1}(\lambda x)J_{-p+1}(\lambda) + J_{-p+1}(\lambda x)J_{p-1}(\lambda)] \right\}$$

$$g_1 = \max_{x\to\infty} \{[J_p(\lambda x)J_{-p+1}(\lambda s) + J_{-p}(\lambda x)J_{p-1}(\lambda s)];$$
$$[J_{p-1}(\lambda x)J_{-p+1}(\lambda s) - J_{-p+1}(\lambda x)J_{p-1}(\lambda s)]\}$$
$$\left\{ \frac{x^{1/2}}{|\Delta|} e^{-t(x-s)} b(s) s^{-2k+1/2} \right\}$$

$$g_2 = \max_{x\to\infty} \{[J_{-p}(\lambda x)J_p(\lambda s) - J_p(\lambda x)J_{-p}(\lambda s)];$$
$$[J_{-p+1}(\lambda x)J_p(\lambda s) + J_{p-1}(\lambda x)J_{-p}(\lambda s)]\}$$
$$\left\{ \frac{x^{1/2}}{|\Delta|} e^{-t(x-s)} d(s) s^{2k+1/2} \right\} .$$

According to the lemma in [3] this can be reduced to the inequality

$$h_1(x, \lambda); h_2(x, \lambda) \leqq M \exp\left\{ \int_1^x (g_1 + g_2)\, ds \right\} .$$

Consequently, it can be concluded that when $b(x)x^{-2k}$ and $d(x)x^{2k}$ are both $L(1, \infty)$, $h_1(x, \lambda)$ and $h_2(x, \lambda)$ will both be bounded for all values of x. Moreover, for large x

$$\phi_1(x, \lambda) = 0\{e^{tx}x^k\}, \quad \phi_2(x, \lambda) = 0\{e^{tx}x^{-k}\} .$$

Furthermore, when $Im\, \lambda = 0$, that is for real λ, and $x \to \infty$ the order properties for ϕ and θ may be written as follows:

$$\begin{aligned}
\phi_1(x, \lambda) &= x^p\{\mu(\lambda)J_p(\lambda x) + \nu(\lambda)J_{-p}(\lambda x) + o(1)\} \\
\phi_2(x, \lambda) &= x^{-p+1}\{+\mu(\lambda)J_{p-1}(\lambda x) - \nu(\lambda)J_{-p+1}(\lambda x) + o(1)\} \\
\theta_1(x, \lambda) &= x^p\{\xi(\lambda)J_p(\lambda x) + \eta(\lambda)J_{-p}(\lambda x) + o(1)\} \\
\theta_2(x, \lambda) &= x^{-p+1}\{\xi(\lambda)J_{p-1}(\lambda x) - \eta(\lambda)J_{-p+1}(\lambda x) + o(1)\}
\end{aligned} \tag{6}$$

where

(7)

$$\mu(\lambda) = \frac{1}{\Delta}\left[J_{-p+1}(\lambda) - \int_1^\infty [-b(s)\phi_2(s)s^{-p+1}J_{-p+1}(\lambda s) + d(s)\phi_1(s)s^pJ_{-p}(\lambda s)]\,ds\right]$$

$$\nu(\lambda) = \frac{1}{\Delta}\left[J_{p-1}(\lambda) - \int_1^\infty [-b(s)\phi_2(s)s^{-p+1}J_{p-1}(\lambda s) + d(s)\phi_1(s)s^pJ_p(\lambda s)]\,ds\right]$$

$$\xi(\lambda) = \frac{1}{\Delta}\left[+J_{-p}(\lambda) - \int_1^\infty [-b(s)\theta_2(s)s^{-p+1}J_{-p+1}(\lambda s) + d(s)\theta_1(s)s^pJ_{-p}(\lambda s)]\,ds\right]$$

$$\eta(\lambda) = \frac{1}{\Delta}\left[-J_p(\lambda) - \int_1^\infty [-b(s)\theta_2(s)s^{-p+1}J_{p-1}(\lambda s) + d(s)\theta_1(s)s^pJ_{-p}(\lambda s)]\,ds\right].$$

The integrals in (7) converge uniformly in λ hence, $\mu(\lambda)$, $\nu(\lambda)$, $\xi(\lambda)$, $\eta(\lambda)$ will be continuous and bounded functions of λ.

When λ is complex and has an imaginary part greater than zero, $Im\,\lambda > 0$, and when $b(x)x^{-2k}$ and $d(x)x^{2k}$ are both $L(1, \infty)$, one can obtain

(8)

$$\begin{aligned}
\phi_1(x, \lambda) &= x^k e^{-i\lambda x}\,[M_1(\lambda) + o(1)]\\
\phi_2(x, \lambda) &= x^{-k} e^{-i\lambda x}\,[M_2(\lambda) + o(1)]\\
\theta_1(x, \lambda) &= x^k e^{-i\lambda x}\,[N_1(\lambda) + o(1)]\\
\theta_2(x, \lambda) &= x^{-k} e^{-i\lambda x}\,[N_2(\lambda) + o(1)]
\end{aligned}$$

where

(9)

$$\begin{aligned}
M_1(\lambda) &= \frac{(2\pi\lambda)^{-1/2}e^{ik\pi/2}}{\Delta}\Big\{iJ_{-p+1}(\lambda) + iJ_{p-1}(\lambda)\cos k\pi\\
&\quad + \int_1^\infty b(s)\phi_2(s)s^{-p+1}\,[iJ_{-p+1}(\lambda s) + J_{p-1}(\lambda s)\cos k\pi]\,ds\\
&\quad + \int_1^\infty d(s)\phi_1(s)s^p\,[J_p(\lambda s)\cos k\pi - J_{-p}(\lambda s)]\,ds\Big\}\\
M_2(\lambda) &= \frac{(2\pi\lambda)^{-1/2}e^{ik\pi/2}}{\Delta}\Big\{+J_{-p+1}(\lambda) - iJ_{p-1}(\lambda)\cos k\pi\\
&\quad + \int_1^\infty b(s)\phi_2(s)s^{-p+1}\,[J_{-p+1}(\lambda s) - iJ_{p-1}(\lambda s)\cos k\pi]\,ds\\
&\quad - \int_1^\infty d(s)\phi_1(s)s^p\,[iJ_p(\lambda s)\cos k\pi + J_{-p}(\lambda s)]\,ds\Big\}\\
N_1(\lambda) &= \frac{(2\pi\lambda)^{-1/2}e^{ik\pi/2}}{\Delta}\Big\{-iJ_{-p}(\lambda) + J_p(\lambda)\cos k\pi\\
&\quad + \int_1^\infty b(s)\theta_2(s)s^{-p+1}\,[iJ_{-p+1}(\lambda s) + J_{p-1}(\lambda s)\cos k\pi]\,ds\\
&\quad + \int_1^\infty d(s)\theta_1(s)s^p\,[J_p(\lambda s)\cos k\pi - iJ_{-p}(\lambda s)]\,ds\Big\}\\
N_2(\lambda) &= \frac{(2\pi\lambda)^{-1/2}e^{ik\pi/2}}{\Delta}\Big\{J_{-p}(\lambda) + iJ_p(\lambda)\cos k\pi\\
&\quad + \int_1^\infty b(s)\theta_2(s)s^{-p+1}\,[J_{-p+1}(\lambda s) - iJ_{p-1}(\lambda s)\cos k\pi]\,ds\\
&\quad - \int_1^\infty d(s)\theta_1(s)s^p\,[iJ_p(\lambda s)\cos k\pi + J_{-p}(\lambda s)]\,ds\Big\}.
\end{aligned}$$

Let $\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda)$ be that solution of (2) mentioned in the Preliminaries where $c|\psi_1|^2 + a|\psi_{-2}|^2$ is $(1, \infty)$. ψ is given by

$$\psi_1(x, \lambda) = \theta_1(x, \lambda) + m(\lambda)\phi_1(x, \lambda) = x^k e^{i\lambda x}\,[N_1(\lambda) + m(\lambda)M_1(\lambda) + o(1)]\,,$$
$$\psi_2(x, \lambda) = \theta_2(x, \lambda) + m(\lambda)\phi_2(x, \lambda) = x^{-k} e^{-i\lambda x}\,[N_2(\lambda) + m(\lambda)M_2(\lambda) + o(1)]\,.$$

It is apparent that in order for $c|\psi_1|^2 + a|\psi_2|^2$ to be in $L(1, \infty)$ we must have

$$m(\lambda) = -\frac{N_1(\lambda)}{M_1(\lambda)} = -\frac{N_2(\lambda)}{M_2(\lambda)} \tag{10}$$

When $Im\,\lambda \to 0$, it follows from (7) and (9)that

$$\begin{aligned} M_1(\lambda) &\to (2\pi\sigma)^{-1/2} e^{ik\pi/2}\,[i\mu(\sigma) + \nu(\sigma)\cos k\pi] \\ M_2(\lambda) &\to (2\pi\sigma)^{-1/2} e^{ik\pi/2}\,[+\mu(\sigma) - i\nu(\sigma)\cos k\pi] \\ N_1(\lambda) &\to (2\pi\sigma)^{-1/2} e^{ik\pi/2}\,[i\xi(\sigma) + \eta(\sigma)\cos k\pi] \\ N_2(\lambda) &\to (2\pi\sigma)^{-1/2} e^{ik\pi/2}\,[+\xi(\sigma) - i\eta(\sigma)\cos k\pi]\,. \end{aligned} \tag{11}$$

Hence,

$$\lim_{t\to 0} m(\lambda) = -\frac{i\xi(\sigma) + \eta(\sigma)\cos k\pi}{i\mu(\sigma) + \nu(\sigma)\cos k\pi}$$

$$= -\cos k\pi\,\frac{\mu\eta - \nu\xi}{\nu^2(\sigma) + \mu^2(\sigma)}\,. \tag{12}$$

From (6) one also obtains

$$W_x[\phi, \theta] = \phi_1\theta_2 - \phi_2\theta_1 = \Delta[\nu\xi - \mu\eta] + o(1)\,.$$

Because $W_0[\phi, \theta] = 1$,

$$\nu\xi - \mu\eta = \frac{1}{\Delta}\,.$$

The substitution of this result in (12) yields

$$\lim_{t\to 0} Im\, m(\lambda) = -\frac{\sigma\pi}{\nu^2(\sigma) + \mu^2(\sigma)}\,.$$

Hence, $Im\, m(\lambda)$ is a nonpositive, nonvanishing continuous and bounded function of λ for both positive and negative λ.

This is easily seen for positive λ since λ, ν^2 and μ^2 are all positive numbers. For negative λ not only is λ negative, but so are ν^2 and μ^2. This can be verified by making use of the formula

$$J_p(i^{2m}z) = i^{2mp}J_p(z)\,.$$

The spectrum is, therefore, continuous over the infinite interval $-\infty < \lambda < \infty$. These results may be summarized in the following theorem,

THEOREM 1. *Consider the system* (2) *where* $a(x) = x^{2k}$ *and* $c(x) = x^{-2k}$ *over the semi-infinite interval* $[l, \infty]$ *and under the boundary conditions*

$$u(l)\cos\alpha + v(l)\sin\alpha = 0 \tag{13}$$

Let $b(x)$ *and* $d(x)$ *be real-valued continuous functions of* x *and let* $b(x)\,x^{-2k}$ *and* $d(x)x^{2k}$ *belong to the class* $L(l, \infty)$. *A solution of the system* (2), (13) *is defined as a vector function* $[u(x, \lambda), v(x, \lambda)]$ *with continuous first derivatives satisfying this system. The values of* λ *for which such solutions exists form a continuous spectrum over the real* λ*-axis* $(-\infty < \lambda < \infty)$.

In order to simplify the appearance of the equations in the preceeding proof and in the following, it was assumed, without loss of generality, that $\alpha = \pi/2$ and $l = 1$.

3. Nature of the spectrum; interval (0, 1).

THEOREM 2. *Consider the system* (2) *where* $a(x) = x^{2k}$, $c(x) = x^{-2k}$ *over the interval* $0 \leqq x \leqq 1$ *and subject to a linear homogeneous boundary condition at* $x = 1$. *The spectrum is discrete provided:*

$b(x)x^{-2(p-1)}$	*and*	$d(x)$ *are* $L(0, 1)$	*for* $p = k + 1/2 > 1$
$b(x)$	*and*	$d(x)$ *are* $L(0, 1)$	*for* $0 < p < 1$
$b(x)$	*and*	$d(x)x^{2p}$ *are* $L(0, 1)$	*for* $p < 0$.

The proof of this theorem follows closely that of Theorem 1. Except for an obvious change of integration limits, the integral equations corresponding to system (2) of Theorem 2, are given by equations (4) and (5).

First, consider the case $p > 0$. For $x \to 0$ one has the well-known asymptotic relations:

$$J_p(\lambda x) \to \frac{(\lambda x)^p}{\Gamma(p+1)2^p} \quad J_{-p}(\lambda x) \to \frac{(-1)^k\Gamma(p)2^p(\lambda x)^{-p}}{\pi}$$

and for $p - 1 > 0$:

$$J_{p-1}(\lambda x) \to \frac{(\lambda x)^{p-1}}{\Gamma(p)2^{p-1}} \quad J_{-p+1}(\lambda x) \to \frac{(-1)^{k-1}\Gamma(p-1)\,(\lambda x)^{-p+1}2^{p-1}}{\pi}\,.$$

Let $h_1(x, \lambda) = \phi_1(x, \lambda)$, $h_2(x, \lambda) = x^{+2(p-1)}\phi_2(x, \lambda)$ and $\lambda = \sigma + it$ where $t > 0$. One obtains:

$$\begin{aligned} h_1(x, \lambda) = \frac{x^p}{\Delta}\Big[& J_{-p+1}(x)J_p(\lambda x) + J_{p-1}(\lambda)J_{-p}(\lambda x) \\ & - \int_x^1 \{bs^{-3(p-1)}h_2(s, \lambda)\,[J_{-p}(x)J_{-p+1}(s) + J_{-p}(x)J_{p-1}(s)] \\ & + dh_1 s^p\,[J_p(\lambda s)J_{-p}(\lambda x)J_{-p}(\lambda x)J_{-p}(\lambda s)]\}\,ds\Big] \end{aligned}$$

$$h_2(x,\lambda) = \frac{x^{+(p-1)}}{\Delta}\Big[J_{-p+1}(\lambda)J_{p-1}(\lambda x) - J_{p-1}(\lambda)J_{-p+1}(\lambda x)$$
$$- \int_x^1 \{ds^{-3(p-1)}h_2(s)\,[J_{p-1}(x)J_{-p+1}(s) - J_{p-1}(s)J_{-p+1}(x)]$$
$$- dh_1 s^p\,[J_p(\lambda s)J_{-p+1}(\lambda x) + J_{-p}(\lambda s)J_{p-1}(\lambda x)]\}\,ds\Big].$$

For $x \to 0$ $h_1(x,\lambda)$ and $h_2(x,\lambda)$ are bounded for all x provided $bx^{-2(p-1)}$ and d are $L(0,1)$ (see lemma [3]). Hence, $\phi_1(x,\lambda) = o(1)$ and $\phi_2(x,\lambda) = o(x^{-2(p-1)})$ for $x \to 0$.

It follows that one may write

$$\phi_1(x,\lambda) = [M_1(\lambda) + o(1)]\,,$$
$$\phi_2(x,\lambda) = x^{-2(p-1)}\,[M_2(\lambda) + o(1)]\,,$$

where

$$M_1(\lambda) = \frac{(-1)^k\Gamma(p)2^p\lambda^{-p}}{\pi\Delta}\Big[J_{p-1}(\lambda) - \int_0^1 [b\phi_2 J_{p-1}(\lambda s)s^{-p+1} + d\phi_1 J_p(\lambda s)s^p]\,ds\Big]$$

and

$$M_2(\lambda) = \frac{(-1)^{k-1}\Gamma(p-1)2^{p-1}\lambda^{-p-1}}{\pi\Delta}$$
$$\Big[-J_{p-1}(\lambda) + \int_0^1 [J_{p-1}(\lambda s)b\phi_2 s^{-p+1} - d\phi_1 J_p(\lambda s)s^p]\,ds\Big].$$

Similarly,

$$\theta_1(x,\lambda) = [N_1(\lambda) + o(1)\,,$$
$$\theta_2(x,\lambda) = x^{-2(p-1)}\,[N_2(\lambda) + o(1)]\,,$$

where

$$N_1(\lambda) = \frac{(-1)^k\Gamma(p)2^p\lambda^{-p}}{\pi\Delta}\Big[+J_{+p}(\lambda) - \int_0^1 \{b\theta_2 s^{-p+1}J_{p-1}(\lambda s) + d\theta_1 s^p J_p(\lambda s)\}\,ds\Big]$$

and

$$N_2(\lambda) = \frac{(-1)^{k-1}\Gamma(p-1)2^{p-1}\lambda^{-(p-1)}}{\pi\Delta}$$
$$\Big[-J_p(\lambda) + \int_0^1 \{b\theta_2 s^{-p+1}J_{p-1}(\lambda s) - d\theta_1 s^p J_p(\lambda s)\}\,ds\Big].$$

A similar argument as in §3, yields

$$m(\lambda) = -\frac{N_1(\lambda)}{M_1(\lambda)} = -\frac{N_2(\lambda)}{M_2(\lambda)}\,.$$

Hence

$$m(\lambda) = \frac{J_p(\lambda) - \int_0^1 \{b\theta_2 s^{-p+1} J_{p-1}(\lambda s) + d\theta_1 s^p J_p(\lambda s)\}\, ds}{J_{p-1}(\lambda) - \int_0^1 \{b\theta_2 s^{-p+1} J_{p-1}(\lambda s) + d\phi_1 s^p J_p(\lambda s)\}\, ds}$$

For $t \to 0$, i.e., $Im\ \lambda \to 0$, $M_1(\lambda)$ and $N_1(\lambda)$ tend to real limits. This is apparent for $\lambda > 0$ and is easily shown for $\lambda < 0$ by using again the relation $J_r(ze^{im\pi}) = e^{im\pi r} J_r(z)$. Hence, $Im\ m(\lambda) \to 0$ except possibly at the zeros of $M_1(\lambda)$. Consequently, the spectrum will be discrete.

For values of the parameter p such that $0 < p < 1$ one has

$$J_p(\lambda x) \to \frac{(\lambda x)^p}{\Gamma(p+1)2^p}, \quad J_{-p}(\lambda x) \to \frac{(-1)^k \Gamma(p) 2^p (\lambda x)^{-p}}{\pi},$$

$$J_{p-1}(\lambda x) \to \frac{(-1)^{k-1}\Gamma(p-1)\,(\lambda x)^{(p-1)} 2^{p-1}}{\pi}, \quad J_{-p+1}(\lambda x) \to \frac{(\lambda x)^{-p+1}}{\Gamma(p) 2^{p-1}}.$$

It is not difficult to verify, applying the same method as discussed above, that in this case, when $b(x)$ and $d(x)$ are $L(0, 1)$ the limit circle case prevails and the spectrum is discrete. k, an integer, is not in this interval.

For $p < 0$ the asymptotic expressions become

$$J_p(\lambda x) \to \frac{(-1)^k \Gamma(p) 2^p (\lambda x)^{+p}}{\pi}, \quad J_{-p}(\lambda x) \to \frac{(\lambda x)^{-p}}{\Gamma(p+1)2^p},$$

$$J_{p-1}(\lambda x) \to \frac{(-1)^{k-1}\Gamma(p-1) 2^{p-1} (\lambda x)^{+(p-1)}}{\pi}, \quad J_{-(p-1)}(\lambda x) \to \frac{(\lambda x)^{-(p-1)}}{\Gamma(p) 2^{p-1}}.$$

In a similar fashion, for the case where $p > 1$ one can obtain that $\phi_1(x, \lambda) = o(x^{2p})$ and $\phi_2(x, \lambda) = o(1)$. In this situation the additional condition that b and $d(x)x^{2p}$ are in $L(0, 1)$ must be imposed. The conclusion that the spectrum will be discrete follows accordingly.

4. Nature of the spectrum; interval $(0, \infty)$.

In this case singularities exist at both ends of the interval and it has been shown [4] that the spectral properties of the system (2) are determined by the imaginary parts of the following functions:

$$\frac{1}{m_0 - m_\infty}, \quad \frac{m_0}{m_0 - m_\infty}, \quad \frac{m_0 m_\infty}{m_0 - m_\infty},$$

where $m_0(\lambda)$ and $m_\infty(\lambda)$ are the previously determined m's at $x = 0$ and ∞. As was shown in the previous sections $m_0(\lambda)$ is a meromorphic function while m_∞ has a nonvanishing imaginary part. It is clear that the imaginary parts of the three functions above tend to finite limits which can vanish at most at discrete points. The associated spectrum will therefore be continuous over the whole real λ-axis.

THEOREM 3. *Consider the system* (2) *where* $a(x) = x^{2k}$ *and* $c(x) = x^{-2k}$

over the interval $0 \leqq x \infty$. *The spectrum will be continuous over the entire real* λ*-axis* $(-\infty < \lambda < \infty)$, *provided* $b(x)$ *and* $d(x)$ *satisfy the conditions given in Theorem* 1 *and Theorem* 2.

REFERENCES

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, Mac Graw-Hill New York, 1955.
2. S. D. Conte, and W. C. Sangren, *An Expansion theorem for a pair of first order equations*, Can. J. Math., **6** (1954), 554-560.
3. B. W. Roos and W. C. Sangren, *Spectra for a pair of singular first order differential equations*, Proc. Amer. Math. Soc., **12** (1961), 468-476.
4. ———, General Atomic Report (unpublished).
5. ———, General Atomic Report GA-1777.
6. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Oxford, 1946.

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