THE SPECTRA OF MINIMAL SELF-ADJOINT EXTENSIONS OF A SYMMETRIC OPERATOR

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1. Introduction. Let T be a closed symmetric operator with domain D_T dense in a Hilbert space $\mathscr{H}$. A (*generalized*) *spectral resolution of* T is a family of bounded self-adjoint operators E_μ defined for $-\infty < \mu < \infty$ and such that:

(a) E_μ is nondecreasing, continuous from the right, and $E_{-\infty} = 0$, $E_\infty = 1$.

(b) For $u \in D_T$ and $v \in \mathscr{H}$,

$$(Tu, v) = \int_{-\infty}^{\infty} \mu d(E_\mu u, v), \| Tu \|^2 = \int_{-\infty}^{\infty} \mu^2 d(E_\mu u, u) .$$

When in particular T is self-adjoint, it possesses only one generalized spectral resolution, namely the *orthogonal* spectral resolution where E_μ is for each μ an orthogonal projection. For an account of the theory of generalized resolutions see [1], Appendix I.

M. A. Naimark has shown that for each generalized resolution E_μ there is at least one self-adjoint extension T^+ of T in a Hilbert space $\mathscr{H}^+ \supset \mathscr{H}$ with the following property: If E_μ^+ is the orthogonal resolution of T^+ and P is the projection onto the subspace $\mathscr{H}$ of $\mathscr{H}^+$, then $E_\mu = PE_\mu^+$. We shall usually require that T^+ be a *minimal* self-adjoint extension of T, i.e. that $\mathscr{H}^+$ be the closed linear hull of the set of vectors $E_\mu^+ \mathscr{H}$, $(-\infty < \mu < \infty)$; (see § 3). The minimal extension T^+ corresponding to a given E_μ is determined by E_μ uniquely, up to unitary equivalence ([8], § 4). We shall denote it by $T^+ = \psi(E_\mu)$.

In this paper we investigate certain questions regarding the *spectrum* Σ of $T^+ = \psi(E_\mu)$. In view of the above mentioned unitary equivalence, the point set Σ depends only upon E_μ; it may in fact be characterized directly as the set of points of increase of E_μ (see § 3). Parts of the spectrum—e.g. eigenvalues and essential spectrum—may likewise be characterized directly in terms of E_μ. It will be convenient to refer to the spectrum of T^+ as the *spectrum of* E_μ.

We are interested in comparing the spectra of various resolutions of a given T. In order to describe the situation precisely, one refers to A. V. Štraus' extension theory of symmetric operators [10]. For any complex λ, let $\Delta_T(\lambda)$ denote the range of $T - \lambda$. By definition, the *defect subspace* $M(\lambda)$ is the orthogonal complement in $\mathscr{H}$ of $\Delta_T(\lambda)$.

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Štraus has associated with each generalized resolution E_μ of T a family of *contraction operators* F_λ, mapping $M(i)$ into $M(-i)$, and such that F_λ is analytic on $\mathscr{I}\lambda > 0$ with $\|F_\lambda\| \leqq 1$ there. Conversely each such family of contractions is associated with some E_μ. A constant unitary F corresponds by this association to an orthogonal resolution E_μ, and for these Štraus' extension theory reduces to that of J. von Neumann. (For a complete description, see § 2)

We characterize the spectral resolutions E_μ of T by the behavior near the real axis of the corresponding F_λ. Specifically we single out two extreme cases, where F_λ satisfies, respectively, conditions α and β or condition γ as defined in § 4. These are local conditions, defined for an open real interval Δ. When E_μ is an orthogonal resolution, conditions α and β hold on the entire real axis.

In §§ 4–6 we consider a symmetric operator T with equal finite defect numbers (n, n). In § 4 we extend to generalized resolutions of T, satisfying conditions α and β on an interval Δ, the theorem of H. Weyl [13] on the invariance of essential spectrum. In § 6 we obtain a parallel theorem on the invariance of absolutely continuous spectrum, proved for T a singular second order ordinary differential operator. This extends a theorem of N. Aronszajn [2]. (The theorems of Weyl and Aronszajn both concern self-adjoint extensions of T *in* $\mathscr{H}$, hence orthogonal resolutions.)

When F_λ is such that α and β fail everywhere on an interval Δ an altogether different pattern emerges, for in this case Δ lies entirely within the spectrum of E_μ. In § 5 we adopt the more stringent assumption that condition γ holds on Δ. In particular, suppose F_λ is a *family of strict contractions*, i.e. satisfies condition γ on the entire real axis. Suppose that $T - \mu$ has a bounded inverse for each real μ. Then $T^+ = \psi(E_\mu)$ is unitarily equivalent to the n-fold direct sum of iD with itself, where D is the differentiation operator in $L_2(-\infty, \infty)$. This generalizes a theorem proved by Coddington and Gilbert ([4], Theorem 14) for T a regular ordinary differential operator of order n. As is indicated in § 6, the situation is more complicated when T is a singular differential operator.

The study of the spectrum of E_μ requires an analysis of the behavior of the *resolvent* R_λ of E_μ near the real axis. The *generalized resolvent* R_λ *of a spectral resolution* E_μ is defined for $\mathscr{I}\lambda \neq 0$ by

$$R_\lambda = \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} dE_\mu \,. \tag{1.1}$$

Thus R_λ is a bounded operator with domain $\mathscr{H}$, analytic on each half plane $\mathscr{I}\lambda > 0$, $\mathscr{I}\lambda < 0$. Inversely, E_μ is determined by R_λ through the formula

$$(E(\Delta)u, u) = \lim_{\varepsilon\to 0+} \frac{1}{\pi} \int_{\Delta} \mathscr{I}(R_{\mu+i\varepsilon}u, u)d\mu \tag{1.2}$$

where Δ is an interval $(\mu_1, \mu_2]$, μ_1 and μ_2 are continuity points of E_μ, and $E(\Delta) = E_{\mu_2} - E_{\mu_1}$. When T is self-adjoint, the (generalized) resolvent R_λ of its orthogonal resolution E_μ coincides with the *resolvent of* T, i.e. $R_\lambda = (T - \lambda)^{-1}$ for $\mathscr{I}\lambda \neq 0$.

Let $T^+ = \psi(E_\mu)$. The resolvents R_λ^+ and R_λ of E_μ^+ and E_μ are related, when $\mathscr{I}\lambda \neq 0$, by (see [1])

$$R_\lambda = PR_\lambda^+ . \tag{1.3}$$

A. V. Štraus [10] has given another characterization of R_λ, when $\mathscr{I}\lambda \neq 0$, as the resolvent of a certain *quasi-self-adjoint extension* T_λ in $\mathscr{H}$ of T. (For precise definition, see § 2). Thus

$$R_\lambda = (T_\lambda - \lambda)^{-1} . \tag{1.4}$$

In § 2 we investigate limit values, as λ tends to the real axis, of R_λ. It is found that in general the interpretation (1.3) fails for limit values while (1.4) retains its meaning. The interpretation (1.3) remains valid on a real interval Δ precisely when R_λ can be continued analytically through Δ, and this is possible precisely when Δ lies in the complement of the spectrum of E_μ (theorem 3.1).

It is a pleasure to express here my indebtedness to E. A. Coddington, who first drew my attention to generalized resolutions and in particular suggested that the theorem of Coddington and Gilbert, referred to above, might be valid in a broader setting. During the course of the work I have had access to his library and frequent benefit of his counsel.

2. Limit values of the resolvent. We shall designate an arbitrary one of the half planes $\mathscr{I}\lambda > 0$, $\mathscr{I}\lambda < 0$ by π^+ and the other by π^-. Choose any $\lambda_0 \in \pi^+$ and any contraction operator F (i.e. $\|F\| \leqq 1$) with domain $M(\lambda_0)$ and values in $M(\bar{\lambda}_0)$. The operator $\hat{T}$, defined by

$$\begin{gathered} T \subset \hat{T} \subset T^* , \\ D_{\hat{T}} = \{u \colon u = u_0 + \phi - F\phi .\ u_0 \in D_T, \phi \in M(\lambda_0)\} \end{gathered} \tag{2.1}$$

has been called by A. V. Štraus a *quasi-self-adjoint extension of* T. The class C^+ of operators $\hat{T}$ obtained by holding λ_0 fixed and varying F is, in fact, independent of the choice of $\lambda_0 \in \pi^+$. (See Štraus [10], Lemma 9 and the discussion preceding it). A second, and in general different, class C^- of quasi-self-adjoint extensions of T is obtained by taking $\lambda_0 \in \pi^-$.

Let R_λ be the resolvent of E_μ of T. Štraus has proved that, to each $\lambda \in \pi^+$ corresponds a quasi-self-adjoint extension $T_\lambda \in C^+$ such that

$$(T_\lambda - \lambda)^{-1} = R_\lambda , \qquad \lambda \in \pi^+ . \tag{2.2}$$

For a fixed choice of $\lambda_0 \in \pi^+$, the corresponding contraction $F_\lambda = F_\lambda(\lambda_0)$ is analytic in λ on π^+. Conversely, any analytic contraction F_λ carrying $M(\lambda_0)$ into $M(\bar{\lambda}_0)$ gives rise, through (2.1) and (2.2) to a resolvent R_λ of T. The relation $R_{\bar{\lambda}} = R_\lambda^*$ (which follows from (1.1) or (1.3)) has as its correspondent the relation

$$F_{\bar{\lambda}}(\bar{\lambda}_0) = [F_\lambda(\lambda_0)]^* \tag{2.3}$$

defining a contraction taking $M(\bar{\lambda}_0)$ into $M(\lambda_0)$.

The following theorem shows that these statements remain valid in a limiting sense on the real axis.

THEOREM 2.1. *Let $\lambda_1, \lambda_2, \cdots$ in π^+ tend to $\hat{\lambda}$ on the real axis.*

(A) *Suppose that for a certain $\lambda_0 \in \pi^+$ the sequence of contractions $F_{\lambda_k}(\lambda_0)$ converges in norm as $k \to \infty$. Then the same is true for every $\lambda_0' \in \pi^+$. The limit, also a contraction taking $M(\lambda_0)$ into $M(\bar{\lambda}_0)$, will be denoted by $F_{\hat{\lambda}+} = F_{\hat{\lambda}+}(\lambda_0)$. It defines a quasi-self-adjoint extension in C^+ of T, and the extension $T_{\hat{\lambda}+}$ so obtained does not depend upon the particular $\lambda_0 \in \pi^+$ figuring in its construction.*

(B) *Necessary and sufficient for the convergence in norm of R_{λ_k} to a limit, denoted by $R_{\hat{\lambda}+}$, is:*

(i) *Convergence in norm of F_{λ_k}, and*

(ii) *Existence of $(T_{\hat{\lambda}+} - \hat{\lambda})^{-1}$ as a bounded operator with domain $\mathscr{H}$. In this case,*

$$R_{\hat{\lambda}+} = (T_{\hat{\lambda}+} - \hat{\lambda})^{-1} . \tag{2.4}$$

(C) *In any subset of [π^+ plus the real axis] in which both R_λ and F_λ are defined (by extension), the single-valuedness and continuity of either implies that of the other.*

(D) *When, as above, R_{λ_k} and $F_{\lambda_k}(\lambda_0)$ tend to limits in norm, then the same is true of $R_{\bar{\lambda}_k}$ and $F_{\bar{\lambda}_k}(\bar{\lambda}_0)$, and*

$$R_{\hat{\lambda}-} = [R_{\hat{\lambda}+}]^* , \qquad F_{\hat{\lambda}-}(\bar{\lambda}_0) = [F_{\hat{\lambda}+}(\lambda_0)]^* . \tag{2.5}$$

Proof. (A) Let $W(\lambda_0)$ denote the Cayley transform of a quasi-self-adjoint extension $\hat{T}$ of T. Thus $W = U \oplus F$, where

$$U(\lambda_0) = (T - \bar{\lambda}_0)(T - \lambda_0)^{-1}$$

is the Cayley transform of T. One easily shows ([10], equation (5.22)), that for λ_0 and λ_0' in π^+,

$$W(\lambda_0') = [(\bar{\lambda}_0' - \bar{\lambda}_0) - (\bar{\lambda}_0' - \lambda_0) W(\lambda_0)][(\lambda_0' - \bar{\lambda}_0) - (\lambda_0' - \lambda_0) W(\lambda_0)]^{-1} \tag{2.6}$$

where the inverse shown is a bounded operator with domain $\mathscr{H}$. Since this equation holds between $W_{\lambda_k}(\lambda_0) = U(\lambda_0) \oplus F_{\lambda_k}(\lambda_0)$ and $W_{\lambda_k}(\lambda_0')$, therefore by continuity $W_{\hat{\lambda}+}(\lambda_0') = \lim W_{\lambda_k}(\lambda_0')$ exists. Furthermore $W_{\hat{\lambda}+}(\lambda_0)$ and $W_{\hat{\lambda}+}(\lambda_0')$ are related by (2.6) and hence are Cayley transforms relative to λ_0 and λ_0', of the same $\hat{T} = T_{\hat{\lambda}+}$. Since $W_{\lambda_k}(\lambda_0') = U(\lambda_0') \oplus F_{\lambda_k}(\lambda_0')$ therefore $F_{\hat{\lambda}+}(\lambda_0') = \lim F_{\lambda_k}(\lambda_0')$ exists, and $W_{\hat{\lambda}+}(\lambda_0') = U(\lambda_0') \oplus F_{\hat{\lambda}+}(\lambda_0')$. Thus $F_{\hat{\lambda}+}(\lambda_0)$ and $F_{\hat{\lambda}+}(\lambda_0')$ define the same extension of T.

$(B)_1$ Here we establish the necessity of the condition. Let $\lambda_0 \in \pi^+$. It follows from (2.2) that, for $\lambda \in \pi^+$,

$$T_\lambda - \lambda_0 = (T_\lambda - \lambda) + (\lambda - \lambda_0) = [1 + (\lambda - \lambda_0)R_\lambda](T_\lambda - \lambda)$$

and therefore that

$$(T_\lambda - \lambda_0)^{-1} = R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1}\,, \qquad \lambda \in \pi^+\,.$$

Here $[1 + (\lambda - \lambda_0)R_\lambda]^{-1}$ is bounded with domain $\mathscr{H}$. ([10], equation (5.30), footnote.). By assumption, $\lambda_k \to \hat{\lambda}$ on the real axis, and $R_{\lambda_k} \to R_{\hat{\lambda}+}$ in norm. By choosing a *special* λ_0 for which $|\hat{\lambda} - \lambda_0| \cdot \| R_{\hat{\lambda}+} \| < 1$, we guarantee that $[1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}+}]^{-1}$ exists, is bounded, and has domain $\mathscr{H}$. Consequently the operator

$$G_\lambda = R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1}$$

is well defined for $\lambda = \hat{\lambda}+$ as well as $\lambda \in \pi^+$, and $G_{\lambda_k} \to G_{\hat{\lambda}+}$ in norm.

The Cayley transform $W_\lambda(\lambda_0)$ of T_λ for $\lambda \in \pi^+$, is given by

$$W_\lambda = (T_\lambda - \bar{\lambda}_0)(T_\lambda - \lambda_0)^{-1} = 1 + (\lambda_0 - \bar{\lambda}_0)(T_\lambda - \lambda_0)^{-1}\,.$$

Hence

$$W_\lambda = 1 + (\lambda_0 - \bar{\lambda}_0)G_\lambda\,, \qquad \text{for } \lambda \in \pi^+\,. \tag{2.7}$$

We define the transformation $W_{\hat{\lambda}+}$ also by this formula, and show that $W_{\hat{\lambda}+}$ is a quasi-unitary extension, with $\| W_{\lambda+} \| \leqq 1$ of the Cayley transform $U(\lambda_0)$ of T. In fact, the statements

$$\| W_\lambda \| \leqq 1\,; \qquad W_\lambda f = U(\lambda_0)f \qquad \text{for } f \in \varDelta_T(\lambda_0)$$

are valid for $\lambda \in \pi^+$ and, since by (2.7) $W_{\lambda_k} \to W_{\hat{\lambda}+}$, are valid for $\lambda = \hat{\lambda}+$ as well. But by [10], Lemma 8, these statements imply that $W_{\hat{\lambda}+}$ is a quasi-unitary extension of $U(\lambda_0)$.

Consequently $W_{\hat{\lambda}+}$ is the Cayley transform of a quasi-self-adjoint extension (of class C^+) of T. From the relation

$$W_\lambda = U \oplus F_\lambda \qquad \text{for } \lambda \in \pi^+$$

it follows, since $W_{\lambda_k} \to W_{\hat{\lambda}+}$, that: $F_{\hat{\lambda}+} = \lim F_{\lambda_k}$ exists and

$$W_{\hat{\lambda}+} = U \oplus F_{\hat{\lambda}+}\,.$$

Thus $W_{\hat{\lambda}+}$ is the Cayley transform of the extension which we have denoted (in A) by $T_{\hat{\lambda}+}$.

From the relation between any quasi-self-adjoint extension and its Cayley transform we have

$$W_{\hat{\lambda}+} = (T_{\hat{\lambda}} - \bar{\lambda}_0)(T_{\hat{\lambda}+} - \lambda_0)^{-1} = 1 + (\lambda_0 - \bar{\lambda}_0)(T_{\hat{\lambda}+} - \lambda_0)^{-1} .$$

Comparing this relation with (2.7), we conclude that

$$(T_{\hat{\lambda}+} - \lambda_0)^{-1} = G_{\hat{\lambda}+} = R_{\hat{\lambda}+}[1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}+}]^{-1} .$$

From this it immediately follows that $R_{\hat{\lambda}+}^{-1}$ exists and that

$$T_{\hat{\lambda}+} - \lambda_0 = [1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}+}]R_{\hat{\lambda}+}^{-1} = R_{\hat{\lambda}+}^{-1} + (\hat{\lambda} - \lambda_0) .$$

Hence

$$T_{\hat{\lambda}+} - \hat{\lambda} = R_{\hat{\lambda}+}^{-1} ,$$

or

$$R_{\hat{\lambda}+} = (T_{\hat{\lambda}+} - \hat{\lambda})^{-1} .$$

This shows the necessity of conditions (i) and (ii) for the special choice of λ_0 made in the course of the argument. But from part (A), already proved, it follows that the conditions hold as well for any other $\lambda_0 \in \pi^+$.

$(B)_2$ In order to prove the sufficiency of the conditions (i) and (ii), we make use of the inverse relation between T_λ and its Cayley transform, namely

$$T_\lambda = (\lambda_0 W_\lambda - \bar{\lambda}_0)(W_\lambda - 1)^{-1} \qquad \text{for } \lambda \in \pi^+ \text{ or } \lambda = \hat{\lambda} + .$$

(For notation, see the proof of part A). Hence

$$T_\lambda - \lambda = [(\lambda_0 - \lambda) W_\lambda + (\lambda - \bar{\lambda}_0)](W_\lambda - 1)^{-1} .$$

and, since $(T_\lambda - \lambda)^{-1}$ exists (condition ii), therefore

$$(T_\lambda - \lambda)^{-1} = (W_\lambda - 1)[(\lambda - \bar{\lambda}_0) - (\lambda - \lambda_0) W_\lambda]^{-1} \qquad \text{for } \lambda \in \pi^+ \text{ or } \lambda = \hat{\lambda} + . \tag{2.8}$$

Furthermore, since the inverse appearing on the left side of this equation is bounded with domain $\mathscr{H}$, the same is true for the inverse appearing on the right side. This fact, together with $W_{\lambda_k} \to W_{\hat{\lambda}+}$ in norm, shows that

$$R_{\hat{\lambda}_k} = (T_{\lambda_k} - \lambda_k)^{-1} \to (T_{\hat{\lambda}+} - \hat{\lambda})^{-1} ,$$

which proves the proposition.

(C) This is a direct consequence of the reciprocal relations (2.7) and (2.8), namely

$$R_\lambda = (W_\lambda - 1)[(\lambda_0 - \lambda) - (\lambda - \bar{\lambda}_0) W_\lambda]^{-1}$$
$$W_\lambda = 1 + (\lambda - \lambda_0)R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1} \,.$$

These are valid in π^+ and under the assumptions of (C), are valid, in the limiting sense, on the entire set considered. Since the inverses displayed are bounded operators with domain $\mathscr{H}$, the assertion regarding continuity is evident.

REMARK 2.1. When $R_{\hat{\lambda}+}$ exists (as a limit in norm) it is, by Theorem 2.1, an extension of $(T - \hat{\lambda})^{-1}$. This implies that $\hat{\lambda}$ is a *point of regular type of* T, i.e. that $(T - \hat{\lambda})^{-1}$ exists and is bounded.
In particular (see [1], Chap. 7), the defect numbers of T are equal.

REMARK 2.2. Necessary and sufficient for the continuity of R_λ across an open interval Δ of the real axis is:

(i) Continuity of R_λ down to Δ in π^+ and

(ii) Self-adjointness of $R_{\lambda+}$ on Δ, i.e. $R_{\lambda+} = R_{\lambda-}$.

In the presence of (i), condition (ii) is equivalent to

(ii)′ Unitariness of $F_{\lambda+}$ on Δ, i.e. $(F_{\lambda+})^{-1} = F_{\lambda-}$.

Under these conditions R_λ is in fact analytic across Δ.
(One has only to consider $(R_\lambda f, f)$, which is analytic in π^+ and π^- and continuous across Δ)

3. **Resolvent set and spectrum.** By the *resolvent set of a spectral resolution* will be meant the points of $\pi^+ \cup \pi^-$ plus any real point λ_0 contained in an open real interval Δ across which R_λ may be continued analytically. The *resolvent* R_λ at $\lambda = \lambda_0$ is the common value of the limits $R_{\lambda 0+}$ and $R_{\lambda 0-}$ there.

In this paragraph we characterize the resolvent set, showing that it is the complementary point set of the *spectrum* of E_μ, described in the introduction.

According to M. A. Naimark, the spectral family E_λ in $\mathscr{H}$ may be regarded as the projection on $\mathscr{H}$ of an orthogonal family E_λ^+ in an enclosing space $\mathscr{H}^+ \supset \mathscr{H}$. Thus $E_\lambda = PE_\lambda^+$, where P is the orthogonal projection onto $\mathscr{H}$: $P\mathscr{H}^+ = \mathscr{H}$. The family E_λ^+ is the spectral resolution of a self-adjoint operator T^+ in $\mathscr{H}^+$. In the following we shall assume that T^+ is a *minimal* self-adjoint extension of T, thus we assume that the set of vectors

$$\{E^+(\Delta)h\colon \Delta \text{ is any interval, } h \in \mathscr{H}\}$$

is fundamental in $\mathscr{H}^+$. In other words, $\mathscr{H}^+$ is the closed linear hull of this set. (See Naimark [8], § 4).

LEMMA 3.1. *Let Δ be a (possibly degenerate) interval of the real* *axis. Then*

(A). *The set of vectors*

$$Z(\Delta) = \{E^+(\Delta')h \colon \Delta' \subset \Delta, h \in \mathscr{H}\}$$

is fundamental in $E^+(\Delta)\mathscr{H}^+$.

(B) $E^+(\Delta) = 0$ if and only if $E(\Delta) = 0$.

Proof. (A) Given $f \in E^+(\Delta)\mathscr{H}^+$. For any $\varepsilon > 0$ there exists $g = \sum_{k=1}^n E^+(\Delta_k)g_k$, for certain intervals Δ_k and certain $g_k \in \mathscr{H}$, such that $\|f - g\| < \varepsilon$. We can write $E^+(\Delta_k)g_k = E^+(\Delta_k \cap \Delta)g_k + E^+(\Delta - \Delta_k)g_k$, and thus $g = g^{(1)} + g^{(2)}$ with $g^{(1)} = \sum_{j=1}^{n_1} E^+(\Delta'_j)g_j^{(1)}$ and $g^{(2)} = \sum_{j=1}^{n_2} E^+(\Delta''_j)g_j^{(2)}$, where $\Delta'_j \subset \Delta$ and $\Delta''_j \cap \Delta = 0$. Thus $g^{(1)} \in E^+(\Delta)\mathscr{H}^+$ while $g^{(2)} \perp E^+(\Delta)\mathscr{H}^+$, and $\|f - g\|^2 = \|f - g^{(1)}\|^2 + \|g^{(2)}\|^2$. It follows that $\|f - g^{(1)}\| < \varepsilon$, proving the proposition.

(B) (i) Suppose $E(\Delta) > 0$. The there exists $h \in \mathscr{H}$ such that $0 < (E(\Delta)h, h) = (PE^+(\Delta)h, h) = (E^+(\Delta)h, h) = \|E^+(\Delta)h\|^2$. Thus $E^+(\Delta) > 0$.

(ii) Suppose $E(\Delta) = 0$. Then for $\Delta' \subset \Delta$, $E(\Delta') = 0$ also. Hence for $h \in \mathscr{H}$, $0 = (E(\Delta')h, h) = (E^+(\Delta')h, h)$, i.e. $E^+(\Delta')h = 0$. By part (A) this implies that $E^+(\Delta) = 0$.

THEOREM 3.1. *A real point $\hat{\lambda}$ of the resolvent set of the spectral family E_λ of T may be characterized in these equivalent ways:*

(A) *R_λ may be continued analytically across some open real interval Δ containing $\hat{\lambda}$.*

(B) *$E(\Delta) = 0$, for some real interval Δ containing $\hat{\lambda}$.*

(C) *$\hat{\lambda}$ is in the resolvent set of a minimal self-adjoint extension $T^+ = \psi(E_\lambda)$ of T.*

In this case, $R_{\hat{\lambda}} = PR_{\hat{\lambda}}^+$, where R_λ^+ is the resolvent of T^+.

Proof. (A → B) This is a consequence of the formula (1.2).

(B → C) By the lemma, $E^+(\Delta) = 0$. Since E_λ^+ is an *orthogonal* resolution of the identity, this implies that the points of Δ are in the resolvent set of T^+.

(C → A) If Δ is in the resolvent set of T^+ then $R_{\hat{\lambda}}^+$ exists for $\hat{\lambda} \in \Delta$, and PR_λ^+ is well defined for points in Δ as well as for nonreal points. Since R_λ^+ is analytic across Δ, the same is true of PR_λ^+. But for nonreal λ, $R_\lambda = PR_\lambda^+$. Hence R_λ can be continued analytically through Δ, and will then equal PR_λ^+ there.

REMARK 3.1. The representation $R_\lambda = PR_\lambda^+$ throughout the resolvent set allows the establishment of a number of formulas already known for nonreal points:

(i) $(R_\lambda f, g) = \int_{-\infty}^{\infty} \frac{d(E_\mu f, g)}{\mu - \lambda}$

(ii) For $f \in \Delta_T(\lambda)$, $(R_\mu - R_\lambda)f = (\mu - \lambda)R_\mu R_\lambda f$

(iii) $\Delta_T(\mu) = [1 + (\lambda - \mu)R_\lambda]\Delta_T(\lambda)$.

We next obtain a result concerning the point spectrum of a minimal self-adjoint extension T^+ of T. In the following theorem, $\dim \mathscr{E}$ denotes the dimension ($\leqq \infty$) of the manifold $\mathscr{E}(\hat{\lambda})$ of solutions of $T^*u = \hat{\lambda}u$. Also $E[\lambda] = E_{\lambda+} - E_{\lambda-}$.

THEOREM 3.2. *Let $M^+(\hat{\lambda})$ be the characteristic manifold in $\mathscr{H}^+$ corresponding to an eigenvalue $\hat{\lambda}$ of T^+, a minimal self-adjoint extension of T. Then* $\dim M^+(\hat{\lambda}) = \dim E[\hat{\lambda}]\mathscr{H} \leqq \dim \mathscr{E}(\hat{\lambda})$.

Proof. (i) $E[\hat{\lambda}]\mathscr{H} \subset \mathscr{E}(\hat{\lambda})$; proving the inequality in the theorem. To verify this let $h \in \mathscr{H}$ and choose $f \in D_T$. Then $Tf = T^+f$, and $(E[\hat{\lambda}]h, Tf) = (E^+[\hat{\lambda}]h, Tf) = (T^+E^+[\hat{\lambda}]h, f) = (\hat{\lambda}E^+[\hat{\lambda}]h, f) = \hat{\lambda}(E[\hat{\lambda}]h, f)$. Thus $E[\hat{\lambda}]h \in D_{T^*}$ and $T^*E[\hat{\lambda}]h = \hat{\lambda}E[\hat{\lambda}]h$.

(ii) By Lemma 3.1, $E^+[\hat{\lambda}]\mathscr{H}$ is dense on $M^+(\hat{\lambda})$. Thus $\dim M^+(\hat{\lambda}) = \dim E^+[\hat{\lambda}]\mathscr{H}$. The theorem will be proved by showing $\dim E^+]\hat{\lambda}]\mathscr{H} = \dim E[\hat{\lambda}]\mathscr{H}$.

Suppose $f_1, \cdots, f_m$ are vectors in $\mathscr{H}$ such that $E^+[\hat{\lambda}]f_1, \cdots, E^+[\hat{\lambda}]f_m$ are linearly independent. Then $E[\hat{\lambda}]f_1, \cdots, E[\hat{\lambda}]f_m$ are also linearly independent. For otherwise there would be constants $c_1, \cdots, c_m$, not all zero, such that

$$P \sum c_k E^+[\hat{\lambda}]f_k = \sum c_k E[\hat{\lambda}]f_k = 0 \ .$$

This would then imply that $f = \sum c_k E^+[\hat{\lambda}]f_k$ was a characteristic vector of T^+ such that $f \in \mathscr{H}^+ \ominus \mathscr{H}$. But that cannot be, since no reducing manifold of a *minimal* extension can lie in $\mathscr{H}^+ \ominus \mathscr{H}$ (see Naimark [8], § 4.)

On the other hand $E^+[\hat{\lambda}]f_1, \cdots, E^+[\hat{\lambda}]f_m$ are obviously independent when their projections $E[\hat{\lambda}]f_1, \cdots, E[\hat{\lambda}]f_m$ are. Thus $\dim E^+[\hat{\lambda}]\mathscr{H} = \dim E[\hat{\lambda}]\mathscr{H}$, proving the theorem.

REMARK 3.2. Because of the unitary equivalence of all minimal self-adjoint extensions T^+ associated with a given spectral resolution E_μ of T, it is natural to associate with E_μ the various aspects of the spectrum of T^+. Thus by the *spectrum, point spectrum, essential spectrum, etc. of* E_μ will be meant the corresponding point sets in the spectrum of T^+. An *eigenvalue of* E_μ will mean an eigenvalue of T^+, with its *multiplicity* the dimension of the corresponding manifold in $\mathscr{H}^+$.

From the theorems of this paragraph it follows that certain aspects of spectrum may be simply characterized *directly in terms of* E_μ. We mention especially:

(i) *Spectrum*: The points of increase of E_μ

(ii) *Eigenvalues*: Points of jump of E_μ. The multiplicity of an eigenvalue $\hat{\lambda}$ is $\dim E[\hat{\lambda}]\mathscr{H}$.

(iii) *Point Spectrum*: Closure of the set of eigenvalues.

(iv) *Essential Spectrum*: Cluster points of the spectrum, plus eigenvalues of infinite multiplicity.

4. Essential spectrum. Let E_μ be a generalized resolution of the identity associated with a symmetric operator T. From Remark 2.2, a *necessary* condition for an open real interval Δ to belong to the resolvent set of E_μ is that the associated family of contractions F_λ from $M(\lambda_0)$ to $M(\bar{\lambda}_0)$ have the properties:

(α) *F_λ is continuous from π^+ down to Δ, and*

(β) *$F_{\lambda+}$ is unitary on Δ.*

These properties obviously cannot hold for any Δ unless the defect spaces $M(\lambda_0)$ and $M(\bar{\lambda}_0)$ have the same dimension. Hence, *when T has unequal defect numbers, the spectrum of any resolution E_μ consists of the entire real axis.*

On the other hand, when T has equal defect numbers the properties (α) and (β) may well hold; in particular, when F_λ is a constant unitary operator, thus when E_μ is an *orthogonal* resolution, the properties are valid for every interval Δ.

In the remainder of the paper we shall consider a symmetric operator A with equal *finite* defect numbers. We recall that the essential spectrum Σ_e is the same point set for all orthogonal resolutions of A, that is, for all self-adjoint extensions *in* $\mathscr{H}$ of A. This is the classical theorem of H. Weyl, ([13] p. 251), proved originally for ordinary differential operators, and later extended to abstract operatars by E. Heinz [6]. The principal theorem of this paragraph extends Weyl's result to generalized resolutions which satisfy (α) and (β).

THEOREM 4.1. *Let the symmetric operator A have defect numbers (n, n) with $n < \infty$, and let Σ_e denote the points of the essential spectrum of any (hence every) orthogonal resolution of A. If E_μ be an arbitrarily chosen (generalized) resolution of A with essential spectrum Σ'_e, then:*

(i) $\Sigma'_e \supset \Sigma_e$.

(ii) *When (α) and (β) hold on Δ for the family of contractions associated with E_μ, then Σ'_e and Σ_e coincide on Δ.*

(iii) *If (α) and (β) fail on every subinterval of Δ, then $\Delta \subset \Sigma'_e$.*

We remark that the hypothesis of (iii) holds in particular under the condition:

(γ) *F_λ is continuous from π^+ down to the open real interval Δ and $\|F_{\lambda+}\| < 1$ on Δ.*

The proof will be based upon two lemmas of independent interest.

For any complex λ, let $\mathscr{E}(\lambda)$ denote the eigenspace of solutions of $T^*u = \lambda u$. Thus $\mathscr{E}(\lambda) = M(\bar{\lambda})$.

LEMMA 4.1. *Let $\hat{T}$ be a quasi-self-adjoint extension of T defined by F: $M(\lambda_0) \to M(\bar{\lambda}_0)$. For $f, g \in D_{T^*}$ introduce the form $\langle f, g\rangle = (T^*f, g) - (f, T^*g)$. Then the domains of $\hat{T}$ and $\hat{T}^*$ have the following characterization*:

$$D_{\hat{T}} = \{u\colon u \in D_{T^*} \text{ and } \langle u, \phi - F^*\phi\rangle = 0 \text{ for all } \phi \in \mathscr{E}(\lambda_0)\}$$
$$D_{\hat{T}^*} = \{u\colon u \in D_{T^*} \text{ and } \langle u, \psi - F\psi\rangle = 0 \text{ for all } \psi \in \mathscr{E}(\bar{\lambda}_0)\} .$$

Proof. The proof of Theorem 1 in Coddington [3] is directly adaptable.

LEMMA 4.2. *Consider a symmetric T with equal finite defect numbers (n, n), and suppose that λ is a real point of regular type of T, i.e. that $T-\lambda$ has a bounded inverse. For any quasi-s.a. extension $\hat{T}$, if $(\hat{T} - \lambda)^{-1}$ exists, it is a bounded operator with domain $\mathscr{H}$.*

Proof. $(\hat{T} - \lambda)^{-1}$ is defined on $\Delta_{\hat{T}}(\lambda) = \Delta_T(\lambda) \oplus [\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)]$. It is bounded on the first since $(T - \lambda)^{-1}$ is bounded at a point of regular type, and bounded on the second since the enclosing subspace $M(\lambda)$ has dimension n. Hence $(\hat{T} - \lambda)^{-1}$ is bounded on the sum of these orthogonal manifolds.

It remains to show that $\Delta_{\hat{T}}(\lambda) = \mathscr{H}$. Since $\Delta_T(\lambda)$ is closed, the problem reduces to showing that $\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)$ is n-dimensional. By (2.1), which gives the domain of $\hat{T}$, and by the existence of $(\hat{T} - \lambda)^{-1}$, it follows that $\Delta_{\hat{T}}(\lambda)$ contains n vectors which are linearly independent mod $\Delta_T(\lambda)$. Their projections onto $\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)$ are therefore linearly independent. Q.E.D.

Proof of Theorem 4.1. The statement that in general $\Sigma'_e \supset \Sigma_e$ follows from a result of Hartman, ([5], § 3, proof of proposition (iii)): He has shown that, when $\hat{\lambda} \in \Sigma_e$ (and n is finite), there exists a sequence $f_n \in D_A$ such that $\|f_n\| = 1, f_n \to 0$ weakly (in $\mathscr{H}$) and $(A - \hat{\lambda})f_n \to 0$ strongly. Consequently for any extension A^+ in $\mathscr{H}^+, f_n \in D_{A^+}, f_n \to 0$ weakly (in $\mathscr{H}^+$), and $(A^+ - \hat{\lambda})f_n \to 0$ strongly. Thus by Weyl's criterion ([9], § 133), $\hat{\lambda}$ is in the essential spectrum of A^+, and (by Remark 3.2) in the essential spectrum of the corresponding E_λ.

Next we show that, under the conditions (ii) on F_λ, when $\hat{\lambda} \notin \Sigma_e$ it cannot belong to Σ'_e. Since $\hat{\lambda} \notin \Sigma_e$, therefore the eigenspace of A at $\hat{\lambda}$ is finite dimensional at most. We can depress $\mathscr{H}$ and every $\mathscr{H}^+$ to the orthogonal complement of this manifold without changing any essential spectrum. Hence it may be assumed from the beginning that

$\hat{\lambda}$ is not an eigenvalue of A. Hence, by [5], § 3, property (ii), there exists a self-adjoint extension $\mathring{A}$ in $\mathscr{H}$ of A for which $\hat{\lambda}$ is not an eigenvalue. Since $\hat{\lambda} \notin \Sigma_e$, it cannot be a cluster point of the spectrum of $\mathring{A}$; consequently $\hat{\lambda}$ is in the resolvent set of $\mathring{A}$. Let Δ about $\hat{\lambda}$ be an open real interval in which $\mathring{R}_\lambda = (\mathring{A} - \lambda)^{-1}$ is analytic. We shall show that R_λ (corresponding to the given E_λ) is analytic in Δ except at isolated points. Since $\hat{\lambda}$ has at most finite multiplicity (by Theorem 3.2) as an eigenvalue of E_λ, it follows that $\hat{\lambda} \notin \Sigma'_e$.

It will be enough to show that $(A_\lambda - \lambda)\varphi = 0$ has a nonzero solution at only isolated points λ in Δ. For, by Lemma 4.2, R_λ will then exist except at these isolated points and, by the conditions of the theorem, and Remark 2.2, will be analytic.

Following M. G. Krein (see [1], § 84), we introduce an analytical basis $\phi_1(\lambda), \cdots, \phi_n(\lambda)$ for $\mathscr{E}(\lambda)$, $\lambda \in \pi^+ \cup \pi^- \cup \Delta$, by

$$\phi_k(\lambda) = [1 + (\lambda - \lambda_0)R_\lambda]\phi_k(\lambda_0) , \qquad k = 1, 2, \cdots, n .$$

Here $\phi_1(\lambda_0), \cdots, \phi_n(\lambda_0)$ form a basis (for convenience assumed *orthonormal*) for $\mathscr{E}(\lambda_0)$, with $\lambda_0 \in \pi^+$.

The solution space of $(A_\lambda - \lambda)\varphi = 0$ is $\mathscr{E}(\lambda) \cap D_{A_\lambda}$. According to Lemma 4.1, this subspace contains a nonzero vector at just those points $\lambda \in \Delta$ which are zeroes in $(\Delta-)$ of

$$\det \langle \phi_j(\lambda), \phi_k(\bar{\lambda}_0) - F_\lambda^* \phi_k(\bar{\lambda}_0) \rangle , \qquad \lambda \in \pi^- \cup (\Delta-) . \tag{4.1}$$

As noted, the expression is meaningful also in π^-, indeed is analytic there and continuous in $\pi^- \cup (\Delta-)$. Thus the theorem can be proved by showing that (4.1), (which is nonvanishing in π^-), can be continued analytically across Δ.

For $\lambda \in \pi^+ \cup (\Delta+)$ we have

$$F_\lambda \phi_k(\bar{\lambda}_0) = \sum_{l=1}^{n} F_{kl}(\lambda)\phi_l(\lambda_0), \text{ where } F_{kl}(\lambda) = (F_\lambda \phi_k(\bar{\lambda}_0), \phi_l(\lambda_0)) .$$

The coefficient determinant, $\det(F_{kl}(\lambda))$ is analytic on π^+ and continuous on $\pi^+ \cup (\Delta+)$. It is non-vanishing wherever F_λ^{-1} exists, hence in particular on $\Delta+$.

We shall show that the expression, defined for $\lambda \in \pi^+ \cup (\Delta+)$,

$$(-1)^n \det(\phi_l(\lambda_0), F_\lambda \phi_k(\bar{\lambda}_0)) \cdot \det \langle \phi_j(\lambda), \phi_k(\lambda_0) - F_\lambda^* \phi_k(\lambda_0) \rangle \tag{4.2}$$

coincides on Δ with (4.1). Since this expression (4.2) is analytic on π^+ and continuous on $\pi^+ \cup \Delta +$, it furnishes the desired continuation of (4.1) across Δ.

Since $F_{\lambda+}^{-1} = F_{\lambda+}^*$ on Δ, therefore

$$\begin{aligned}\phi_k(\bar\lambda_0) - F_{\lambda+}\phi_k(\bar\lambda_0) &= F^*_{\lambda+}F_{\lambda+}\phi_k(\bar\lambda_0) - F_{\lambda+}\phi_k(\bar\lambda_0) \\ &= \sum_l F_{kl}(\lambda+)[F^*_{\lambda+}\phi_l(\lambda_0) - \phi_l(\lambda_0)]\ .\end{aligned}$$

Noting that $F^*_{\lambda-} = F_{\lambda+}$, this permits writing the limit value of (4.1) in the form

$$(4.3)\qquad (-1)^n \det \overline{(F_{kl}(\lambda+))} \det \langle \phi_j(\lambda), \phi_l(\lambda_0) - F^*_{\lambda+}\phi_l(\lambda_0)\rangle\ .$$

But

$$\begin{aligned}\overline{F_{kl}(\lambda+)} &= \overline{(F_{\lambda+}\phi_k(\bar\lambda_0), \phi_l(\lambda_0))} \\ &= (\phi_l(\lambda_0), F_{\lambda+}\phi_k(\bar\lambda_0))\end{aligned}$$

so that (4.3) is identical with the limit value on Δ of (4.2).

The theorem is proved.

5. Strict contractions. In this paragraph we shall examine spectral resolutions of a symmetric operator A satisfying the conditions

(I) *A has equal finite defect numbers (n, n).*

(II) *Every point $\hat\lambda$ on a real interval Δ is of regular type for A, i.e. $(A - \hat\lambda)^{-1}$ exists and is bounded.*

Condition II can be stated in the following equivalent form:

(II′) *Any self-adjoint extension in $\mathscr{H}$ of A has in Δ only isolated points of its spectrum. No point* of *Δ is common to the spectra of all such extensions.*

The equivalence of II and II′ follows from Hartman ([5], prop. (ii))

Let E_μ be a spectral resolution of A, and F_λ be the associated family of contractions of $M(\lambda_0)$ into $M(\bar\lambda_0)$. It follows from theorem 4.1 that, on any sub-interval of Δ where (α) and (β) hold, the spectrum of E_μ will contain only isolated points.

Our interest here, however, will be in resolutions for which condition (γ) of § 4 holds on Δ. In this case, by Theorem 4.1 (iii), the spectrum of E_μ includes Δ. We first state a result valid when Δ is the entire real axis $\mathscr{R}$. When (γ) holds on $\mathscr{R}$ we shall describe F_λ as a *family of strict contractions.*

THEOREM 5.1. *Suppose that A satisfies* (I), *and* (II) *on $\mathscr{R}$. Let E_μ be a resolution of A for which the associated family of contractions F_λ is strict. Then:*

The associated minimal self-adjoint extension of A is unitarily equivalent to the n-fold direct sum of iD with itself, D being the differential operator d/dx on $\mathscr{L}_2(-\infty, \infty)$.

REMARK 5.1. This theorem generalizes results of Coddington and Gilbert [4] for ordinary differential operators on a closed bounded interval. Their method of proof appears to be adaptable to handle certain other ordinary differential operators satisfying I and II, in particular, singular operators in Weyl's limit circle case.

REMARK 5.2. Condition II on $\mathscr{R}$ of course implies that A has no eigenvalues. However it is easy to analyze the more general situation in which eigenvalues do occur, provided II on $\mathscr{R}$ holds for the restriction of A to the manifold orthogonal to the eigenvectors. In that case the minimal self-adjoint extension is equivalent to the direct sum of the discrete part of A with the operator described in Theorem 5.1.

We shall prove Theorem 5.1 as a special case of a more general theorem. We now suppose that I, II, and (γ) hold on Δ. By assumption, $F_{\lambda+}$, and hence $A_{\lambda+}$, exists for every $\lambda \in \Delta$. The assumptions that $\| F_\lambda \| < 1$ and that A has no eigenvalues on Δ imply that $D_{A_{\lambda+}} \cap \mathscr{E}(\lambda) = \{0\}$ for $\lambda \in \Delta$, and hence that $(A_{\lambda+} - \lambda)^{-1}$ exists. This statement follows from the fact, noted by Hartman [5], that when $f \in \mathscr{E}(\lambda)$, for $\lambda \in \mathscr{R}$, is written in the form

$$f = f_0 + f^+ + f^-, \quad \text{where} \quad f_0 \in D_A,\ f^+ \in \mathscr{E}(\lambda_0),\ f^- \in \mathscr{E}(\bar{\lambda}_0)$$

for $\lambda_0 \in \pi^+$, then $\|f^+\| = \|f^-\|$. Then, by assumption I and Lemma 4.2, R_λ exists, and is continuous in λ, on $\pi^+ \cup (\Delta+)$.

One may define a basis for $\mathscr{E}(\lambda)$, $\lambda \in \pi^+ \cup \Delta$, by

$$\phi_k(\lambda) = [1 + (\lambda - \lambda_0)R_\lambda]\phi_k(\lambda_0)\ , \qquad k = 1, 2, \cdots, n\ . \tag{5.1}$$

Here $\lambda_0 \in \pi^+$, and $\phi_1(\lambda_0), \cdots, \phi_n(\lambda_0)$ form a basis for $\mathscr{E}(\lambda_0)$. That $\phi_k(\lambda)$ is in $\mathscr{E}(\lambda)$ follows from $(A^* - \lambda)R_\lambda = 1$. That $\phi_1(\lambda), \cdots, \phi_n(\lambda)$ are independent follows from the fact that

$$1 + (\lambda - \lambda_0)R_\lambda = (A_\lambda - \lambda_0)(A_\lambda - \lambda)^{-1}$$

has an inverse.[1]

We shall henceforth identify π^+ with the half-plane $\mathscr{I}(\lambda) > 0$. The basis (5.1) allows a simple representation for $\mathscr{I} R_{\lambda+} = 1/2i\,[R_{\lambda+} - R_{\lambda-}]$:

LEMMA 5.1. *Assume that A satisfies* I, II, *and F_λ satisfies* (γ). *Then for every $\lambda \in \Delta$ and every $f \in \mathscr{H}$,*

$$\mathscr{I} R_{\lambda+} f = \sum_{j,k=1}^{n} \Phi_{jk}(\lambda)(f, \phi_j(\lambda))\phi_k(\lambda)\ . \tag{5.2}$$

The matrix $\Phi(\lambda)$ is positive definite and continuous in λ. Here π^+ has been identified with the half plane $\mathscr{I}(\lambda) > 0$.

[1] In what follows only the *existence* of a continuous basis is needed, not its relation (5.1) to R_λ.

Proof. Since for every $f \in \mathscr{H}$, $(A^* - \lambda)R_{\lambda+}f = f$, therefore $\mathscr{I} R_{\lambda+}f \in \mathscr{E}(\lambda)$. In terms of an orthonormal basis $\tilde{\phi}_1, \cdots, \tilde{\phi}_n$ for $\mathscr{E}(\lambda)$, $\mathscr{I} R_{\lambda+}f = \Sigma C_k\tilde{\phi}_k$, where

$$C_k = (\mathscr{I} R_{\lambda+}f, \tilde{\phi}_k) = (f, \mathscr{I} R_{\lambda+}\tilde{\phi}_k) = (f, \psi_k)$$

for some $\psi_k \in \mathscr{E}(\lambda)$. Writing $\tilde{\phi}_1, \cdots, \tilde{\phi}_n, \psi_1, \cdots, \psi_n$ as linear combinations of $\phi_1(\lambda), \cdots, \phi_n(\lambda)$ establishes the form of (5.2).

From the known relation

$$(\mathscr{I} R_\lambda f, f) \geqq 0 \quad \text{for} \quad \mathscr{I}\lambda > 0$$

follows

$$(5.3) \qquad (\mathscr{I} R_{\lambda+}f, f) \geqq 0 \quad \text{for} \quad \mathscr{I}\lambda = 0 .$$

Recalling that $\mathscr{E}(\lambda)$ is invariant under $\mathscr{I} R_{\lambda+}$, let $\{\mathscr{I} R_{\lambda+}\}$ denote the restriction of $\mathscr{I} R_{\lambda+}$ to $\mathscr{E}(\lambda)$. We assert that

$$(5.4) \qquad (\{\mathscr{I} R_{\lambda+}\}\phi, \phi) > 0 \quad \text{when} \quad \|\phi\| > 0 , \qquad \phi \in \mathscr{E}(\lambda) .$$

In view of (5.3) it is sufficient to show that

$$(5.5) \qquad \{\mathscr{I} R_\lambda\}\psi = 0 \quad \text{implies} \quad \psi = 0 .$$

Suppose that $\{\mathscr{I} R_\lambda\}\psi = 0$. Then $g = R_{\lambda+}\psi = R_{\lambda-}\psi$ belongs to $D(A_{\lambda+}) \cap D(A_{\lambda-})$. Writing g in the form

$$g = g_0 + g^+ + g^-, \quad g_0 \in D(A),\ g^+ \in \mathscr{E}(\lambda_0),\ g^- \in \mathscr{E}(\bar{\lambda}_0)$$

then, by the definition of $D(A_{\lambda\pm})$,

$$-g^+ = F_{\lambda+}g^- , \qquad -g^- = F_{\lambda-}g^+ .$$

Since $\|F_{\lambda+}\|, \|F_{\lambda-}\| < 1$, this implies that $g^+ = g^- = 0$, i.e. $g \in D(A)$. Since $R_{\lambda+}\psi \in D(A)$ therefore $\psi \in \Delta_A(\lambda)$, the orthogonal complement of $\mathscr{E}(\lambda)$. Thus $\psi = 0$. This proves (5.4).

Now let ϕ be an arbitrary element of $\mathscr{E}(\lambda)$ and put

$$\xi_k = (\phi, \phi_k(\lambda)) , \qquad k = 1, 2, \cdots, n .$$

In view of the independence of $\phi_1(\lambda), \cdots, \lambda_n(\phi)$, this relation is a one-to-one linear mapping of $\mathscr{E}(\lambda)$ onto the n-dimensional space of vectors $\xi = (\xi_1, \cdots, \xi_n)$. Thus relation (5.4) is equivalent, because of the form of (5.2), to

$$\Sigma\Phi_{jk}(\lambda)\xi_j\bar{\xi}_k > 0 \quad \text{when} \quad \|\xi\| \neq 0 .$$

That is, the matrix $\Phi(\lambda)$ is positive-definite.

It remains only to prove the continuity of $\Phi(\lambda)$. This follows directly from the relation

$$(\mathscr{I} R_{\lambda+}\phi_\mu(\lambda), \phi_\nu(\lambda)) = \sum_{j,k} \Phi_{jk}(\lambda)(\phi_\mu(\lambda), \phi_j(\lambda))\overline{(\varphi_\nu(\lambda), \varphi_k(\lambda))} ,$$

since $\det(\phi_\mu(\lambda), \phi_j(\lambda)) \neq 0$.

THEOREM 5.2. *Suppose that on an interval Δ the operator A satisfies* I, II *and that E_μ is a spectral resolution of A for which the corresponding mapping F_λ satisfies* (γ). *Let A^+ in $\mathscr{H}^+$ be a minimal s.a. extension of A with orthogonal resolution E^+_μ satisfying $E_\mu = PE^+_\mu$. For $\mu \in \Delta$ define $\rho(\mu) = 1/\pi\int\Phi(\mu)d\mu$. Then the part of A^+ on $E^+(\Delta)\mathscr{H}^+$ is unitarily equivalent to the multiplication operator on $\mathscr{L}_2(\rho(\mu))$, $\mu \in \Delta$.*

Proof of the Theorems. It is pointed out by Coddington and Gilbert [4] that the multiplication operator in $\mathscr{L}_2(\rho)$ (where ρ is strictly increasing and continuous in λ on $\mathscr{R}$) is unitarily equivalent to the n-fold direct product of iD with itself, D being the differential operator d/dx on $\mathscr{L}_2(-\infty, \infty)$. Thus Theorem 5.1 is a corollary of Theorem 5.2.

For every $f \in \mathscr{H}$ and every bounded real interval $\Delta' \subset \Delta$,

$$\begin{aligned}(E(\Delta')f, f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi}\int_{\Delta'}(\mathscr{I} R_{\lambda+i\varepsilon}f, f)d\lambda \\ &= \frac{1}{\pi}\int_{\Delta'}(\mathscr{I} R_{\lambda+}f, f)d\lambda .\end{aligned}$$

Here we have used the continuity of R_λ on $\pi^+ \cup (\Delta+)$ and of E_λ on Δ. Let $g(\lambda) = \{g_k(\lambda)\}_{k=1}^n$ be defined by $g_k(\lambda) = (f, \phi_k(\lambda))$. Hence

$$(E(\Delta')f, f) = \frac{1}{\pi}\int_{\Delta'}\Sigma\Phi_{jk}(\lambda)g_j(\lambda)\bar{g}_k(\lambda)d\lambda ;$$

$$\| E^+(\Delta')f\|^2 = \int_{\Delta'}\Sigma g_j(\lambda)\bar{g}_k(\lambda)d\rho_{jk}(\lambda) . \tag{5.6}$$

Now suppose $f \in \mathscr{H}$ is in $E^+(\Delta)\mathscr{H}^+$. Thus $E(\Delta)f = f$. Consider V: $E^+(\Delta')f \to \chi_{\Delta'}(\lambda)g(\lambda)$, where $\chi_{\Delta'}(\lambda)$ is the characteristic function of the interval $\Delta' \subset \Delta$. From (5.6) V is an isometric mapping of $Z(\Delta)$ (see Lemma 3.1) into $\mathscr{L}_2(\rho(\lambda))$, $(\lambda \in \Delta)$, which carries $E^+(\Delta')$ into the operation of multiplication by $\chi_{\Delta'}(\lambda)$. Since $Z(\Delta)$ is fundamental on $E^+(\Delta)\mathscr{H}^+$, Theorem 5.2 follows.

6. Differential operators. Let $Lu = -(pu')' + qu$ be an ordinary differential expression on the positive axis $0 \leqq x \leqq \infty$, with p and q real measurable functions such that $p(x) > 0$,

$$\int_0^b p(x)^{-1}dx < \infty , \quad \int_0^b | q(x) | \, dx < \infty$$

for any $b > 0$. With a suitably prescribed[2] minimal domain in $\mathscr{L}_2(0, \infty)$, L defines a symmetric quasi-differential operator L_0 with defect numbers (1, 1) or (2, 2). It is easily seen that L_0 has no eigenvalues. When the defect numbers are (2, 2), the Conditions I and II of § 5 are automatically satisfied on $\mathscr{R}$, so that the results of that section hold.

We shall assume that L_0 has defect numbers (1, 1), i.e. is in the limit point case, and shall study the absolutely continuous spectrum of a minimal self-adjoint extension L_0^+ of L_0. As before (§ 3), L_0^+ operates in a space $\mathscr{H}^+$ containing $\mathscr{H}$ and has a spectral family of projections denoted by E_μ^+.

Let M_a and M_s be the absolutely continuous and singular subspaces of $\mathscr{H}^+$ with respect to L_0^+ (see [7] for definitions). Thus M_a and M_s reduce L_0^+, are orthogonal, and $\mathscr{H}^+ = M_a \oplus M_s$. For any $u \in M_a[u \in M_s]$, the function $(E_\mu^+ u, u)$ is absolutely continuous [singular] with respect to Lebesque measure on $-\infty < \mu < \infty$. Let E_μ be a generalized resolution of L_0 for which $E_\mu = PE_\mu^+$. By the *absolutely continuous spectrum of* L_0^+ *(or of* E_μ*)* will be meant the spectrum of the part of L_0^+ in M_a. The *singular spectrum* is defined similarly.

It has been proved by N. Aronszajn [2] that the absolutely continuous spectrum is the same point set for all orthogonal resolutions of the differential operator L_0. The following theorem extends Aronszajn's result to generalized resolutions in a way parallel to Theorem 4.1 for essential spectrum. Clause (iii) contains a partial extension, for differential operators, of Theorem 5.1.

THEOREM 6.1. *Let L_0 be a quasi-differential operator as described above and let Σ_a denote the points of the absolutely continuous spectrum of any (hence every) orthogonal resolution of L_0. If E_μ is an arbitraily chosen (generalized) resolution of L_0 with absolutely continuous spectrum Σ_a' and singular spectrum Σ_s', then:*

(i) $\Sigma_a' \supset \Sigma_a$

(ii) *When* (α) *and* (β) *hold on* Δ *for the family of contractions associated with* E_μ, *then* Σ_a' *and* Σ_a *coincide on* Δ.

(iii) *When* (γ) *holds on* Δ, *then* $\Delta \subset \Sigma_a'$, *while* $\Delta \cap \Sigma_s' = 0$.

The proof depends upon

LEMMA 6.1. *Let $\rho(\mu) = [\rho_{jk}(\mu)]_{j,k=1}^2$ be a nondecreasing Hermitian matrix ($-\infty < \mu < \infty$) and Λ the multiplication operator with maximal domain in $\mathscr{L}_2(\rho)$. Let*

$$\rho(\mu) = \rho_a(\mu) + \rho_s(\mu)$$

be the Lebesque decomposition of ρ into its absolutely continuous and

[2] A precise specification may be found in [1], Appendix II.

singular parts, defined by the corresponding decomposition of the components of ρ.

Then ρ_a *and* ρ_s *are nondecreasing Hermitian matrices.* $\mathscr{L}_2(\rho_a)$ *and* $\mathscr{L}_2(\rho_s)$ *are, respectively, the absolutely continuous and singular subspaces of* $\mathscr{L}_2(\rho)$ *with respect to* Λ. *Thus the absolutely continuous spectrum of* Λ *consists of the points of increase of* ρ_a, *or equivalently of its trace* tr ρ_a. *A similar statement holds for the singular spectrum.*

We shall omit the proof of Lemma 6.1.

Proof of Theorem 6.1. For $\mathscr{I}\lambda \neq 0$, let $\psi_\lambda = \psi(x, \lambda)$ denote the $\mathscr{L}_2$ solution of $L\psi = \lambda\psi$ which is determined by

$$[p(x)\psi'(x, \lambda)]_{x=0} = -1 .$$

Put $\psi(0, \lambda) = m(\lambda)$. Each generalized resolution E_μ of L_0 is now specified by a family of contractions F_λ: $M(i) \rightarrow M(-i)$ of the form

$$F_\lambda \psi_{-i} = W(\lambda)\psi_i$$

where $W(\lambda)$ is analytic and $|W(\lambda)| \leqq 1$ for $\mathscr{I}\lambda > 0$. Define

$$\theta(\lambda) = \frac{W(\lambda)m(i) - m(-i)}{1 - W(\lambda)} , \qquad \mathscr{I}\lambda > 0 .$$

Since $\mathscr{I} m(i) > 0$ and $m(-i) = \bar{m}(i)$, therefore $\mathscr{I}\theta(\lambda) \geqq 0$ (with $\theta = \infty$ when $W = 1$).

A. V. Štraus [11] has associated with each E_μ a *spectral matrix* $\rho(\mu) = [\rho_{jk}(\mu)]_{j,k=1,2}$, $-\infty < \mu < \infty$, which is Hermitian nondecreasing, and such that

$$\operatorname{tr} \rho(\mu) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \int_0^\mu \mathscr{I} \phi(\gamma + i\varepsilon) d\gamma$$

where

$$\phi(\lambda) = \frac{m(\lambda)\theta(\lambda) - 1}{\theta(\lambda) + m(\lambda)} , \qquad \mathscr{I}\lambda > 0 . \tag{6.1}$$

In particular, when $W(\lambda) \equiv 1$, $\phi(\lambda)$ reduces to $m(\lambda)$.

Let Λ be the multiplication operator with maximum domain in $\mathscr{L}_2(\rho)$ and let L_0^+ be a minimal self adjoint extension of L_0, with $E_\mu = PE_\mu^+$. By the reasoning in [4], § 4, Λ is unitarily equivalent to L_0^+.

Therefore, by Lemma 6.1, the problem is reduced to a consideration of the absolutely continuous and singular parts of tr ρ. But such consideration is possible along the lines of [2].

A set G is a *support* of a real measure ν when $\nu(\mathscr{R} - G) = 0$. It is a *minimal* support when for every support $G_1 \subset G$, the Lebesque measure $|G - G_1| = 0$. It is easy to prove that when ν and ν' are absolutely continuous measures with minimal supports $G \subset G'$ then $\nu \prec \nu'$ (i.e. $\nu'(s) = 0$ implies $\nu(s) = 0$).

The following disjoint sets G_a and G_s are minimal supports for, respectively, the absolutely continuous and singular parts of $\operatorname{tr} \rho$ (compare [2]):

$$G_a = \{\mu \in \mathscr{R}\colon \lim_{\lambda\to\mu} \phi(\lambda) \text{ exists finitely and } \lim_{\lambda\to\mu} \mathscr{I}\phi(\lambda) > 0\}$$
$$G_s = \{\mu \in \mathscr{R}\colon \mathscr{I}\phi(\lambda) \to \infty \text{ when } \lambda \to \mu\} .$$

(Here it is understood that $\lambda \to \mu$ with the constraint that $\varepsilon < \operatorname{Arg}(\lambda - \mu) < \pi - \varepsilon$ for some fixed $\varepsilon > 0$.)

We shall compare the sets G_a, G_s corresponding to an arbitrarily chosen resolution E_μ with the special sets G_a^0, G_s^0 corresponding to the orthogonal resolution for which $W \equiv 1$. Thus in the definitions of $G_a^0, G_s^0, \phi(\lambda)$ is replaced by $m(\lambda)$.

We note first that $\mathscr{I}\theta, \mathscr{I}\phi$ and $\mathscr{I}m$ are all $\geqq 0$ when $\mathscr{I}\lambda > 0$.

Since $\lim_{\lambda\to\mu} \theta(\lambda)$ exists finitely except for λ on a certain set S_0 of Lebesgue measure zero, inspection of (6.1) and the formula

$$\mathscr{I}\phi = \frac{(1 + |\theta|^2)\mathscr{I}m + (1 + |m|^2)\mathscr{I}\theta}{|m + \theta|^2}$$

reveals that $G_a \supset G_a^0 - S_0$. Since these are minimal support it follows that $\operatorname{tr} \rho_a \succ \operatorname{tr} \rho_a^0$, implying the statement (i) of the theorem.

Next, assume that (α) and (β) hold on Δ. Therefore $\theta(\lambda)$ may be continued down to Δ with $\mathscr{I}\theta(\mu+) = 0$ on Δ. Inverting (6.1) one obtains the formulas

$$m = \frac{\phi\theta + 1}{\theta - \phi} , \qquad \mathscr{I}m = \frac{(1 + |\theta|^2)\mathscr{I}\phi - (1 + |\phi|^2)\mathscr{I}\theta}{|\theta - \phi|^2} ,$$

which show on inspection that $G_a \cap \Delta \subset G_a^0$. Together with the earlier obtained inclusion, this implies that G_a and G_a^0 coincide on Δ. Since these are minimal supports, (ii) follows.

Finally, assume (γ) holds on Δ. In this case $\theta(\lambda)$ may be continued down to Δ with $\mathscr{I}\theta(\mu+) > 0$ for $\mu \in \Delta$. Equation (6.1) shows that $\phi(\lambda)$ remains bounded as $\lambda \to \mu$ on Δ and hence that $G_s \cap \Delta = 0$. Thus $\Sigma_s' \cap \Delta = 0$. At the same time, by Theorem 4.1 (iii), Δ does belong to the spectrum of E_μ, and hence must belong to the absolutely continuous spectrum Σ_a'. Q.E.D.

REMARK 6.1. A. V. Štraus [12] has shown that when $\theta(\lambda)$ may be continued to real limit values on the entire real axis—equivalent to the assertion that (α) and (β) hold on $\mathscr{R}$—then E_μ^+ has simple spectrum, This, together with Theorem 6.1 (ii), implies the *unitary equivalence of the absolutely continuous parts* of minimal self-adjoint extensions corresponding to resolutions E_μ satisfying (α) and (β) on $\mathscr{R}$.

REMARK 6.2. Assume (γ) holds on $\mathscr{R}$. If conditions I and II of § 5 hold for L_0 then, by Theorem 5.1, the multiplicity of spectrum of L_0^+ will be 1, and the operator equivalent to iD. Simple examples show that in general (i.e. without Conditions I and II) the multiplicity of spectrum may well be 2 (the maximum consistent with ρ being a 2×2 matrix) and that L_0^+ may even be equivalent to $iD \oplus iD$.

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