A CHARACTERIZATION OF $C(X)$

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It is a classical fact that there exist harmonic functions u in the unit disk with conjugate harmonic function v such that u has continuous boundary values on the unit circumference, while v does not. Let us restate this fact as follows:

Denote by A_0 the algebra of functions analytic in $|z| < 1$ with continuous boundary values on $|z| = 1$ and write $\mathbf{Re}\, A_0$ for the space of all real parts of functions in A_0. Then we may say: there exists a harmonic function u in $|z| < 1$ with continuous boundary values such that u does not lie in $\mathbf{Re}\, A_0$. On the other hand, u is certainly a uniform limit of functions in $\mathbf{Re}\, A_0$ on $|z| = 1$, for all finite real trigonometric polynomials on $|z| = 1$ are in $\mathbf{Re}\, A_0$. Thus we see: $\mathbf{Re}\, A_0$ is not closed under uniform convergence on $|z| = 1$. In this paper, we shall show that this phenomenon is a special case of a very general property of algebras of functions.

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X. Let A be a complex linear subalgebra of $C(X)$ such that

(1) A is closed under uniform convergence;

(2) A contains the constant functions;

(3) A separates the points of X.

We write $\mathbf{Re}\, A$ for the set of functions $\mathbf{Re} f$ with f in A, that is, for the set of real parts of the functions in A. Clearly $\mathbf{Re}\, A$ is a (real) vector space of real-valued continuous functions on X. The purpose of this paper is to prove the following.

THEOREM. *If* $\mathbf{Re}\, A$ *is closed under uniform convergence, then* $A = C(X)$.

COROLLARY 1. *If* $\mathbf{Re}\, A$ *contains every real-valued continuous function on* X, *then* $A = C(X)$.

COROLLARY 2. (*Stone-Weierstrass*) *If* A *is closed under complex conjugation, then* $A = C(X)$.

Corollary 1 is an evident consequence of the theorem, and Corollary 2 follows upon observing that, if A is closed under complex conjuga-

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tion, $\boldsymbol{Re}\, A$ is simply the collection of real-valued functions which are contained in A. The proof of the theorem proceeds by reducing it to the case when A is *anti-symmetric*, i.e., every real-valued function in A is constant. Let us first settle this case.

LEMMA. *If* $\boldsymbol{Re}\, A$ *is closed and* A *is anti-symmetric, then the space* X *contains not more than one point.*

Proof. Suppose that X contains at least two points. Fix a point x_0 in X, and let $(\boldsymbol{Re}\, A)_0$ be the class of all u in $\boldsymbol{Re}\, A$ with $u(x_0) = 0$.

Suppose u is in $(\boldsymbol{Re}\, A)_0$. Let f be a function in A such that $u = \boldsymbol{Re} f$. Since the constants are in A, we may assume that $v = \boldsymbol{Im} f$ vanishes at x_0. Since $v = \boldsymbol{Re}(-if)$, we then have $v \in (\boldsymbol{Re}\, A)_0$. Now given u, the function v in $(\boldsymbol{Re}\, A)_0$ such that $(u + iv)$ is in A is uniquely determined. For, if v' is another such function, $(v - v')$ is a real-valued function in A. Since A is anti-symmetric $v - v'$ is constant, and the condition $v(x_0) = v'(x_0) = 0$ tells us that $v = v'$. Put $v = Tu$.

Then T is a linear transformation of $(\boldsymbol{Re}\, A)_0$ into itself. Since we are assuming that $\boldsymbol{Re}\, A$ is closed under uniform convergence, $(\boldsymbol{Re}\, A)_0$ is a Banach space with the norm

$$\| u \| = \sup_X | u | \,.$$

We claim that T is a bounded operator on this Banach space. To prove this, it will suffice to show that the graph of T is closed. Suppose we have a sequence of elements u_n in $(\boldsymbol{Re}\, A)_0$ such that $u_n \to u$ and $Tu_n \to v$ uniformly. Then the functions $(u_n + iTu_n)$ lie in A and converge uniformly to $(u + iv)$. Thus $(u + iv)$ is in A, and since it is apparent that $v(x_0) = 0$, we have $v = Tu$. We conclude that T is bounded.

Since X contains at least two points, we may choose a nonconstant function $g = s + it$ in A such that $g(x_0) = 0$. Let R denote the rectangle in the complex plane defined by

$$-\| s \| \leqq x \leqq \| s \| \,, \qquad -\| t \| \leqq y \leqq \| t \| \,.$$

Then $g(X)$ is a compact subset of R. Since g is nonconstant, we cannot have $s = 0$ or $t = 0$. In particular, there is a point $x_1 \neq x_0$ in X such that $| t(x_1) | = \| t \|$. Let $z_0 = g(x_1)$, so that z_0 is a boundary point of R.

Fix any integer $N > 0$. There exists a conformal map ϕ of the interior of R onto the interior of the rectangle R_N:

$$-\| s \| \leqq x \leqq \| s \| \,, \qquad -N \leqq y \leqq N$$

such that $\phi(0) = 0$ and $\theta(z_0) = iN$. Since R and R_N are rectangles, the conformal map ϕ extends to a homeomorphism of the boundaries of R and R_N. In particular, ϕ is a uniform limit of polynomials on R. Therefore, the function $h = \phi(g)$ is in the algebra A, and $h(x_0) = \phi(0) = 0$. If $h = u + iv$ we have

$$\| u \| \leqq \| s \|$$
$$\| v \| = N .$$

Since N was arbitrary and $v = Tu$, we have contradicted the fact that T is bounded. Thus X cannot contain more than one point.

Proof of theorem. A theorem of Bishop [1] states the following. If A is a subalgebra of $C(X)$ satisfying (1), (2), (3), there exists a partition P of the space X into nonempty disjoint closed sets, such that

(i) for each S in P the algebra A_S, obtained by restricting A to S, is anti-symmetric;

(ii) A_S is a uniformly closed subalgebra of $C(S)$;

(iii) the algebra A consists of all continuous functions f on the space X such that the restriction of f to S is in A_S for each S in the partition P.

Glicksberg [2] proved that we may also arrange that

(iv) if S is a fixed element of P and T is a closed subset of X disjoint from S, there exists a function g in A such that

$$\| g \| \leqq 1 , \qquad g = 1 \text{ on } S , \qquad | g | < 1 \text{ on } T .$$

Actually, (ii) is a consequence of (iv). What we shall show now is that (iv), together with the assumption that $\mathbf{Re}\, A$ is closed, implies that $\mathbf{Re}\, A_S$ is uniformly closed for each set S in the partition P. This will prove the theorem. For A_S is an anti-symmetric closed algebra on the space S, and the above lemma shows that S consists of one point. By (iii) we then have $A = C(X)$.

Fix S in P. We show that $\mathbf{Re}\, A_S$ is closed. We first assert the following. If $f \in A$ and $\varepsilon > 0$, we can find $F \in A$ such that

$$(4) \qquad \sup_X | \mathbf{Re}\, F | \leqq \sup_S | \mathbf{Re} f | + 2\varepsilon , \quad \text{and} \quad \mathbf{Re}\, F = \mathbf{Re} f \text{ on } S .$$

Let Ω be the region in the w-plane ($w = u + iv$) defined by

$$| w | < 1 , \qquad -\varepsilon < v < \varepsilon .$$

Let τ be a conformal map of $| z | < 1$ on Ω with $\tau(0) = 0$ and $\tau(1) = 1$. Choose $\delta > 0$ such that τ maps $| z | < \delta$ into $| w | < \varepsilon$. Choose a neighborhood U of S in X with

$$| \mathbf{Re} f | \leqq \sup_S | \mathbf{Re} f | + \varepsilon , \quad \text{on } U.$$

By (iv) above there is a $g \in A$ such that $\| g \| \leqq 1$, $g = 1$ on S, $| g | < 1$ on $X - U$. Choose a positive integer n large enough that $| g^n | < \delta$ on

$X - U$. Put $h = \tau(g^n)$. Then $h \in A$, $h = 1$ on S, and $|\mathbf{Im}\, h| \leqq \varepsilon$ on all of X. Also $|\mathbf{Re}\, h| < \varepsilon$ on $X - U$ and $|\mathbf{Re}\, h| \leqq 1$ on all of X. Now define $F = fh$. Then $F \in A$ and

$$\mathbf{Re}\, F = \mathbf{Re}\, f\, \mathbf{Re}\, h - \mathbf{Im}\, f\, \mathbf{Im}\, h \ .$$

Therefore

$$(5) \qquad \mathbf{Re}\, F = \mathbf{Re}\, f \ \text{ on } \ S$$

$$(6) \qquad |\mathbf{Re}\, F| \leqq (\sup_S |\mathbf{Re}\, f| + \varepsilon) + \varepsilon, \ \text{ on } \ U$$

$$(7) \qquad |\mathbf{Re}\, F| \leqq \varepsilon + \varepsilon, \ \text{on } \ X - U \ .$$

In particular, F satisfies (4). (For (6) and (7) we have used $||f|| \leqq 1$.)

We finish the proof with a standard closure argument. Let R_S denote the subspace of $\mathbf{Re}\, A$ consisting of all functions in $\mathbf{Re}\, A$ which vanish on S. With norm given by maximum modulus over X, $\mathbf{Re}\, A$ is a Banach space, and R_S is a closed subspace. The quotient space $Q = \mathbf{Re}\, A/R_S$ is therefore complete in the norm

$$||\mathbf{Re}\, f + R_S|| = \inf_F ||\mathbf{Re}\, F|| \ , \qquad \mathbf{Re}\, F = \mathbf{Re}\, f \ \text{ on } \ S \ .$$

But by (4)

$$\sup |\mathbf{Re}\, f| = \inf ||\mathbf{Re}\, F|| \ , \qquad \mathbf{Re}\, F = \mathbf{Re}\, f \ \text{ on } \ S \ .$$

We conclude that $\mathbf{Re}\, A_S$, which is clearly isomorphic to Q, is complete in the maximum norm on S. We are done.

The theorem of this paper was proved independently by H. Rossi and H. Bear.

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