ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS

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1. Introduction. As is well known, a Blaschke product $f(z)$ in $\{|z| < 1\}$ has radial limits $f(e^{i\theta})$ of modulus one almost everywhere on $\{|z| = 1\}$. The object of the present paper is to give a partial answer to the question: how many times does $f(z)$ assume a given radial limit? We shall prove the following theorem.

THEOREM A. *Let E be a given closed set on $\{|w| = 1\}$ and let E' be the complement of E relative to $\{|w| = 1\}$. Then there exists a Blaschke product $f(z)$, all of whose radial limits are of modulus one, and such that the set*

$$L(\beta) = \{\theta \,|\, f(e^{i\theta}) = e^{i\beta}\}$$

has the power of the continuum for $e^{i\beta} \in E$ and is countable for $e^{i\beta} \in E'$.

Theorem A is a condensed statement of what we shall actually prove; Theorems 1, 2, and 3 contain somewhat more information on $f(z)$. The method of proof is to construct a suitable regularly-branched covering $\mathscr{W}$ of $\{|w| < 1\}$, corresponding to an automorphic function $w = f(z)$, and then use the geometry of $\mathscr{W}$ to obtain our results.

The question naturally arises as to whether one could prove Theorem A directly. That is: could one produce an $f(z)$ with the desired properties by exhibiting its zeros instead of defining $f(z)$ by means of a surface $\mathscr{W}$? The answer to this question does not seem to be obvious.

2. The surface $\mathscr{W}$. Let E be a given *nonvoid* closed subset of $\{|w| = 1\}$ and let $\{a_n\}_1^\infty$ be an infinite sequence of points in $\{|w| < 1\}$ whose derived set is E. Clearly, we may assume that $a_n \neq 0$ and

$$\arg a_m \neq \arg a_n \qquad (m \neq n) . \tag{1}$$

Let $\mathscr{W}$ be the simply-connected unbordered covering of $\{|w| < 1\}$ which is regularly-branched over the points $\{a_n\}$ with all branch points of multiplicity 2. It is well known [2, 3, 6] that such a covering, with any specified multiplicity or signature for each a_n, exists and is unique. Instead of appealing to the general theory of regularly-branched coverings, we shall construct the surface $\mathscr{W}$ directly, since the details of the construction play a role in the proof of Theorem A.

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Let C_n be the radial segment $\arg w = \arg a_n$, $|a_n| \leqq |w| < 1$. The C_n are disjoint because of (1). We make cuts in $\{|w| < 1\}$ along each C_n and so obtain a alit disc W, copies of which are joined together, according to the following specifications, to form the surface.

0^{th} *level.* The surface $\mathscr{W}_0$ consists of just one slit disc W. Note that $\mathscr{W}_0$ is simply-connected.

1^{st} *level.* The surface $\mathscr{W}_1$ is obtained by adjoining an infinite sequence of *distinct* copies of W, namely $W(1)$, $W(2)$, $\cdots$, to $\mathscr{W}_0$. $W(n_1)$ is joined to $\mathscr{W}_0$ along C_{n_1} so as to form a first-order branch-point over a_{n_1}. The surface $\mathscr{W}_1 = \mathscr{W}_0 \cup \bigcup_n W(n)$ is simply-connected; for by adjoining the $W(n)$ one at a time we obtain an increasing sequence of simply-connected surfaces which exhaust $\mathscr{W}_1$. We denote by $\chi(n_1)$ the curve in $\mathscr{W}_1$ along which $W(n_1)$ and $\mathscr{W}_0$ are identified.

2^{nd} *level.* Along each free slit on the boundary of $\mathscr{W}_1$ we adjoin a copy of W. More precisely, the sheet $W(n_1, n_2)$ is adjoined to $W(n_1)$ along the cut C_{n_2} in $W(n_1)$. The added sheets correspond one-to-one with all pairs (n_1, n_2) of positive integers such that $n_1 \neq n_2$. Again we see that the surface $\mathscr{W}_2 = \mathscr{W}_1 \cup \bigcup W(n_1, n_2)$ is simply-connected. The curve over C_{n_2} along which $W(n_1)$ and $W(n_1, n_2)$ are joined is denoted by $\chi(n_1 n_2)$.

k^{th} *level.* Continuing the construction, the surface $\mathscr{W}_k$ consists of $\mathscr{W}_{k-1}$ and copies of W denoted by $W(n_1, n_2, \cdots, n_k)$, $n_i \neq n_{i+1}$, which are joined to $\mathscr{W}_{k-1}$; $W(n_1, \cdots, n_k)$ is adjoined to $W(n_1, \cdots, n_{k-1})$ along the cut C_{n_k} in $W(n_1, \cdots, n_{k-1})$. Denote the curve along which those two sheets are joined by $\chi(n_1, n_2, \cdots, n_k)$. Clearly $\mathscr{W}_k$ is simply-connected.

We take the surface $\mathscr{W}$ to be $\lim \mathscr{W}_k$ as $k \to \infty$; it is clear that $\mathscr{W}$ is simply-connected as $\mathscr{W}_k \uparrow \mathscr{W}$. With the natural projection map onto $\{|w| < 1\}$ it is clear that $\mathscr{W}$ is a regularly-branched, unbordered, covering of $\{|w| < 1\}$. All points of $\mathscr{W}$ over the a_n are branch-points of multiplicity 2, and $\mathscr{W}$ has no other branch-points.

3. The function $f(z)$. Since $\mathscr{W}$ is a covering of $\{|w| < 1\}$ it is hyperbolic. Let $w = f(z)$ be the holomorphic function which maps $\{|z| < 1\}$ onto $\mathscr{W}$, with $f(0) = 0 \in \mathscr{W}_0$ and $f'(0) > 0$. Clearly $|f(z)| < 1$. The radial limits of $f(z)$ are *all* of modulus one, since if this were not the case a boundary point of $\{|z| < 1\}$ would correspond to an interior point of $\mathscr{W}$ which is unbordered. Thus $f(z)$ is of class U [5, p. 32]. Applying Frostman's theorem [5, p. 33] we see that $f(z)$ is a Blaschke product.

Also, $f(z)$ is an automorphic function with respect to a Fuschian group F, since the decktransformations of $\mathscr{W}$ correspond to linear transformations preserving $\{|z| < 1\}$. It is easily shown that if $E = \{|w| = 1\}$ then F is of the first kind: the limit points of F fill $\{|z| = 1\}$. If $E \neq \{|w| = 1\}$ then F is of the second kind: the set of limit points of F is a perfect nowhere dense subset of $\{|z| = 1\}$.

The sheets $W(n_1, n_2, \cdots, n_k)$ of $\mathscr{W}$ correspond to a set of fundamental regions $R(n_1, \cdots, n_k)$ of F. These are the fundamental regions which play a role in the proof; since these are defined via the function f it is not clear that they are the same as the fundamental regions obtained by any of the usual constructions in terms of F. Hence we must derive some properties of these regions.

4. Properties of the fundamental regions. For convenience we reduce the notations $W(n_1, \cdots, n_k)$, $R(n_1, \cdots, n_k)$, and $\chi(n_1, \cdots, n_k)$ to W, R, and χ respectively. To each curve χ in $\mathscr{W}$ there corresponds a simple arc X in $\{|z| < 1\}$. It is evident that the fundamental regions R are bounded by the X's and points of $\{|z| = 1\}$. We proceed with an investigation of the X's.

First, *each X ends at two distinct points of* $\{|z| = 1\}$. The two linear pieces of χ correspond to two simple arcs X' and X'', and $f(z)$ tends to a limit as $|z| \to 1$ on X' and X''. Then by Koebe's lemma [1, p. 213] each of X' and X'' must tend to a definite point of $\{|z| = 1\}$. The end points of X' and X'' must be distinct. If not, let D be that part of $\{|z| < 1\}$ bounded by X and a single point b on $\{|z| = 1\}$. Then the part of $\mathscr{W}$ corresponding to D will contain an infinite number of sheets W joined along various χ's, which correspond to X's, all ending at b. Thus $f(z)$ would have infinitely many distinct asymptotic values, namely $\exp(i \arg a_n)$, at b; but this would contradict the theorem of Lindelöf [4, p. 9] to the effect that a bounded holomorphic function can have at most one asymptotic value at a given point.

Thus each X is a crosscut of $\{|z| < 1\}$. A second property is that *no two X's have a common endpoint*. To see this, suppose X_1 and X_2 are two distinct X's with a common endpoint b on $\{|z| = 1\}$. Let the corresponding curves χ_1 and χ_2 in $\mathscr{W}$ end at points α_1 and α_2, respectively, over $\{|w| = 1\}$. If $\alpha_1 \neq \alpha_2$ then we would again have a contradiction of Lindelöf's theorem. Now suppose $\alpha_1 = \alpha_2$. We may construct a sequence of arcs Δ_n in $\{|z| < 1\}$, each joining a point of X_1 to a point of X_2, such that diam $\Delta_n \to 0$. Since by Lindelöf's theorem $f(z) \to \alpha_1$ uniformly between X_1 and X_2 we may also require diam $\{f(\Delta_n)\} < 1/n$. But from the structure of $\mathscr{W}$ it is clear that there exists a curve χ on $\mathscr{W}$, with endpoint $\neq \alpha_1$, such that *any* curve on $\mathscr{W}$, joining a point of χ_1 to a point of χ_2, must intersect χ. Since the projection of χ into $\{|w| < 1\}$ and the common projection of χ_1 and χ_1 are a positive distance δ apart, we must have diam $\{f(\Delta_n)\} \geqq \delta$, which is incompatible with diam $\{f(\Delta_n)\} < 1/n$.

Next, for any $\varepsilon > 0$, *the set* $S = \{X \mid \operatorname{diam} X > \varepsilon\}$ *is finite.* For, any disc $\{|z| < 1 - \delta\}$ intersects only a finite number of the X's. Hence if S were infinite there would exist an infinite sequence $\{X_n\}_1^\infty$ of distinct crosscuts and a nondegenerate arc Λ on $\{|z| = 1\}$ such that the radius joining $z = 0$ to an arbitrary point of Λ crosses every X_n. Now any radial limit $f(e^{i\theta}) = e^{i\alpha}$, $e^{i\theta} \in \Lambda$, forces the χ_n, corresponding to X_n and ending at $e^{i\alpha_n}$, to satisfy $\alpha_n \to \alpha$. But then $f(e^{i\theta}) = e^{i\alpha}$ for almost all $e^{i\theta} \in \Lambda$, which contradicts the theorem of F. and M. Riesz. The point of this paragraph is that if b is a limit point of F, then any neighborhood of b contains infinitely many complete fundamental regions R. There are at least some examples of Fuchsian groups possessing a set of fundamental regions (connected) whose diameters are bounded away from zero.

5. Properties of $f(z)$ on the boundary.

THEOREM 1. *Let b be a limit point of F, U a neighborhood of b, and let $e^{i\alpha} \in E$. Then the set*

$$U \cap L(\alpha)$$

has the power of the continuum.

Proof. There exists a cross-cut X, corresponding to the curve χ in $\mathscr{W}$, which separates $\{|z| < 1\}$ into two domains, one of which, D, is contained in U. The corresponding part, $\mathscr{D}$, of $\mathscr{W}$ contains infinitely many sheets. In $\{|w| < 1\}$ we may select among the arcs C_n two sequences, $\{C_n(0)\}_1^\infty$, and $\{C_n(1)\}_1^\infty$, which satisfy either the following three conditions

(2) *the lengths of the* $C_n(0)$ *and* $C_n(1)$ *tend to zero,*

(3) $\arg C_n(0) \downarrow \alpha$, *and* $\arg C_n(1) \downarrow \alpha$,

(4) $\arg C_{n+1}(0) < \arg C_n(1) < \arg C_n(0)$;

or the same conditions with the arrows in (3) and the inequalities in (4) reversed. Such sequences $\{C_n(\varepsilon)\}$, $\varepsilon = 0, 1$ exist because of $e^{i\alpha} \in E$, the initial choice of $\{\alpha_n\}$, and (1).

Now let $\Gamma(\varepsilon) = \Gamma(\varepsilon_1, \varepsilon_2, \cdots)$, $\varepsilon_i = 0, 1$, be an arc in $\mathscr{D}$ with the properties:

(5) $\Gamma(\varepsilon)$ *crosses, in order, curves* χ *in* $\mathscr{D}$ *over the arcs* $C_1(\varepsilon_1), C_2(\varepsilon_2), C_3(\varepsilon_3), \cdots$, *and meets no other* χ's.

(6) $\Gamma(\varepsilon)$ *tends to a point on the boundary of* $\mathscr{D}$ *over* $e^{i\alpha}$.

This construction of $\Gamma(\varepsilon)$ is possible by (2), (3), (4), and since all the curves χ over $\alpha < \arg w < \alpha + \delta$, $\delta = \delta(\eta)$, are of length $< \eta$. $\Gamma(\varepsilon)$ corresponds to an arc $\Delta(\varepsilon)$ in $\{|z| < 1\}$ which tends to a definite point $b(\varepsilon) \in U \cap \{|z| = 1\}$, since $f(z) \to e^{i\alpha}$ on $\Delta(\varepsilon)$. By a well-known theorem of Lindelöf [4, p. 10] then the radial limit of $f(z)$ exists at $b(\varepsilon)$ and has the value of $e^{i\alpha}$.

By associating $b(\varepsilon)$ with the dyadic expansion $0.\ \varepsilon_1\varepsilon_2\varepsilon_3\cdots$, we see that we have found a set of points $b(\varepsilon)$ in $U \cap \{|z| = 1\}$, associated with the radial limit $e^{i\alpha}$, having the power of the continuum, provided that distinct sequences of ε's correspond to distinct points $b(\varepsilon)$. To show that, let $\{\varepsilon_i\}$ and $\{\varepsilon'_i\}$ be two distinct sequences and let p be the smallest integer for which $\varepsilon_p \neq \varepsilon'_p$. Then $C_p(\varepsilon_p)$ and $C_p(\varepsilon'_p)$ are distinct and the corresponding crosscuts $X_p(\varepsilon_p)$ and $X_p(\varepsilon'_p)$ subtend two *disjoint* (recall the structure of $\mathscr{W}$) closed arcs Λ_p and Λ'_p on $U \cap \{|z| = 1\}$. But $b(\varepsilon) \in \Lambda_p$ and $b(\varepsilon') \in \Lambda'_p$ and so $b(\varepsilon) \neq b(\varepsilon')$.

THEOREM 2. *Let b be a limit point of F and let U be a neighborhood of b. The set*

$$\{\theta \mid e^{i\theta} \in U, f(e^{i\theta}) \text{ does not exist}\}$$

has the power of the continuum.

Proof. Select three distinct arcs, $C(0)$, $C(1)$, $C(2)$, from among the arcs C_n. Suppose a curve Γ in $\mathscr{W}$ meets, in succession, curves χ over the arcs in the sequence

$$C(\varepsilon_1), C(\varepsilon_2), C(\varepsilon_3), \cdots \qquad (\varepsilon_i = 0, 1, 2; \quad \varepsilon_i \neq \varepsilon_{i+1})$$

and crosses no other χ's. To those curves χ in $\mathscr{W}$ which Γ meets there corresponds a sequence of crosscuts $X_1, X_2, X_3, \cdots$, which subtend arcs $\Lambda_1, \Lambda_2, \Lambda_3, \cdots$ on $\{|z| = 1\}$ satisfying the condition $\Lambda^-_{n+1} \subset \Lambda^\circ_n$. Also we choose $\varepsilon_1 = 0$ and X_1 fixed, in U, so that the image of Γ lies in U. The sequence $\{\varepsilon_n\}$ then determines a unique point $b(\varepsilon) = \bigcap \Lambda^\circ_n \in U$. The radius to $b(\varepsilon)$ intersects all X_n; hence $f(z)$ has no radial limit at $b(\varepsilon)$, for $C(0)$, $C(1)$, $C(2)$ are all distinct and $\varepsilon_i \neq \varepsilon_{i+1}$. Now given the start of the sequence, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$, there are two possible choices for ε_{p+1} and the two possible arcs Λ_{p+1} are disjoint. Thus distinct sequences $\{\varepsilon_n\}$ yield distinct points $b(\varepsilon)$. The set of sequences $\{\varepsilon_n\}$ has the power of the continuum.

THEOREM 3. *Let $e^{i\alpha} \in E'$. Then the set $L(\alpha)$ is countable.*

Proof. Let U be a neighborhood of $e^{i\alpha}$ containing none of the points a_n. Then $\mathscr{W}$ contains a countable number of schlicht components $\mathscr{U}_1, \mathscr{U}_2, \cdots$ over $U \cap \{|w| < 1\}$. Each $\mathscr{U}_n$ maps onto $V_n \subset \{|z| < 1\}$, where V_n is bounded by an arc Λ_n of $\{|z| = 1\}$ and a crosscut of $\{|z| < 1\}$ The function $f(z)$ is holomorphic on Λ_n and there is just one radius, ending on Λ_n, associated with the radial limit $e^{i\alpha}$. Since $\mathscr{W}$ contains only this countable collection of components over U, the result is clear.

We remark that if E is void, then the use of a two-point set $\{a_1, a_2\}$ leads to a Blaschke product satisfying Theorem 3. With a three-point set we can satisfy both Theorem 2 and Theorem 3. Theorem 1 is of course vacuous.

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