# ON ALMOST-COMMUTING PERMUTATIONS

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Suppose $A$ and $B$ are two permutations on a finite set $X$ which commute on almost all of the points of $X$. Under what circumstances can we conclude that $B$ is approximately equal to a permutation which actually commutes with $A$? The answer to this question depends strongly upon the order of the centralizer, $C(A)$, of $A$ in the symmetric group on $X$; and this varies greatly according to the cycle structure of $A$, being comparatively small when $A$ is either a product of few disjoint cycles or a product or a large number of disjoint cycles of different lengths and being comparatively large when $A$ is a product of many disjoint cycles, all of the same length. We shall show by example that when the order of $C(A)$ is small there may exist a permutation $B$ which commutes with $A$ "almost everywhere" yet is not approximated by any element of $C(A)$. On the other hand, when $A$ is a product of many disjoint cycles of the same length, we shall see that for any such permutation $B$, there must exist a permutation in $C(A)$ which agrees closely with $B$.

It is clear that if $B$ is a permutation leaving fixed almost all points of $X$, then no matter what permutation $A$ is given, $B$ will commute with $A$ on almost all points of $X$, and at the same time $B$ can be closely approximated by an element of $C(A)$—namely, the identity. However, the examples we shall give will show that only when all (or nearly all) of the cycles of $A$ are of the same length can we hope to approximate *every* $B$ which nearly commutes with $A$ by an element in $C(A)$. Accordingly, the bulk of this paper will be taken up with the study of the case in which $A$ is a product of many disjoint cycles, all of the same length.

1. In order to get a satisfactory notation and a more compact way of discussing the problem, we begin by making the symmetric group $S_N(X)$ on the space $X$ into a metric space. Here $N$ denotes the cardinality of $X$, and it is to be understood that $N$ is finite. Define, for any $A$ in $S_N(x)$,

$$(1) \qquad \| A \| = \frac{N - f_A}{N}$$

where $f_A$ is the number of fixed points of $A$ on $X$. Now define the distance $d(A, B)$ between two elements $A$ and $B$ of $S_N(X)$ to be

$$(2) \qquad d(A, B) = \| AB^{-1} \| .$$

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Under these definitions, the identity is the only permutation of norm 0, every permutation has norm $\leqq 1$, and a permutation has norm $p$ if and only if it moves $pN$ points of $X$. In particular, the permutations $A$ and $B$ commute if and only if $\|[A, B]\| = 0$, or equivalently, if and only if $d(AB, BA) = 0$.

In order to see that these definitions make $S_N(X)$ into a metric space, we need only verify the triangle inequality, since the other properties are trivial. But the points of $X$ displaced by $AB$ are clearly among those which are displaced by either $A$ or $B$. Hence $N - f_{AB} \leqq (N - f_A) + (N - f_B)$ and consequently $\|AB\| \leqq \|A\| + \|B\|$. We thus have the following lemma.

LEMMA 1. *With the norm defined above, $S_N(X)$ forms a metric space.*

When no restriction is placed upon the cycle structure of $A$, we have the following result:

PROPOSITION 1. For any $\varepsilon > 0$, there exists an integer $N$ and permutations $A$ and $B$ in $S_N(X)$ such that $\|[A, B]\| < \varepsilon$ and such that $d(B, D) = 1$ for every $D$ in $C(A)$.

*Proof*. We shall give two examples of permutations $A$ and $B$ which satisfy the conditions of the proposition; in the first, $A$ will be a product of cycles of relatively prime lengths, and in the second, a product of cycles of lengths $n$ and $2n$.

EXAMPLE 1. Let $X = \{1, 2, \cdots, N\}$, where $N = 2n > 4/\varepsilon$. Let $A$ be the permutation

$$(1\,2 \cdots n-1)(n)(n+1\; n+2 \cdots 2n)$$

and $B$ the permutation $xB = x + n$ if $x \leqq n$, and $xB = x - n$ if $x > n$. By direct verification, we find that $A$ and $B$ commute except on the points $n-1, n, 2n-1, 2n$. Thus $f_{[A,B]} = N - 4$ and hence $\|[A, B]\| = 4/N < \varepsilon$.

On the other hand, any element $D$ of $C(A)$ must map each cycle of $A$ into itself, since these cycles are of different lengths. But, for any $x$ in $X$, $xB$ and $x$ lie in distinct cycles of $A$. It follows that for any $D$ in $C(A)$, $BD^{-1}$ displaces every point of $X$ and hence that $d(B, D) = 1$.

EXAMPLE 2. Let $X = \{1, 2, \cdots, N\}$, where $N = 4nm$ and $n > 1/\varepsilon$. Let $A$ be the permutation with $m$ cycles of length $2n$ and $2m$ cycles of length $n$, defined as follows:

$$(1\ 2 \cdots 2n)(2n+1 \cdots 4n) \cdots (2n(m-1)+1 \cdots 2nm)$$
$$(2nm+1 \cdots 2nm+n)(2nm+n+1 \cdots 2nm+2n) \cdots$$
$$(4nm-n+1 \cdots 4nm)\ .$$

Let $B$ be the permutation $xB = x + 2nm$ if $x \leqq 2nm$, and $xB = x - 2nm$ if $x > 2nm$.

Again, by direct computation, we find that $A$ and $B$ commute on all points $x$ of $X$ except when $x \equiv 0 \pmod{n}$. Thus $f_{[A,B]} = 4nm - 4m$ and hence $\|[A, B]\| = 1/n < \varepsilon$. On the other hand, if $D \in C(A)$, $D$ must permute the cycles of $A$ of length $n$ among themselves and must permute the cycles of $A$ of length $2n$ among themselves. But if $x$ is in a cycle of length $n$, then $xB$ is in a cycle of length $2n$, and vice versa. It follows that $BD^{-1}$ displaces every point of $X$ and hence that $d(B, D) = 1$, for any $D$ in $C(A)$.

2. The two examples given in Proposition 1 indicate that unless severe restrictions are placed on the cycle structure of $A$, the fact that $B$ comes very close to commuting with $A$ does not necessarily imply that $B$ can be approximated by an element in $C(A)$. In fact, it seems that unless $A$ consists almost entirely of cycles of the same length, little can be said in general of the relation between $\|[A, B]\|$ and the distance from $B$ to $C(A)$.

In order to be able to make as exact statements as possible, we shall assume in the balance of the paper that $A$ is the product of $m$ disjoint cycles, each of length $n$. In this case our statements about the distance from $B$ to $C(A)$ will depend only upon $\|[A, B]\|$ and $n$.

We may take $X = \{1, 2, \cdots, N\}$, where now $N = nm$. Let $x, k$ be integers such that $1 \leqq x \leqq N$, $0 \leqq k \leqq n$, and write $x = in + r$, where $1 \leqq r \leqq n$. We shall adopt the following notation:

$$\overline{x+k} = in + s, \text{ where } 1 \leqq s \leqq n \text{ and } s \equiv r + k \pmod{n}\ . \tag{3}$$

Without loss of generality we may assume that $A$ is the mapping

$$xA = \overline{x+1},\ x \in X\ . \tag{4}$$

We shall say that $B$ in $S_N(X)$ *transforms* the cycle $a$ of $A$ into the cycle $a'$ if, for some $x$ in $a$, $xB$ is in $a'$ and

$$\overline{(x+k)}B = \overline{xB+k}, \qquad k = 0, 1, \cdots, n-1\ . \tag{5}$$

We shall write $(a)B = a'$ if $B$ transforms $a$ into $a'$. We shall also say that $B$ commutes with $A$ on a cycle $a$ if it commutes with $A$ on each point of $a$.

LEMMA 2. (a) *A permutation $B$ commutes with $A$ on a cycle $a$ if and only if $B$ tranforms $a$ into a cycle $a'$.*

(b) *if $B$ commutes with $A$ on $n-1$ points of a cycle $a$, then $B$ commutes with $A$ on $a$.*

(c) *If $B$ transforms $r$ cycles of $A$ into cycles of $A$, there exists an element $D$ in $C(A)$ which agrees with $B$ on these $r$ cycles.*

*Proof.* For $A$ and $B$ to commute on a point $x$ of $X$ we must have $xBA = xAB$, and hence

$$\overline{xB+1} = (\overline{x+1})B\,. \tag{6}$$

Suppose $(a)B = a'$; then (6) follows at once from (5) for any $x$ in $a$. Conversely if (6) holds for all $x$ in $a$, (5) follows at once by induction on $k$.

To prove (b), suppose $B$ and $A$ commute on $x$, $\overline{x+1}, \cdots, \overline{x+n-2}$. Again by induction on $k$, (5) holds for $k = 0, 1, \cdots, n-2$. In particular, $(\overline{x+n-2})B = \overline{xB+n-2}$. Now using (6) with $x$ replaced by $\overline{x+n-2}$, we obtain

$$\begin{aligned}(\overline{x+n-1})B &= \overline{(\overline{x+n-2})B+1} \\ &= \overline{\overline{xB+n-2}+1} = \overline{xB+n-1}\,.\end{aligned}$$

Thus (5) holds for all $k$, and hence $A$ and $B$ commute on $a$ by part (a).

Finally suppose $B$ transforms the cycles $a_1, \cdots, a_r$ into the cycles $a'_1, \cdots, a'_r$. Denote by $a'_{r+1}, \cdots, a'_m$ the remaining cycles of $A$. Let $D$ be a permutation which agrees with $B$ on $a_1, \cdots, a_r$ and transforms $a_i$ into $a'_i$, $i = r+1, \cdots, m$. By (a) $D$ is in $C(A)$.

3. We shall now begin the analysis of the relationship between $\|[A, B]\|$ and the minimum distance from $B$ to $C(A)$, under the assumption that $A$ is the product of $n$-cycles. We shall denote this minimum distance by $d_A(B)$. Thus

$$d_A(B) = \min_{D \in C(A)} d(B, D)\,. \tag{7}$$

Then following estimate for $d_A(B)$ is easily obtained.

PROPOSITION 2. For any $B$ in $S_N(X)$,

$$d_A(B) \leqq \frac{n\,\|[A, B]\|}{2}\,.$$

*Proof.* If $\|[A, B]\| \geqq 2/n$, the proposition is vacuously true since $d_A(B) \leqq 1$. Hence we may assume that $\|[A, B]\| < 2/n$.

Now $N = nm$, where $m$ is the number of cycles in $A$. It suffices to show that $B$ transforms at least

$$m - \frac{N \| [A, B] \|}{2}$$

cycles of $A$ into cycles of $A$. For then by Lemma 2(c) we can find an element $D$ in $C(A)$ which agrees with $B$ on these cycles and hence on at least

$$N - \frac{nN}{2} \cdot \| [A, B] \|$$

points of $X$. It follows that

$$d(B, D) \leqq \frac{n \| [A, B] \|}{2} .$$

By the definition of $\| [A, B] \|$, $N \cdot \| [A, B] \|$ is the number of points displaced by $[A, B]$ and hence on which $A$ and $B$ do not commute. But by Lemma 2(b) any cycle of $A$ which is not transformed by $B$ into a cycle of $A$ contains at least 2 points on which $A$ and $B$ do not commute. Thus there are at most

$$\frac{N \| [A, B] \|}{2}$$

cycles of $A$ which are not transformed by $B$ into cycles of $A$, and hence $B$ transforms at least

$$m - \frac{N \| [A, B] \|}{2}$$

cycles of $A$ into cycles of $A$.

Proposition 2 gives an upper bound for $d_A(B)$, which depends only upon $\| [A, B] \|$ (and $n$), but not upon the particular structure of $B$. Our main concern in the paper will be in improving this upper bound. The next proposition shows the limit to which this estimate can be improved.

PROPOSITION 3. *If $A$ contains at least two distinct cycles, then there exists a permutation $B$ in $S_N(X)$ such that*

$$d_A(B) = \frac{n \| [A, B] \|}{4}$$

*when $n$ is even, and such that*

$$d_A(B) = \frac{n - 1}{4} \| [A, B] \|$$

when $n$ is odd. Furthermore for any $\varepsilon > 0$, $N$ and $B$ can be chosen so that $\|[A, B]\| < \varepsilon$.

*Proof.* Assume first that $n$ is even. Set $m = m_1 + m_2$, where $m_1 \geqq 0$ and $m_2 \geqq 2$. Define the permutation $B$ as follows: $xB = x$ if $1 \leqq x \leqq nm_1$; if $x > nm_1$, write $x = in + k$ where $1 \leqq k \leqq n$, and define $xB = x$ if $k \leqq n/2$, $xB = x + n$ if $i \neq m - 1$ and $k > n/2$, and $xB = nm_1 + k$ if $i = m - 1$ and $k > n/2$.

Thus $B$ leaves the first $m_1$ cycles of $A$ pointwise fixed, one half of each of the remaining $m_2$ cycles pointwise fixed, and permutes the other halves of these $m_2$ cycles cyclically. From its definition, we see that $B$ commutes with $A$ except on the points $x > nm_1$ for which $x \equiv 0 \pmod{n/2}$. Thus

$$\|[A, B]\| = \frac{2m_2}{N}\,. \tag{8}$$

Since $N = n(m_1 + m_2)$, $2m_2/N$ can be made arbitrarily small by making $m_1$ sufficiently large. Thus, to prove the proposition, we have only to show that

$$d_A(B) = \frac{n\,\|[A, B]\|}{4}\,.$$

Observe, first of all, that the identity, $I$, is in $C(A)$ and agrees with $B$ on

$$nm_1 + \frac{nm_2}{2}$$

points of $X$, whence

$$d(I, B) = \frac{N - nm_1 - \dfrac{nm_2}{2}}{N} = \frac{nm_2}{2N} = \frac{n}{4}\,\|[A, B]\|\,. \tag{9}$$

On the other hand, by Lemma 2, any $D$ in $C(A)$ must transform each cycle $a_i$ of $A$ into some other cycle $a_j$. Since $B$ transforms the two halves of the cycles $a_i$ into distinct cycles of $A$, $m_1 \leqq i \leqq -1$, $D$ and $B$ can agree on at most half of the $nm_2$ points in these cycles. Hence $DB^{-1}$ displaces at least $nm_2/2$ points of $X$, which implies that

$$d(D, B) \geqq \frac{nm_2}{2N} = \frac{n}{4}\,\|[A, B]\|$$

for any $D$ in $C(A)$.

When $n$ is odd, the construction of $B$ is entirely analogous.

4. If we set

$$d_A = \max_{\substack{B \in S_N(X) \\ B \notin C(A)}} \frac{d_A(B)}{\| [A, B] \| n},$$

then $d_A$ is a measure of the extent to which *every* permutation in $S_N(X)$ can be approximated by elements in $C(A)$. Propositions 2 and 3 show that

$$\frac{1}{4} \leqq d_A \leqq \frac{1}{2} \text{ or } \frac{n-1}{4n} \leqq d_A \leqq \frac{1}{2} \tag{10}$$

according as $n$ is even or odd.

In the balance of the paper we shall sharpen these inequalities by lowering the upper bound for $d_A$. Our next result will show that in considering this problem, we may restrict our attention to those cycles of $A$ on which $B$ commutes with $A$ on exactly $n, n-2$, or $n-3$ points. Let $U_B$, $V_B$, $W_B$ be the set of points in those cycles of $A$ on which $B$ commutes with $A$ on $n, n-2$, and $n-3$ points respectively; and let $u_B = |U_B|$, $v_B = |V_B|$, $w_B = |W_B|$.

THEOREM 1. *Suppose there exists an element $D$ in $C(A)$ which agrees with $B$ on at least $u_B + (1/2)v_B + (1/3)w_B$ points of $X$. Then*

$$d_A(B) \leqq \| [A, B] \| \frac{n}{4}.$$

*Proof.* For simplicity of notation, we drop the subscript $B$, and define

$$t = N - u - v - w. \tag{11}$$

Thus $t$ is the number of points in those cycles of $A$ on which $A$ and $B$ commute on no more than $n-4$ points. Then by definition of $u, v, w, t$, we have

$$u + \frac{n-2}{n}v + \frac{n-3}{n}w + \frac{n-4}{n}t \geqq f_{[A,B]}. \tag{12}$$

Now, by hypothesis,

$$d(B, D) \leqq \frac{N - \left(u + \frac{1}{2}v + \frac{1}{3}w\right)}{N} = \frac{\frac{1}{2}v + \frac{2}{3}w + t}{N}. \tag{13}$$

We must show that

$$\frac{\frac{1}{2}v + \frac{2}{3}w + t}{N} \leqq \frac{n}{4} \| [A, B] \|. \tag{14}$$

But using (1), we can rewrite (14) as:

$$(15) \qquad f_{[A,B]} \leqq u + \frac{n-2}{n}v + \left(1 - \frac{8}{3n}\right)w + \left(\frac{n-4}{n}\right)t\ .$$

Since (15) is an immediate consequence of (12), the theorem follows.

5. In this section, we prove that $d_A \leqq 1/4$, by proving that for any $B$ in $S_N(X)$, there exists a permutation $D$ in $C(A)$ which satisfies the conditions of Theorem 1.

To treat our problem, we need an additional concept: By a *block* of a cycle $a$ of $A$, we shall mean a maximal sequence $x, \overline{x+1}, \cdots, \overline{x+r-1}$ of points of $a$ such that $A$ and $B$ commute on every point of the sequence except $\overline{x+r-1}$. The integer $r$ will denote the *length* of the block. According to the definition, if $A$ and $B$ commute on every point of $a$ then $a$ contains no blocks. When $B$ and $A$ do not commute on every point of $a$, we have the following obvious lemma:

LEMMA 3. *If $A$ and $B$ commute on exactly $n-k$ points of a cycle $a$ of $A$, $k > 0$, then $A$ contains exactly $k$ blocks, the sum of whose lengths is $n$.*

Thus when a cycle $a$ of $A$ lies in $V_B$, $a$ consists of 2 blocks which we denote by $p_1, p_2$; and when $a$ lies in $W_B$, $a$ consists of 3 blocks which we denote by $q_1, q_2, q_3$. We define $|p_j|$, $|q_j|$ to be the lengths of $p_j, q_j$, respectively. Furthermore we order the blocks so that $|p_1| \geqq |p_2|$ and $|q_1| \geqq |q_2| \geqq |q_3|$. Since $|p_1| + |p_2| = n$,

$$(16) \qquad |p_1| \geqq \frac{n}{2}$$

and likewise

$$(17) \qquad |q_1| \geqq \frac{n}{3}\ .$$

Let $x, \overline{x+1}, \cdots, \overline{x+r-1}$ be a block contained in a cycle $a$. If $xB = y$, then, it follows from (6) as in the proof of Lemma 2, that

$$(18) \qquad (\overline{x+k})B = \overline{y+k}\ , \qquad 0 \leqq k \leqq r-1\ ;$$

and

$$(19) \qquad (\overline{x+r})B \neq \overline{y+r}\ .$$

Thus the image of the block is a consecutive sequence of points in a cycle $a'$. It follows that there exist permutations which transform $a$

into $a'$ and agree with $B$ on the block $b = \{x, \overline{x+1}, \cdots, \overline{x+r-1}\}$. In fact, any $D$ in $C(A)$ for which $xD = y$ has this property. If $D$ is such a permutation, we shall write simply $(a)D = a'$; $(b)D = (b)B$.

From this fact, we easily derive the following lemma:

LEMMA 4. *Let $a_1, \cdots, a_k$ be distinct cycles of $A$ containing the blocks $b_1, \cdots, b_k$ respectively. If the images of $b_i$ under $B$ lie in distinct cycles $a'_i$ of $A$, $i = 1, 2, \cdots, k$, then there exist permutations $D$ in $C(A)$ such that $(a_i)D = a'_i$; $(b_i)D = (b_i)B$, $i = 1, 2, \cdots, k$.*

We are now in a position to prove the following result:

THEOREM 2. *Given any $B$ in $S_N(X)$, there exists an element $D$ in $C(A)$ which agrees with $B$ on at least*

$$u_B + \frac{1}{2}v_B + \frac{1}{3}w_B$$

*points of $X$.*

*Proof.* Let $a_1, a_2, \cdots, a_m$ be the cycles of $A$. For any $i, j, 1 \leqq i, j \leqq m$, let $b_{ij}$ be the maximal number of elements of $a_i$ on which a permutation $D$ in $C(A)$ mapping $a_i$ into $a_j$ can agree with $B$. Thus if $B$ transforms $a_i$ into $a_j$, $b_{ij} = n$. If $(a_i)B \cap a_j = \phi$, then $b_{ij} = 0$. Now, to any $m \times m$ permutation matrix $(e_{ij})$ there corresponds a permutation $D$ in $C(A)$ which agrees with $B$ on

$$\sum_{i,j} e_{ij} b_{ij} \tag{20}$$

points, where $D$ is defined to transform $a_i$ into $a_j$ if $e_{ij} = 1$, and to map $a_i$ so as to agree with $B$ on $b_{ij}$ points.

We wish to show

$$\max_{(e_{ij})} \sum e_{ij} b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w\,, \tag{21}$$

where $(e_{ij})$ ranges over all permutation matrices. To do this, consider the set of all real $m \times m$ matrices $(x_{ij})$ such that

$$x_{ij} \geqq 0\,; \qquad 1 \leqq i, j \leqq m \tag{22}$$

$$\sum_i x_{ij} = 1\,; \qquad 1 \leqq j \leqq m \tag{23}$$

$$\sum_j x_{ij} = 1\,; \qquad 1 \leqq i \leqq m. \tag{24}$$

This is the set of doubly stochastic matrices and is a convex, bounded set whose vertices consist of exactly the permutation matrices (see [1], pp. 132–3).

The following lemma will be useful in proving the theorem.

LEMMA 5. *If* $(x_{ij})$ *is any doubly stochastic matrix, then there exists a permutation matrix* $(e_{ij})$ *such that*

$$\sum_{i,j} e_{ij}b_{ij} \geqq \sum_{i,j} x_{ij}b_{ij} . \tag{25}$$

*Proof.* See [1], p. 134.

If we can now demonstrate a doubly stochastic matrix such that

$$\sum_{i,j} x_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w , \tag{26}$$

we will clearly be finished since, by Lemma 5, there must then be some permutation matrix $(e_{ij})$ such that

$$\sum_{i,j} e_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w ,$$

and this permutation matrix will yield the desired mapping $D$.

To find a matrix satisfying (26), define

$$x_{ij} = \frac{n_{ij}}{n} , \tag{27}$$

where $n_{ij}$ is the number of points of $a_i$ which $B$ maps into $a_j$. The matrix $(x_{ij})$ is clearly doubly stochastic, so we must show that (26) holds. But if $a_i \subseteqq U_B$, then

$$\sum_j x_{ij}b_{ij} = n ,$$

since $(a_i)B = a_{j_1}$ for some $j_1$. If $a_i \subseteqq V_B$, there exist indices $j_1$ and $j_2$ such that $(p_1)B \subset a_{j_1}$ and $(p_2)B \subset a_{j_2}$. Note that $j_1 \neq j_2$, or else $a_i$ would be transformed by $B$ into $a_{j_1}$. In this case, then,

$$\sum_j x_{ij}b_{ij} = \frac{|\,p_1\,|^2}{n} + \frac{|\,p_2\,|^2}{n} \geqq \frac{n}{2}$$

$$(\text{remember } |\,p_1\,| + |\,p_2\,| = n) .$$

Finally, when $a_i \subseteqq W_B$, one of three things can happen:

(a) $q_1, q_2, q_3$ can be mapped by $B$ into three distinct cycles of $A$.

(b) $q_1, q_2, q_3$ can be mapped by $B$ into only two cycles of $A$,

(c) $q_1, q_2, q_3$ can be mapped into one cycle of $A$.

In the first case,

$$\sum_j x_{ij}b_{ij} = \sum_{k=1}^{3} \frac{|\,q_k\,|^2}{n} .$$

In the second case,

$$\sum_j x_{ij}b_{ij} = \frac{|q_{k_1}|^2}{n} + \frac{(|q_{k_2}| + |q_{k_3}|)}{n} |q_{k_2}|$$
$$\text{where } |q_{k_2}| \geqq |q_{k_3}| .$$

Finally, in case $c$,

$$\sum_j x_{ij}b_{ij} = |q_1| \frac{(|q_1| + |q_2| + |q_3|)}{n} ,$$

where $|q_1| \geqq |q_2|, |q_3|$.

Since $|q_1| + |q_2| + |q_3| = n$, it follows at once in all three cases that

$$\sum_j x_{ij}b_{ij} \geqq \frac{n}{3} .$$

We have thus demonstrated the existence of a doubly stochastic matrix $(x_{ij})$ with the property

$$\sum_{i,j} x_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w .$$

Together with Lemma 5, this completes the proof of the theorem.

As an immediate corollary of Theorems 1 and 2, we obtain our main result:

THEOREM 3. *Let $A$ contain at least two distinct cycles. If $n$ is even, $d_A = 1/4$. If $n$ is odd,*

$$\frac{n-1}{4n} \leqq d_A \leqq \frac{1}{4} .$$

## REFERENCE

1. S. Karlin, *Mathematical Methods and Theory in Games, Programming and Economics*, vol. I (1959).

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