A REMARK ON THE NIJENHUIS TENSOR

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The vanishing of the Nijenhuis tensor of the almost complex structure is known to give the integrability of the almost complex structure [3, 7]. In order to generalize this fact, we consider a vector 1-form h on a manifold M[4], whose Jordan canonical form at all points on M is equal to a fixed matrix μ. Following the idea of E. Cartan, we say that such a vector 1-form is 0-deformable [2]. The frames z at x such that $z^{-1}h_x z = \mu$ define a subbundle of the frame bundle over M, as x runs through M, and the subbundle is called a G-structure defined by h [1]. We find that for a certain type of 0-deformable h, the vanishing of the Nijenhuis tensor of h is sufficient for the G-structure to be integrable (Theorem, § 2). In § 5 we give an example of a 0-deformable derogatory nilpotent vector 1-form, whose Nijenhuis tensor vanishes, but whose G-structure is not integrable.

1. Vector forms and distributions. As usual, we begin by stating, that all the objects we encounter in this paper are assumed to be C^∞.

Let M be a manifold, T_x the tangent space at point x of M, T the tangent bundle over M, $T^{(p)}$ the vector bundle of tangential covariant p-vectors of M. A vector p-form is a cross-section of $T \otimes T^{(p)}$. The collection of all vector p-forms over M is denoted by Ψ_p. We notice that a vector 1-form is nothing but a law that assigns a linear transformation to each tangent space T_x at point x of M.

We list some definitions and lemmas of the theory of vector forms [4], which we use in the sequel.

If $P \in \Psi_p$, $Q \in \Psi_q$, then $P \barwedge Q \in \Psi_{p+q-1}$ is defined by

$$(1) \qquad (p \barwedge Q)(u_1, \cdots, u_{p+q-1}) = \frac{1}{(p-1)!\, q!} \sum_\alpha |\alpha| P(Q(u_{\alpha_1}, \ldots, u_{\alpha_p}), u_{\alpha_{p+1}}, \ldots, u_{\alpha_{p+q-1}})$$

where α runs through all the permutations of $(1, 2, \cdots, p+q-1)$, and $|\alpha|$ denotes the signature of the permutation α.

If h is a vector 1-form and P is a vector p-form, we write hP instead of $h \barwedge p$. In particular if $p = h$, we write $h \barwedge h$ as h^2. In general, $\underbrace{h \barwedge h \cdots \barwedge h}_{k \text{ times}}$ is written as h^k, and this agrees with the usual notation,

Received November 13, 1961. This research was supported in part by National Science Foundation grant G 14736. The author wishes to thank Professor H. C. Wang for first suggesting this problem and for the subsequent discussion we had on this paper. The author is also obliged to the referee whose comments helped to correct and clarify some arguments in this paper.

when we consider h as a linear transformation of the tangent space at each point of the manifold M.

Let h and k be two vector 1-forms. The bracket $[h, k]$ of h and k is a vector 2-form defined by

$$(2)\qquad [h, k](u, v) = [hu, kv] + [ku, hv] - k[hu, v] - h[ku, v] \\ -k[u, hv] - h[u, kv] + kh[u, v] + hk[u, v]\,,$$

where u and v are vector fields over M. If $h = k$, we obtain the tensor $[h, h]$, generally known as the Nijenhuis tensor:

$$(3)\qquad \frac{1}{2}[h, h](u, v) = [hu, hv] - h[hu, v] - h[u, hv] + h^2[u, v]\,.$$

If h, k and l are vector 1-forms, using (2), we can obtain

$$(4)\qquad [hl, k] + [h, kl] - [h, k] \barwedge l = h[l, k] + k[l, h]$$

(cf. (6.7) [4]).

LEMMA 1.1. *Let h be a vector 1-form, then*

$$(5)\qquad [h^k, h^l] = \frac{1}{2} \sum_{\substack{a+b+c=k+l-2 \\ 0\leqq b\leqq k-1 \\ 0\leqq c\leqq l-1}} h^a\{([h, h] \barwedge h^b) \barwedge h^c - [h, h] \barwedge h^{b+c}\}\,.$$

Proof. By replacing h, k and l by h, h and h^k in (4), we obtain

$$(6)\qquad [h^k, h] = h[h^{k-1}, h] + \frac{1}{2}[h, h] \barwedge h^{k-1}\,,$$

which gives us

$$(7)\qquad [h^k, h] = \frac{1}{2}\sum_{i=1}^{k} h^{i-1}[h, h] \barwedge h^{k-i}\,.$$

Again, replacing h, k and l in (4) by h^k, h and h^{l-1}, we obtain

$$(8)\qquad [h^{k+l-1}, h] + [h^k, h^l] - [h^k, h] \barwedge h^{l-1} = h^k[h^{l-1}, h] + h[h^{l-1}, h^k]\,.$$

Using (7) and (8) yields

$$(9)\qquad [h^k, h^l] = h[h^k, h^{l-1}] \\ + \frac{1}{2}\sum_{i=1}^{k} h^{i-1}\{([h, h] \barwedge h^{k-i}) \barwedge h^{l-1} - [h, h] \barwedge h^{k-i+l-1}\}\,,$$

and repeating the reduction we obtain (5).

LEMMA 1.2. *Let h be a vector 1-form on M, whose rank is constant* *in a neighbourhood of each point x of M. If $[h, h] = 0$, the distribution $x \to h_x T_x$ is completely integrable.*

Proof. By Frobenius' theorem we have to show that the bracket of any two vector fields of the form hu, hv belongs to the distribution. This follows from $[h, h] = 0$ and (3):

$$[hu, hv] = h[hu, v] + h[u, hv] - h^2[u, v] .$$

We recall that a necessary and sufficient condition for a distribution to be completely integrable can be given as follows:

Let θ be an r-dimensional distribution $x \to \theta(x)$ on an m-dimensional manifold M. For each $x_0 \in M$, let U be a neighbourhood of x_0 and $L_1, \cdots, L_r$ be vector fields on U such that $(L_1)_x, \cdots, (L_r)_x$ span $\theta(x)$ for each $x \in U$. Then θ is completely integrable if and only if for each $x_0 \in M$, there exist $m - r$ independent functions $\psi^1, \cdots, \psi^{m-r}$ defined on a neighbourhood $V \subset U$ of x_0 such that

$$L_i \psi^j = 0, \text{ for } 1 \leqq i \leqq r,\ 1 \leqq j \leqq m - r \text{ on } V .$$

Using this it is easy to prove,

LEMMA 1.3. *If $\theta_1, \cdots, \theta_g$ are completely integrable distributions of dimensions $r_1, \cdots, r_g$ on M, such that*

$$\theta_1(x) + \theta_2(x) + \cdots + \theta_g(x) = T_x \ (\textit{direct sum})$$

for each $x \in M$, then for each point $x_0 \in M$, there exists a coordinate neighbourhood U of x_0 with coordinate functions $x^1, \cdots, x^m$ such that for each j

$$x^1 = \xi^1, \cdots, x^{r_1+\cdots+r_{j-1}} = \xi^{r_1+\cdots+r_{j-1}}, x^{r_1+\cdots+r_j+1} = \xi^{r_1+\cdots+r_j+1}, \cdots, x^m = \xi^m$$

gives an integral manifold of θ_j contained in U.

2. The integrability of a 0-deformable vector 1-form. Let h be a vector 1-form, defined on M, whose characteristic polynomial has constant coefficients on M. Let the decomposition of the characteristic polynomial be

$$\{p_1(\lambda)\}^{d_1}\{p_2(\lambda)\}^{d_2} \cdots \{p_g(\lambda)\}^{d_g}$$

where $p_i(\lambda)$, $i = 1, \cdots, g$ are polynomials in λ, irreducible over the reals, and $(p_i(\lambda), p_j(\lambda)) = 1$, if $i \neq j$. It is easy to verify [5, pp 130–132], that we can get polynomials $e_1(\lambda), e_2(\lambda), \cdots, e_g(\lambda)$ in λ, with constant coefficients, such that $\sum_{i=1}^g e_i(h) = I$, $\{e_i(h)\}^2 = e_i(h)$, $e_i(h) \cdot e_j(h) = 0$ for $i \neq j$, and

$$e_i(h_x)T_x = \{u_x \in T_x \mid \{p_i(h_x)\}^{a_i}u_x = 0\} .$$

Let θ_i denote the distribution $x \to e_i(h_x)T_x$. If we assume $[h, h] = 0$, then by Lemma 1.1, because $e_i(h)$ is a polynomial in h with constant coefficients, we see that $[e_i(h), e_i(h)] = 0$. Hence, by Lemma 1.2, θ_i is completely integrable.

DEFINITION. A vector 1-form h on M is said to be *0-deformable*, if for all $x \in M$, the Jordan canonical form of h_x is equal to a fixed matrix μ [2].

Note that a 0-deformable vector 1-form has a characteristic polynomial with constant coefficients.

A frame at $x \in M$ is an isomorphism z from R^m onto T_x, where m is the dimension of M. For a 0-deformable vector 1-form h, the frames z at x such that $z^{-1}h_x z = \mu$ define a subbundle H of the frame bundle over M, as x runs through M. H is called the G-structure defined by h [1].

DEFINITION. A G-structure H defined by h is said to be *integrable*, if for each point x of M there exists a coordinate neighbourhood U of x with a coordinate system $\{x^1, \cdots, x^m\}$ such that the frame $\{(\partial/\partial x^1)_{x'}, \cdots, (\partial/\partial x^m)_{x'}\}$ belongs to the subbundle H for all $x' \in U$. We shall say that these coordinate functions are associated with the integrable G-structure H.

Clearly, H is integrable if and only if, for each point x of M, we can find a local coordinate system around x, in which the coordinate expression of h is μ.

We are interested in finding a sufficient condition for a G-structure defined by a 0-deformable vector 1-form h to be integrable. We now assume $[h, h] = 0$. By the argument above we know that the distributions θ_i associated to the irreducible factors $p_i(\lambda)$ are all completely integrable, so by Lemma 1.3, for each point x_0 of M there is a coordinate system $\{x^1, \cdots, x^m\}$ on a neighbourhood U of x_0, and the integral manifolds of θ_i contained in U are given by coordinate slices.

In U take a point given by coordinates $(\xi^1, \cdots, \xi^m)$. For each i, let $x^1 = \xi^1, \cdots, x^{r_{i-1}} = \xi^{r_{i-1}}, x^{r_i+1} = \xi^{r_i+1}, \cdots, x^m = \xi^m$ give an integral manifold M_i of θ_i in U, where $r_i = m_1 + m_2 + \cdots + m_i$ and $m_i =$ dimension of θ_i. Consider the restriction h_i of h on M_i. Notice that we can view h_i as a vector 1-form on an open set of M_i, depending on $m - m_i$ parameters $x^1, \cdots, x^{r_{i-1}}, x^{r_i+1}, \cdots, x^m$ in such the way that h_i is C^∞ with respect to the coordinates on M_i and the parameters together. The characteristic polynomial of h_i is $\{p_i(\lambda)\}^{a_i}$ and the minimum polynomial of h_i is $\{p_i(\lambda)\}^{v_i}$, where $\prod_{i=1}^{g}\{p_i(\lambda)\}^{v_i}$ is the minimum polynomial of h; h_i is a 0-deformable vector 1-form on M_i, and $[h_i, h_i] = 0$. If for each i, the G_i-structure defined by h_i on M_i is integrable, and if coordinate functions $y^{r_{i-1}+1}, \cdots, y^{r_i}$ associated to the integrable G_i-structure around the point $(x^{r_{i-1}+1}, \cdots, x^{r_i}) = (\xi^{r_{i-1}+1}, \cdots, \xi^{r_i})$ are dependent on coordinates $x^{r_{i-1}+1}, \cdots, x^{r_i}$ and on parameters $x^1, \cdots, x^{r_{i-1}}, x^{r_i+1}, \cdots, x^m$ jointly in a C^∞-manner, then we can replace $\{x^1, \cdots, x^m\}$ in a neighbourhood of the point $(x^1, \cdots, x^m) = (\xi^1, \cdots, \xi^m)$ by a new coordinate system $\{y^1, \cdots, y^m\}$, so that h takes the matrix form μ, i.e. H is integrable.

Hence we consider the case where h has characteristic polynomial $\{p(\lambda)\}^d$ and minimum polynomial $\{p(\lambda)\}^v$, where $p(\lambda)$ is irreducible over the reals, and suppose that h jointly depends on the coordinates of M and some parameters in a C^∞-manner. We have the following results:

Case I. $\deg p(\lambda) = 1$.

(i) If $v = 1$, then h is a constant multiple of the identity vector 1-form I on M, hence the G-structure is integrable.

(ii) If $v = d = m$, consider the nilpotent part n of h. n is a polynomial in h with constant coefficients on M, so from $[h, h] = 0$, we get $[n, n] = 0$, by Lemma 1.1. Moreover $n^m = 0$ but $n^l \neq 0$ for $l < m$, for all points of M. In § 3 we prove a proposition which shows that the G-structure defined by n (which is the same as that defined by h) is integrable, and that the associated coordinate functions depend on the parameters of h and on the point in M jointly in a C^∞-manner.

Case II. $\deg p(\lambda) = 2$. In § 4 we shall show that the semi-simple part s of h gives rise to a complex manifold structure $\tilde{M}$ in this case, and that for the $\tilde{G}$-structure given by h which is induced from h on $\tilde{M}$, (i) and (ii) of Case I has a straightforward parallel on $\tilde{M}$; hence coming back to the real manifold, we have: if $v = 1$, or $v = d = m/2$, then the G-structure defined by h is integrable, and the associated coordinate functions are C^∞ with respect to the coordinates on M and the parameters jointly.

By the preceding arguments and the results in § 3 and 4, we can conclude the following:

THEOREM. *Let h be a* 0*-deformable vector* 1*-form on a manifold M, with characteristic polynomial*

$$\prod_{i=1}^{g} p_i(\lambda)^{d_i}$$

where $p_i(\lambda)$ are polynomials in λ, irreducible over the reals, and $(p_i(\lambda), p_j(\lambda)) = 1$ for $i \neq j$, and the minimum polynomial

$$\prod_{i=1}^{g} p_i(\lambda)^{v_i} .$$

Suppose for each i, $v_i = 1$ or d_i. Then the G-structure defined by h is integrable if $[h, h] = 0$.

REMARK. If $v_i = 1$ for all i, we say that h is semi-simple. If $v_i = d_i$ for all i, we say that h is nonderogatory, and otherwise derogatory [6, p. 21].

3. The integrability of a nonderogatory nilpotent vector 1-form.

PROPOSITION. *Let h be a nilpotent vector 1-form on an m-dimensional manifold M, and suppose $h^m = 0$ but $h^l \neq 0$ for $l < m$, for all points on M. Then $[h, h] = 0$ implies that the G-structure defined by h is integrable. Moreover, if h depends on some parameters and is C^∞ with respect to the local coordinates $x^1, \cdots, x^m$ on M and the parameters jointly, then the local coordinates $y^1, \cdots, y^m$ associated to the integrable G-structure are C^∞ with respect to $x^1, \cdots, x^m$ and the parameters jointly.*

Proof. (1) Let $m = 2$. Denoting the tangent space at $x \in M$ by T_x, we have a one dimensional distribution given by $x \to h_x T_x$. For each point x_0 of M we can find a neighbourhood U of x_0 and a coordinate system $\{x^1, x^2\}$ on U, such that $x^2 = \xi^2$ is an integral manifold of this distribution in U. Let h take the matrix form in this coordinate system

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

β_{ij} being functions of x^1, x^2. As $\partial/\partial x^1$ at $x \in U$ spans $h_x T_x$, we have $\beta_{21} = \beta_{22} = 0$, and as h restricted to integral manifold $x^2 = \xi^2$ is given by β_{11}, and as $h^2 = 0$, we have $\beta_{11} = 0$. We claim, that we can choose a new coordinate system $\{y^1, y^2\}$ such that in this new coordinate system h takes the matrix form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In fact, let the vector fields $\partial/\partial x^1$ and $\partial/\partial x^2$ be denoted by X_1 and X_2, and choose new vector fields Y_1 and Y_2 by

$$\begin{cases} Y_1 = \alpha_1 X_1 \\ Y_2 = \alpha_0 X_1 + X_2 \end{cases}$$

where α_1 and α_0 are to be determined so that $hY_2 = Y_1$ and $[Y_1, Y_2] = 0$. Let then π^1, π^2 be the 1-forms dual to Y_1, Y_2; we have $d\pi^1 = 0, d\pi^2 = 0$, so that y^1, y^2 can be determined from $dy^1 = \pi^1, dy^2 = \pi^2$. To prove that Y_1 and Y_2 can be found we observe that the condition $hY_2 = Y_1$ leads to

$$\alpha_1 = \beta_{12}$$

and that the condition $[Y_1, Y_2] = 0$ leads to

$$(\alpha_0 X_1 + X_2)\alpha_1 - \alpha_1 X_1 \alpha_0 = 0$$

which is a first order linear differential equation for α_0:

$$\alpha_1 \frac{\partial}{\partial x^1}\alpha_0 - \alpha_0\left(\frac{\partial}{\partial x^1}\alpha_1\right) - \frac{\partial}{\partial x^2}\alpha_1 = 0 .$$

α_1 is clearly C^∞ with respect to x^1, x^2 and the parameters. α_0 is obtained as a solution of the above differential equation, so α_0 depends on x^2 and the parameters in a C^∞ manner. By differentiating this differential equation repeatedly, we see that α_0 is C^∞ with respect to x^1, x^2 and the parameters. Hence π^1 and π^2 are C^∞ with respect to x^1, x^2 and the parameters, and finally y^1 and y^2 are C^∞ with respect to x^1, x^2 and the parameters.

(2) We assume that our proposition is true for $(m-1)$-dimensional manifolds and proceed to prove it for an m-dimensional manifold $(m \geqq 3)$.

Because $[h, h] = 0$, we know that the distribution $x \to h_x T_x$, given by the image of h at each point x of M is integrable; hence, locally, there exists a coordinate system $\{x^1, \cdots, x^m\}$ such that

(i) $x^m = \xi^m$ gives the integral manifolds of this distribution, and

(ii) in this coordinate system h takes the matrix form

$$\begin{pmatrix} & & \beta_{1\,m} \\ & H & \vdots \\ & & \beta_{m-1\,m} \\ 0 & \cdots\, 0 & 0 \end{pmatrix} \tag{1}$$

We further claim that $x^1, \cdots, x^{m-1}, x^m$ can be chosen so that

(iii) H takes the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \tag{2}$$

In fact, if H is not in the form (2) already, we view the restriction h_1 of h to an integral manifold $x^m = \xi^m$ as a vector 1-form on an open set V of R^{m-1}, depending on parameter x^m, and consider H to be the matrix form of h_1 with respect to the coordinate system $\{x^1, \cdots, x^{m-1}\}$. From the inductive assumption, there are coordinate functions $z^1, \cdots, z^{m-1}$ on an open set $V_1 \subset V$ depending on $x^1, \cdots, x^{m-1}$ and x^m in a C^∞-manner, such that h_1 has matrix form (2) with respect to the coordinate system

$\{z^1, \cdots, z^{m-1}\}$. Now, if we take $\{z^1, \cdots, z^{m-1}, x^m\}$ as the local coordinate system on M, then (iii) will be satisfied.

So let us suppose that we are in a coordinate system where (i) (ii) and (iii) are satisfied. For simplicity we write $\beta_1, \beta_2, \cdots, \beta_{m-1}$ instead of $\beta_{1m}, \beta_{2m}, \cdots, \beta_{m-1m}$. Note that $\beta_{m-1} \neq 0$. We want to prove that we can find a new coordinate system $\{y^1, \cdots, y^m\}$ such that in this coordinate system h takes the matrix form (1), H being of the form (2) and $\beta_1 = \beta_2 = \cdots = \beta_{m-2} = 0$, $\beta_{m-1} = 1$. In order to do this, as in the case $m = 2$, we find vector fields $Y_1, \cdots, Y_m$ satisfying $hY_i = Y_{i-1}$ $(i = 2, \cdots, m)$, $hY_1 = 0$ and $[Y_i, Y_j] = 0$ for all i, j; let the dual of $Y_1, \cdots, Y_m$ be $\pi^1, \cdots, \pi^m$ and obtain $y^1, \cdots, y^m$ from $dy^1 = \pi^1, \cdots, dy^m = \pi^m$. If we denote by $X_1, \cdots, X_m$ the vector fields $\partial/\partial x^1, \cdots, \partial/\partial x^m$ and set

$$(3) \qquad \begin{cases} Y_1 = \alpha_{m-1}X_1 \\ Y_2 = \alpha_{m-2}X_1 + \alpha_{m-1}X_2 \\ \quad \cdots\cdots\cdots\cdots\cdots \\ Y_{m-1} = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_{m-1}X_{m-1} \\ Y_m = \alpha_0 X_1 + (\alpha_1 - \beta_1)X_2 + \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m \end{cases}$$

where $\alpha_{m-1} = \beta_{m-1}$, then the problem reduces to finding the α's so that $[Y_i, Y_j] = 0$ are satisfied for all i, j.

First we shall obtain all the relations on the derivatives of $\beta_1, \cdots, \beta_{m-1}$ imposed by the condition $[h, h] = 0$. We see that

$$[h, h](X_i, X_j) = 0$$

gives us no relations for $i, j \leqq m - 1$, but

$$\frac{1}{2}[h, h](X_i, X_m) = [X_{i-1}, \beta_1 X_1 + \cdots + \beta_{m-1}X_{m-1}] - h[X_i, \beta_1 X_1 + \cdots + \beta_{m-1}X_{m-1}]$$

from which we obtain

$$(4) \qquad X_{i-1}\beta_{j-1} = X_i \beta_j \qquad i, j \leqq m - 1$$

and

$$(5) \qquad X_i \beta_{m-1} = 0 \qquad i \leqq m - 2 .$$

To make this relation clear, we write this result in Table 1.

$$
\begin{array}{llllll}
0 = X_1\beta_{m-1} & & & & & \\
0 = X_1\beta_{m-2} = X_2\beta_{m-1} & & & & & \\
\quad \cdots\cdots\cdots & & & & & \\
0 = X_1\beta_3 & = X_2\beta_4 = & \cdots\cdots\cdots & = X_{m-3}\beta_{m-1} & & \\
0 = X_1\beta_2 & = X_2\beta_3 = & \cdots\cdots\cdots & = X_{m-3}\beta_{m-2} & = X_{m-2}\beta_{m-1} & \\
\quad X_1\beta_1 & = X_2\beta_2 = & \cdots\cdots\cdots & = X_{m-3}\beta_{m-3} & = X_{m-2}\beta_{m-2} & = X_{m-1}\beta_{m-1} \\
 & \quad X_2\beta_1 = & \cdots\cdots\cdots & = X_{m-3}\beta_{m-4} & = X_{m-2}\beta_{m-3} & = X_{m-1}\beta_{m-2} \\
 & & & \cdots\cdots\cdots & & \\
 & & & X_{m-3}\beta_1 & = X_{m-2}\beta_2 & = X_{m-1}\beta_3 \\
 & & & & X_{m-2}\beta_1 & = X_{m-1}\beta_2
\end{array}
$$

TABLE 1

Now let us examine $[Y_i, Y_j] = 0$ for $i < j \leqq m - 1$. We see that this is equivalent to the set of equations (6),

$$
(6) \quad \begin{cases}
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-1} = 0 \\
\qquad \cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j+i} = 0 \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j+i-1} \\
\qquad\qquad - (\alpha_{m-j}X_1 + \alpha_{m-j+1}X_2 + \cdots + \alpha_{m-1}X_j)\alpha_{m-1} = 0 \\
\qquad\qquad \cdots\cdots\cdots \\
(\alpha_{m-i}X_1 + \alpha_{m-i+1}X_2 + \cdots + \alpha_{m-1}X_i)\alpha_{m-j} \\
\qquad\qquad - (\alpha_{m-j}X_1 + \alpha_{m-j+1}X_2 + \cdots + \alpha_{m-1}X_j)\alpha_{m-i} = 0
\end{cases}
$$

where $i < j \leqq m - 1$. Using $X_1\alpha_{m-1} = X_1\beta_{m-1} = 0$ from Table 1, we see that (6) is equivalent to the following Table 2.

$$
\left.\begin{array}{llllll}
0 = X_1\alpha_{m-1} & & & & & \\
0 = X_1\alpha_{m-2} = X_2\alpha_{m-1} & & & & & \\
\qquad \cdots\cdots\cdots & & & & & \\
0 = X_1\alpha_3 & = X_2\alpha_4 = & \cdots = X_{m-3}\alpha_{m-1} & & & \\
0 = X_1\alpha_2 & = X_2\alpha_3 = & \cdots = X_{m-3}\alpha_{m-2} & = X_{m-2}\alpha_{m-1} & & \\
\quad X_1\alpha_1 & = X_2\alpha_2 = & \cdots = X_{m-4}\alpha_{m-3} & = X_{m-2}\alpha_{m-2} & = X_{m-1}\alpha_{m-1} &
\end{array}\right\} \text{(a)}
$$

$$
\left.\begin{array}{llll}
X_2\alpha_1 = \cdots = X_{m-3}\alpha_{m-4} & = X_{m-2}\alpha_{m-3} & = X_{m-1}\alpha_{m-2} \\
\qquad \cdots\cdots\cdots & & \\
X_{m-3}\alpha_1 & = X_{m-2}\alpha_2 & = X_{m-1}\alpha_3 \\
 & X_{m-2}\alpha_1 & = X_{m-1}\alpha_2
\end{array}\right\} \text{(b)}
$$

TABLE 2

Next consider $[Y_i, Y_m] = 0$, $i \leqq m - 1$. This is equivalent to the following (7a, b, c),

$$
\text{(7a)}\begin{cases}(\alpha_{m-i}X_1+\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)(\alpha_{m-2}-\beta_{m-2})=0\\ \cdots\cdots\cdots\cdots\\ (\alpha_{m-i}X_1+\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)(\alpha_i-\beta_i)=0\end{cases}
$$

$$
\text{(7b)}\begin{cases}(\alpha_{m-i}X_1+\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)(\alpha_{i-1}-\beta_{i-1})\\ \quad -\{\alpha_0X_1+(\alpha_1-\beta_1)X_2+\cdots+(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-1}=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (\alpha_{m-i}X_1+\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)(\alpha_1-\beta_1)\\ \quad -\{\alpha_0X_1+(\alpha_1-\beta_1)X_2+\cdots+(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-i+1}=0\end{cases}
$$

$$
\text{(7c)}\quad \begin{aligned}&(\alpha_{m-i}X_1+\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)\alpha_0\\ &\quad -\{\alpha_0X_1+(\alpha_1-\beta_1)X_2+\cdots+(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-i}=0\end{aligned}
$$

where $i \leqq m-1$.

Because of Table 1, we see that (7a) is equivalent to part (a) of Table 2. Using part (a) of Table 2, we see that (7b) reduces to a simpler system (7b′),

$$
\text{(7b}'\text{)}\begin{cases}(\alpha_{m-1}X_i)(\alpha_{i-1}-\beta_{i-1})-\{(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-1}=0\\ (\alpha_{m-2}X_{i-1}+\alpha_{m-1}X_i)(\alpha_{i-2}-\beta_{i-2})-\{(\alpha_{m-3}-\beta_{m-3})X_{m-2}\\ \qquad\qquad +(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-2}=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (\alpha_{m-i+1}X_2+\cdots+\alpha_{m-1}X_i)(\alpha_1-\beta_1)\\ \quad -\{(\alpha_{m-i}-\beta_{m-i})X_{m-i+1}+\cdots+(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-i+1}=0\end{cases}
$$

Using Table 1 again, we can show that (7b′) is equivalent to part (b) of Table 2 plus the following equations which are obtained from (7b′) by letting $i=m-1$:

$$
\begin{cases}(\alpha_{m-1}X_{m-1})(\alpha_{m-2}-\beta_{m-2})-\{(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_{m-1}=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (\alpha_2X_2+\cdots+\alpha_{m-1}X_{m-1})(\alpha_1-\beta_1)\\ \qquad -\{(\alpha_1-\beta_1)X_2+\cdots+(\alpha_{m-2}-\beta_{m-2})X_{m-1}+X_m\}\alpha_2=0\end{cases}
$$

Using Table 1 and part (b) of Table 2, these equations can be written as (8),

$$
\text{(8)}\qquad (\alpha_{m-1})^2X_{m-1}\frac{\alpha_{m-k}-\beta_{m-k}}{\alpha_{m-1}}+(\alpha_{m-2})^2X_{m-1}\frac{\alpha_{m-k+1}-\beta_{m-k+1}}{\alpha_{m-2}}
$$

$$
+\cdots+(\alpha_{m-k+1})^2X_{m-1}\frac{\alpha_{m-2}-\beta_{m-2}}{\alpha_{m-k+1}}-X_m\alpha_{m-k+1}=0\,,^{1)}\quad k=2,\cdots,m-1\,.
$$

[1] For simplicity we write $(\alpha_{m-1-j})^2X_{m-1}(\alpha_{m-k+j}-\beta_{m-k+j}/\alpha_{m-1-j})$, $1\leqq j\leqq k-2$, for $\alpha_{m-1-j}X_{m-1}(\alpha_{m-k+j}-\beta_{m-k+j})-(X_{m-1}\alpha_{m-1-j})X_{m-1}(\alpha_{m-k+j}-\beta_{m-k+j})$, although at some point α_{m-1-j} might vanish.

We can now obtain $\alpha_{m-2}, \alpha_{m-3}, \cdots, \alpha_1$ succesively by integrating (8) with respect to x^{m-1}; in fact, start from $k = 2$, and integrate to get α_{m-2}, then use this α_{m-2} in (8) for $k = 3$ and integrate to get α_{m-3}, in general

$$\alpha_{m-k} - \beta_{m-k} = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (\alpha_{m-2})^2 X_{m-1} \frac{\alpha_{m-k+1} - \beta_{m-k+1}}{\alpha_{m-2}} + \cdots + (\alpha_{m-k+1})^2 X_{m-1} \frac{\alpha_{m-2} - \beta_{m-2}}{\alpha_{m-k+1}} - X_m \alpha_{m-k+1} \Big\} dx^{m-1} . \tag{9}$$

We still have to show that $\alpha_{m-2}, \alpha_{m-3}, \cdots, \alpha_1$ thus obtained satisfy Table 2. For simplicity let us write (8) in the form

$$(\alpha_{m-1})^2 X_{m-1} \frac{\alpha_{m-k} - \beta_{m-k}}{\alpha_{m-1}} + A_{m-k+1} = 0 . \tag{8_k}$$

Then (9) becomes

$$\alpha_{m-k} - \beta_{m-k} = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} A_{m-k+1} dx^{m-1} . \tag{9_k}$$

To show that the α's do satisfy Table 2, it suffices to show (10_k),

$$X_{m-q}(\alpha_{m-k} - \beta_{m-k}) = X_{m-q+1}(\alpha_{m-k+1} - \beta_{m-k+1}) \tag{10_k}$$

for $k, q = 2, \cdots, m-1$. We shall prove (10_k) inductively. For $k = 2$ it is easy to check. Suppose $(10_2), \cdots, (10_{k-1})$ are true; using this assumption, we differentiate (9_k) and get (11),

$$X_{m-q}(\alpha_{m-k} - \beta_{m-}) = a_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (X_{m-q m-2}\alpha)^2 X_{m-1} \frac{\alpha_{m-k+1} - \beta_{m-k+1}}{X_{m-q}\alpha_{m-2}} + X_{m-q+1} A_{m-k+2} + (\alpha_{m-k+1})^2 X_{m-1} \frac{X_{m-q}(\alpha_{m-2} - \beta_{m-2})}{\alpha_{m-k+1}} \Big\} dx^{m-1} . \tag{11}$$

If $q > 2$, then $X_{m-q}\alpha_{m-2} = 0$, so (11) gives us (10_k). If $q = 2$, we observe first that differentiating (8_{k+1}) with respect to x^{m-1} gives us (12),

$$(X^2_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1}))\alpha_{m-1} - (\alpha_{m-k+1} - \beta_{m-k+1}) X^2_{m-1}\alpha_{m-1} + X_{m-1} A_{m-k+2} = 0. \tag{12}$$

Using (12) and $X_{m-2}(\alpha_{m-2} - \beta_{m-2}) = 0$ in (11) for $q = 2$, we obtain

$$X_{m-2}(\alpha_{m-k} - \beta_{m-k}) = \alpha_{m-1} \int \frac{-1}{(\alpha_{m-1})^2} \Big\{ (X_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1})) X_{m-1}\alpha_{m-1} - (X^2_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1}))\alpha_{m-1} \Big\} dx^{m-1} = X_{m-1}(\alpha_{m-k+1} - \beta_{m-k+1})$$

which completes the proof (10_k).

Finally to obtain α_0, we examine (7c), and find that the same type of argument employed to obtain (8) enables us to show that (7c) is equivalent to

$$(13)\quad \begin{cases} X_1\alpha_0 = X_{m-1}(\alpha_{m-2} - \beta_{m-2}) \\ \quad \cdots\cdots\cdots \\ X_{m-2}\alpha_0 = X_{m-1}(\alpha_1 - \beta_1) \\ (\alpha_1 X_1 + \cdots + \alpha_{m-1}X_{m-1})\alpha_0 - \{\alpha_0 X_1 + (\alpha_1 - \beta_1)X_2 + \\ \qquad\qquad \cdots + (\alpha_{m-2} - \beta_{m-2})X_{m-1} + X_m\}\alpha_{m-1} = 0 . \end{cases}$$

Using the first $m-2$ equations of (13) in the last one, gives us (8_k) for $k = m$, where we agree that $\beta_0 = 0$. Hence we obtain α_0 from (9_m). To check that the first $m-2$ equations in (13) are satisfied by this α_0, we check (10_k) for $k = m$. The same argument in (11) holds for $k = m$, and it is even simpler than before, because in this case the first term in the integrand vanishes.

If h depends on $x^1, \cdots, x^m$ and some parameters jointly in a C^∞-manner, then it is clear that $\alpha_{m-2}, \cdots, \alpha_1, \alpha_0$ obtained above depend on $x^1, \cdots, x^m$ and the parameters in a C^∞-manner, hence we can claim the same for $y^1, \cdots, y^m$.

4. The complex case. For Case II in § 2, where $\deg p(\lambda) = 2$, we have $\dim M = m = 2n$. Let the roots of $p(\lambda) = 0$ be $\sigma \pm i\tau$ $(\tau \neq 0)$. Because the semi-simple part s of h is a polynomial in h with constant coefficients, from $[h, h] = 0$, via Lemma 1.1, we get $[s, s] = 0$. The vector 1-form J_s defined by

$$J_s = \frac{1}{\tau}(s - \sigma I)$$

satisfies $\lambda^2 + 1 = 0$, because s satisfies $p(\lambda) = 0$. So we have an almost complex structure J_s on M, and as $[J_s, J_s] = 0$ (because $[s, s] = 0$), this almost complex structure is integrable [7]. Hence we can introduce a new real local coordinate system $\{x^1, \cdots, x^m\}$ such that $z^k = x^{2k-1} + ix^{2k}$ $(k = 1, \cdots, n)$ gives a local complex coordinate system, with which M becomes the underlying C^∞-manifold of complex manifold $\tilde{M}$. As h is C^∞ with respect to the coordinates on M and the parameters jointly, so is the almost complex structure J_s. Hence the new coordinate functions $x^1, \cdots, x^m$ are also C^∞ with respect to the coordinates on M and the parameters jointly [7].[2] h is now C^∞ with respect to $x^1, \cdots, x^m$ and the parameters jointly. The vector 1-forms on M induce vector 1-forms on $\tilde{M}$ in a natural way. The vector 1-form $\tilde{s}$ on $\tilde{M}$ induced by s is equal to $\rho\tilde{I}$, where $\rho = \sigma + i\tau$ and $\tilde{I}$ is the identity vector 1-from on $\tilde{M}$. We shall show that polynomials in h with constant coefficients induce holomorphic vector 1-forms on M. In particular, the nilpotent part n of h induces the nilpotent holomorphic vector 1-form $\tilde{n}$ on M.

[2] The author wishes to thank Professor L. Nirenberg for communicating the proof of this fact to him. The dependence on parameters is stated without proof in [7].

Let T_C and $T_C^{(p)}$ be the vector bundles over M, which are obtained by complexifying the tangent space T_x and the space of tangential covariant p-vectors $T_x^{(p)}$ respectively at each point x of M. Then any p-form P on M, i.e. any cross-section of $T\otimes T^{(p)}$, extends in a natural way to a cross-section P_C of $T_C\otimes T_C^{(p)}$. If k and l are two vector 1-forms on M, then k_C, l_C and $[k, l]_C$ are defined. If we define the bracket of two cross-sections of T_C in a natural way, and if we define $[k_C, l_C]$ by (2) of § 1, where we replace h, k by k_C, l_C acd u, v by cross-sections of T_C, then we have $[k, l]_C = [k_C, l_C]$.

Denote $\partial/\partial\bar{z}^i$, $\partial/\partial z^i$ by $Z_i, \bar{Z}_i$ for $i = 1, \cdots, n$. $(Z_1)_x, \cdots, (Z_n)_x, (\bar{Z}_1)_x, \cdots, (\bar{Z}_n)_x$ span the complexification of T_x. $(Z_1)_x, \cdots, (Z_n)_x$ span the eigenspaces of eigenvalue ρ. This eigenspace can be identified with the tangent space of $\tilde{M}$ at x. $(\bar{Z}_1)_x, \cdots, (\bar{Z}_n)_x$ span the eigenspace of $(s_C)_x$ of eigenvalue $\bar{\rho}$. If k is a polynomial in h with constant coefficients, by Lemma 1.1 we have $[s, k] = 0$, and hence $[s_C, k_C] = [s, k]_C = 0$. On the other hand we have

$$[s_C, k_C](Z_i, \bar{Z}_j) = (\rho - s)[Z_i, k_C\bar{Z}_j] + (\bar{\rho} - s)[k_C Z_i, \bar{Z}_j] \;.$$

s_C and k_C are polynomials in h_C with constant coefficients, so s_C and k_C commute; hence k_C leave the eigenspaces of s_C invariant, so using the coordinate expression for k_C, the equation above can be written as

$$[s_C, k_C](Z_i, \bar{Z}_j) = (\rho - \bar{\rho})\sum_{k=1}^{n}\{(Z_i(k_C)_{\bar{k}\bar{j}})\bar{Z}_k + (\bar{Z}_j(k_C)_{ki})Z_k\}$$

from which we get

$$(\partial/\partial\bar{z}^j)(k_C)_{ki} = 0 \;. \qquad (1)$$

$(k_C)_{ki}$ is the matrix form of $\tilde{k}$ on $\tilde{M}$ (induced by k) with respect to the coordinate system $\{z^1, \cdots, z^n\}$, and (1) expresses the fact that $\tilde{k}$ is holomorphic.[3]

(i) If $v = 1$ in Case II of § 2, then $\tilde{h}$ induced by h on $\tilde{M}$, is equal to $\tilde{s} = \rho\tilde{I}$. So in the real coordinate system $\{x^1, \cdots, x^m\}$ h takes the matrix form

$$\begin{pmatrix} A & & & \\ & A & 0 & \\ & & \cdot & \\ & 0 & & \cdot \\ & & & & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix} ,$$

[3] The author is indebted to Professor H. C. Wang for this proof.

so that G-structure is integrable.

(ii) If $v = d = n$ in Case II of § 2, then $\tilde{n}$ satisfies $\tilde{n}^n = 0$ but $\tilde{n}^l \neq 0$ for $l < n$ for all points on $\tilde{M}$. As $\tilde{n}$ is holomorphic, it is meaningful to define the Nijenhuis tensor $[\tilde{n}, \tilde{n}]$ of $\tilde{n}$, using (3) of § 1 as the defining formula, where u, v should be holomorphic vector fields on $\tilde{M}$. As $[n_\sigma, n_\sigma] = [n, n]_\sigma = 0$, we have $[\tilde{n}, \tilde{n}] = 0$.

Now following the method in § 3, it is easy to see that we have a complex version of the Proposition in § 3, i.e.

"Let $\tilde{k}$ be a holomorphic nilpotent vector 1-form on an n-dimensional complex manifold, and suppose $\tilde{k}^n = 0$ but $\tilde{k}^l = 0$ for $l < n$, for all points. Then $[\tilde{k}, \tilde{k}] = 0$ implies that the $\tilde{G}$-structure defined by $\tilde{k}$ is integrable. Moreover, if $\tilde{k}$ depends on some complex [real] parameters and is holomorphic $[C^\infty]$ with respect to the local coordinates $z^1, \cdots, z^n$ [the real coordinates $x^1, \cdots, x^m$, where $z^k = x^{2k-1} + ix^{2k}$] and the parameters jointly, then the local coordinates $w^1, \cdots, w^n$ associated to the integrable $\tilde{G}$-structure [the real coordinates $y^1, \cdots, y^m$ obtained from $w^k = y^{2k-1} + iy^{2k}$] are holomorphic $[C^\infty]$ with respect to $z^1, \cdots, z^n[x^1, \cdots, x^m]$ and the parameters jointly."

Using this complex version, for each point of $\tilde{M}$, we have a neighbourhood with a local complex coordinate system $w^1, \cdots, w^n$, with respect to which $\tilde{h} = \tilde{s} + \tilde{n}$ takes the matrix form

$$\begin{pmatrix} \rho & 1 & & & \\ & \rho & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \rho \end{pmatrix}$$

Passing back to the real coordinate system $\{y^1, \cdots, y^m\}(w^k = y^{2k-1} + iy^{2k})$, h takes the matrix form

$$\begin{pmatrix} A & B & & & \\ & A & B & & 0 \\ & & \ddots & \ddots & \\ & 0 & & A & B \\ & & & & A \end{pmatrix}$$

where

$$A = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The G-structure defined by h is thus integrable. The associated local coordinates $y^1, \cdots, y^m$ are C^∞-functions of the coordinates of M and the parameters jointly.

5. An example.[4]

Let M be the euclidean space of dimension 4, and

[4] The author is indebted to Professor H. C. Wang for this example.

suppose x, y, z, t are the coordinates. Let

$$X_1 = \partial/\partial x,\ X_2 = \partial/\partial y,\ X_3 = \partial/\partial z,\ X_4 = (\partial/\partial t) + (1 + z)(\partial/\partial x)\ ,$$

and define h by $hX_1 = X_2$, $hX_i = 0$ for $i = 2, 3, 4$. It is easy to check that

(i) $h^2 = 0$,

(ii) $[h, h] = 0$,

and (iii) $[X_3, X_4] = X_1$.

Now, if the G-structure defined by h would be integrable, so would the distributions intrinsically given by h. However, (iii) shows that the distribution given by the kernel of h at each point of M is not integrable, hence we conclude that the G-structure is not integrable.

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