FAMILIES OF INDUCED REPRESENTATIONS

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In [11], Mackey constructed certain representations (the *induced* representations) of a group G. If the group is acting on a measure space X then the construction also gives a projection valued measure P on X which is a system of imprimitivity for the representation U of G. ($P(\sigma E) = U(\sigma)P(E)U(\sigma^{-1})$.) In this paper we determine the topology in the set of equivalence classes of induced pairs U, P whose joint action is irreducible, provided certain restrictions are imposed on G and X. This set of pairs is (homeomorphic to) a space W/G of orbits, where W consists of fibers over X as a base space and G acts on W. The fiber over x is $\hat{G}_x$, the space of equivalence classes of irreducible representations of $G_x = \{\gamma: \gamma x = x\}$. The principal restriction on G and X is equivalent to assuming that G_x is a continuous function of x. (See the Appendix.) One might hope that in interesting cases X could be expressed as a finite disjoint union of subsets upon which our assumptions are satisfied.

One of the motivations for this paper was the hope of introducing in certain cases a differentiable or real analytic structure into W/G. If W is a manifold (except perhaps for a set of singular points), if G is an analytic group and if G acts smoothly on W then W/G is a manifold, except perhaps for a set of singular points, if W/G is countably separated (if there are Borel sets $W_1, W_2, \cdots$ in W which are G invariant and which separate points of W/G). This is a simple consequence of [14, Theorem 8, page 19] and [6, Theorem 1] and does not depend upon the special nature of W. In particular it applies equally well to a closed subset K of W which is a manifold and upon which G acts smoothly. As might be expected, K/G being countably separated is equivalent to all representations of a certain C^*-algebra being of type I. The assumption that W is a manifold except for singular points is unsatisfactory. One would like to assume that X is a manifold and that G acts on X smoothly and conclude that W is a manifold (except perhaps for singular points) if all the G_x are type I groups. Whether this is true is not known even when X is a point. The results of this paper presumably have implications for the representations of analytic groups which have closed normal subgroups.

The group G and the topological space X considered in the paper will be assumed to satisfy the second axiom of countability. This is not used until § 2 and in view of [10, 1], it would not be surprising

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if Theorem 2.1 were true without this assumption. That φ is a representation of a group (resp. $*$ algebra $\mathfrak{R}$) means that the representation space $\mathfrak{H}(\varphi)$ is a Hilbert space and that φ is a unitary representation (resp. $*$ representation and $\varphi(\mathfrak{R})\mathfrak{H}(\varphi)$ is dense in $\mathfrak{H}(\varphi)$). For any locally compact space Y, $C_0(Y)$ denotes the set of complex valued continuous functions on Y with compact support.

1. Group algebras. In this section we study $*$-algebras which are fields of group algebras and which are associated with a locally compact group G acting as a topological transformation group on a locally compact T_2 space X. That G is a topological transformation group means that there is a jointly continuous map $(\gamma, x) \to \gamma x$ from $G \times X$ into X such that $(\beta^{-1}\gamma)x = \beta^{-1}(\gamma x)$ and $ex = x$. Suppose a left invariant Haar measure $d(x, \sigma) = d\sigma$ can be chosen on the isotropy subgroups G_x "continuously," that is so that for each f in $C_0(G)$, the function $x \to \int_{G_x} f(\sigma)d\sigma$ defined on X is continuous. Let $Y = \{(x, \sigma): x \in X \text{ and } \sigma \in G_x\}$. Then Y is a closed subspace of $X \times G$ and so is locally compact.

The continuity requirement of the Haar measures could also be expressed by saying that $x \to d(x, \sigma)$ is a w^*-continuous map from X to regular Borel measures on G.

LEMMA 1.1. *Let $x \to d\mu(x, \sigma)$ be a w^*-continuous map from X to the regular Borel measures on G. For each compact subset K of $X \times G$ there is a constant $M = M(K)$ such that $\left|\int f(x, \sigma)d\mu(x, \sigma)\right| \leqq M \|f\|_\infty$ for all f in $C_0(K)$ and x in X.*

There are compact subsets K_1 and K_2 of X and G respectively such that $K \subset K_1 \times K_2$. If $g \in C_0(G)$ and $g = 1$ on K_2, let M be the supremum of $\int |g(\sigma)| d\mu(x, \sigma)$ as x varies in K_1. If $f \in C_0(K)$ then $\left|\int f(x, \sigma)d\mu(x, \sigma)\right|$ is dominated by $\|f\|_\infty \int |g(\sigma)| d\mu(x, \sigma) \leqq \|f\|_\infty M$ if $x \in K_1$ and is equal to zero if $x \notin K_1$.

It follows from Lemma 1.1 that $\int_{G_x} f(x, \sigma)d\sigma$ is a jointly continuous function of f in $C_0(K)$ and x in X.

Let Δ_x be the modular function for G_x, $d(x, \sigma\tau) = d(x, \sigma)\Delta_x(\tau)$. For a suitably chosen f in $C_0(G)$,

$$\Delta_x(\tau) = \int_{G_x} f(\sigma\tau^{-1})d\sigma \Big/ \int_{G_x} f(\sigma)d\sigma$$

and so as a function on Y, $\Delta_x(\tau)$ is continuous. If $f, g \in C_0(Y)$ define

$$f * g(x, \sigma) = \int_{G_x} f(x, \rho)g(x, \rho^{-1}\sigma)d\rho$$

$$f^*(x, \sigma) = f(x, \sigma^{-1})^- \Delta_x(\sigma^{-1}) .$$

Then $f*g$ and $f^* \in C_0(Y)$ and $C_0(Y)$ is a $*$-algebra. It is also an algebra of vector fields defined on X and having values in the $C_0(G_x)$ If $f \in C_0(Y)$, let $\|f\|_1 = \sup_{x \in X} \int_{G_x} |f(x, \sigma)| \, d\sigma$ and let $\|f\|$ be the supremum of $\|\varphi(f)\|$, for φ a representation of $C_0(Y)$ which is continuous in the inductive limit topology on $C_0(Y)$ (the topology which is the inductive limit of the uniform topologies on the $C_0(K)$ for K compact). The next lemma shows that $\|f\| < \infty$. It then follows that the completion $\mathfrak{K}$ of $C_0(Y)$ in $\|\cdot\|$ is a C^*-algebra.

LEMMA 1.1A[1]. $\|\cdot\| \leqq \|\cdot\|_1$. *If φ is an irreducible representation of $\mathfrak{K}$ then there is a unique x in X and a unique representation φ_x of G_x such that*

$$\varphi(f) = \varphi_x(f(x, \cdot)), f \in C_0(Y) ,$$

and x is determined uniquely by the kernel of φ. Furthermore $\mathfrak{K}$ is closed under multiplication by bounded continuous functions on X.

Let φ be a continuous irreducible representation of $C_0(Y)$ on a Hilbert space $\mathfrak{H}$. Let $X_0 = \{x \colon x \in X$ and for some neighborhood N_x of x, kernel φ contains all f in $C_0(Y)$ which vanish off N_x (or more precisely, off $(N_x \times G) \cap Y)\}$. Then $X_0 \neq X$. If x and y are distinct elements of $X \sim X_0$ then there are disjoint neighborhoods N_x and N_y of x and y respectively and elements f_x and f_y of $C_0(Y) \sim$ kernel φ which vanish off N_x and N_y respectively. Then $\varphi(C_0(Y))\varphi(f_x)\mathfrak{H}$ and $\varphi(C_0(Y))\varphi(f_y)\mathfrak{H}$ are orthogonal nonzero invariant subspaces of $\mathfrak{H}$. This contradicts the irreducibility of φ and so $X_0 = X \sim \{x\}$ for some x. It is now evident from the definition of X_0 that if $f(x, \cdot) \equiv 0$ then $f \in$ kernel φ. Hence there is a representation φ_x of $C_0(G_x)$ for which $\varphi(f) = \varphi_x(f(x, \cdot))$, and one can check that φ_x is continuous. Thus φ_x comes from a representation, also called φ_x, of G_x and this implies $\|\varphi(f)\| \leqq \int_{G_x} |f(x, \sigma)| \, d\sigma$. The first two statements of the lemma follow immediately. If h is a bounded continuous function on X then $\|\varphi(hf)\| = |h(x)| \, \|\varphi(f)\| \leqq \|h\|_\infty \|f\|$, and so multiplication by h is an operator on $C_0(Y)$ which is continuous in $\|\cdot\|$. It thus has a unique continuous extension to all of $\mathfrak{K}$. If we regard $\mathfrak{K}$ as functions from X to the C^*-group algebras of the G_x then this extension of multiplication by h is still multiplication by h.

If $f \in C_0(G_{\gamma^{-1}x})$ then the functional

$$f \longrightarrow \int_{G_x} f(\gamma^{-1}\sigma\gamma) d\sigma$$

defines a left invariant integral on $G_{\gamma^{-1}x}$. Thus there exists a unique positive number $c(x, \gamma)$ for which

[1] This is based in part upon a lemma supplied by R. Blattner.

$$(1.1) \qquad c(x, \gamma)\int_{G_x} f(\gamma^{-1}\sigma\gamma)d\sigma = \int_{G_{\gamma^{-1}x}} f(\sigma)d\sigma .$$

If we choose f to be a nonnegative element of $C_0(G)$ which is positive at e then (1.1) implies that $c(x, \gamma)$ is jointly continuous in x and γ. It is easy to see that the identities

$$c(x, \beta\gamma) = c(x, \beta)c(\beta^{-1}x, \gamma)$$
$$c(x, \tau) = \Delta_x(\tau) ; \qquad c(x, e) = 1$$

are true for $\beta, \gamma \in G, \tau \in G_x$. Also $\Delta_{\gamma^{-1}x}(\gamma^{-1}\tau\gamma) = \Delta_x(\tau)$ since if f is a suitable element of $C_0(G_{\gamma^{-1}x})$ then

$$\begin{aligned}\Delta_x(\tau) &= \int_{G_x} f(\gamma^{-1}\sigma\tau^{-1}\gamma)d\sigma \Big/ \int_{G_x} f(\gamma^{-1}\sigma\gamma)d\sigma \\ &= \int_{G_{\gamma^{-1}x}} f(\sigma\gamma^{-1}\tau^{-1}\gamma)d\sigma \Big/ \int_{G_{\gamma^{-1}x}} f(\sigma)d\sigma \\ &= \Delta_{\gamma^{-1}x}(\gamma^{-1}\tau\gamma) .\end{aligned}$$

PROPOSITION 1.2. *If $f \in C_0(Y)$ then $\gamma_K(f) \in C_0(Y)$, where*

$$\gamma_K(f)(x, \sigma) = f(\gamma^{-1}x, \gamma^{-1}\sigma\gamma)c(x, \gamma) .$$

γ_K has a unique extension to an automorphism γ_K of $\mathfrak{K}$ and $\gamma \to \gamma_K$ is a strongly continuous representation of G on $\mathfrak{K}$.

There is no difficulty in seeing that $\gamma_K(f) \in C_0(Y)$. If $f, g \in C_0(Y)$ then

$$\begin{aligned}\gamma_K(f * g)(x, \sigma) &= \int_{G_{\gamma^{-1}x}} f(\gamma^{-1}x, \rho)g(\gamma^{-1}x, \rho^{-1}\gamma^{-1}\sigma\gamma)c(x, \gamma)d\rho \\ &= \int_{G_x} f(\gamma^{-1}x, \gamma^{-1}\rho\gamma)g(\gamma^{-1}x, \gamma^{-1}\rho^{-1}\sigma\gamma)c(x, \gamma)^2 d\rho \\ &= (\gamma_K(f) * \gamma_K(g))(x, \sigma) ; \\ \gamma_K(f^*)(x, \sigma) &= f^*(\gamma^{-1}x, \gamma^{-1}\sigma\gamma)c(x, \gamma) \\ &= f(\gamma^{-1}x, \gamma^{-1}\sigma^{-1}\gamma)^- \Delta_{\gamma^{-1}x}(\gamma^{-1}\sigma^{-1}\gamma)c(x, \gamma) \\ &= \gamma_K(f)(x, \sigma^{-1})^- \Delta_x(\sigma^{-1}) = (\gamma_K(f))^*(x, \sigma)\end{aligned}$$

and γ_K is an automorphism of $C_0(Y)$. γ_K is continuous in the inductive limit topology and so $\varphi \circ \gamma_K$ is a continuous representation of $C_0(Y)$ if φ is. γ_K is thus continuous in $\|\cdot\|$. Hence it has a unique continuous extension to $\mathfrak{K}$, and the extension is an automorphism. Also

$$\begin{aligned}(\beta_K(\gamma_K f))(x, \sigma) &= f(\gamma^{-1}\beta^{-1}x, \gamma^{-1}\beta^{-1}\sigma\beta\gamma)c(\beta^{-1}x, \gamma)c(x, \beta) \\ &= ((\beta\gamma)_K f)(x, \sigma),\end{aligned}$$

so $\gamma \to \gamma_K$ is a representation. If $f \in C_0(Y)$ and $\gamma \to \gamma_0$ then $\gamma_K(f) \to \gamma_{0K}(f)$ uniformly with support contained in a fixed compact set and so in the norm $\|\cdot\|$. It follows that γ_K is strongly continuous.

G acts on the dual $\hat{\mathfrak{K}}$ of $\mathfrak{K}$ as a topological transformation group, in fact more generally we have the following lemma; we do not claim that this result is original.

LEMMA 1.3. *Let $\mathfrak{A}$ be a C^*-algebra with dual $\hat{\mathfrak{A}}$ and let there be a strongly continuous representation of a topological group G as automorphisms of $\mathfrak{A}$. Then the map $(\gamma, \varphi) \to \gamma\varphi = \varphi \circ \gamma^{-1}$ from $G \times \hat{\mathfrak{A}}$ into $\hat{\mathfrak{A}}$ makes G into a topological transformation group acting on $\hat{\mathfrak{A}}$.*

$\hat{\mathfrak{A}}$ is the set of equivalence classes of irreducible representations of $\mathfrak{A}$ with the hull kernel topology, which is the topology which has as a subbasis for closed sets the sets of the form $\{\varphi\colon \text{kernel } \varphi \supset \mathfrak{J}\}$ where $\mathfrak{J}$ is an ideal (closed two sided) in $\mathfrak{A}$. It is evident that $(\beta^{-1}\gamma)\varphi = \beta^{-1}(\gamma\varphi)$ and that $\gamma\{\varphi\colon \text{kernel } \varphi \supset \mathfrak{J}\} = \{\varphi \cdot \gamma^{-1}\colon \text{kernel } \varphi \supset \mathfrak{J}\} = \{\varphi\colon \gamma^{-1}(\text{kernel } \varphi) \supset \mathfrak{J}\} = \{\varphi\colon \text{kernel } \varphi \supset \gamma\mathfrak{J}\}$ so each γ in G acts by homeomorphisms of $\hat{\mathfrak{A}}$. Thus we have only to show the joint continuity of the map $(\gamma, \varphi) \to \gamma\varphi$ at $\gamma = e$. A subbasic neighborhood of φ is given by $N = \{\psi\colon \text{kernel } \psi \not\supset \mathfrak{J}\}$ where $\mathfrak{J}$ is an ideal which is not contained in kernel φ. There is a positive A in $\mathfrak{J}$ which is not in kernel φ, by Lemma 2.3 of [16]. Let $M = \{\psi\colon \|\psi(A)\| > \|\varphi(A)\|/2\}$. Let f be a continuous function which is zero on $[0, \|\varphi(A)\|/2]$ and positive elsewhere. M is open since $M = \{\psi\colon \psi(f(A)) \neq 0\}$. For all γ sufficiently near e, $\|\gamma^{-1}(A) - A\| < \|\varphi(A)\|/2$ and for such γ and for ψ in M, $\|\psi \cdot \gamma^{-1}(A)\| > 0$ so $\gamma\psi \in N$ and the proof is complete.

If Z is the structure space of $\mathfrak{A}$ (the set of kernels of irreducible representations of $\mathfrak{A}$) with the hull kernel topology then the map $(\gamma, z) \to \gamma z = \{\gamma(A)\colon A \in z\}$ form $G \times Z$ into Z makes G into a topological transformation group on Z. This follows from Lemma 1.3 and from the facts that γ kernel $\varphi =$ kernel $\gamma\varphi$ and that $\varphi \to$ kernel φ is an open continuous map of $\hat{\mathfrak{A}}$ onto Z.

Let Z be the structure space of $\mathfrak{K}$, let φ be a representation of G. By a system of imprimitivity for φ based on X (resp. Z) we mean a regular countably additive projection valued measure P defined on the Borel subsets of X (resp. Z) with values acting on $\mathfrak{H}(\varphi)$ such that $P(X)$ (resp. $P(Z)) = I$ and $\varphi(\gamma)P(E)\varphi(\gamma^{-1}) = P(\gamma E)$ for all γ in G and all Borel sets E in X (resp. Z), cf. [11]. We shall call the pair (φ, P) a representation of G, X (resp. G, Z). Here the Borel sets are the elements of the smallest σ-ring containing the open sets and regular means that for open U, $P(U) = \bigvee \{P(C)\colon C$ is a compact Borel set contained in $U\}$.

There is a $*$-algebra associated with representations of G, X. It is the set $C_0(X \times G)$ with multiplication and involution defined by

$$f * g(x, \gamma) = \int_G f(x, \beta) g(\beta^{-1}x, \beta^{-1}\gamma) d\beta \tag{1.2}$$

$$f^*(x, \gamma) = f(\gamma^{-1}x, \gamma^{-1}) - \Delta(\gamma^{-1}) \tag{1.3}$$

for $f, g \in C_0(X \times G)$, $d\beta$ a left invariant Haar measure and Δ the modular function ($d\beta\gamma = \Delta(\gamma)d\beta$) of G. This definition is essentially that of [2, p. 310]. There is also a multiplication between elements f of $C_0(Y)$ (resp. $C_0(X)$, $C_0(G)$) and elements g of $C_0(X \times G)$ given by

$$f * g(x, \gamma) = \int_{G_x} f(x, \sigma) g(x, \sigma^{-1}\gamma)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2} d\sigma \tag{1.4}$$

$$fg(x, \gamma) = f(x) g(x, \gamma) \tag{1.5}$$

$$f * g(x, \gamma) = \int_G f(\beta) g(\beta^{-1}x, \beta^{-1}\gamma) d\beta \tag{1.6}$$

and there is a norm on $C_0(X \times G)$ given by

$$\| g \|_1 = \int_G \sup \{| g(x, \gamma) | : x \in X\} d\gamma \ . \tag{1.7}$$

THEOREM 1.4. *$C_0(X \times G)$ is a normed $*$-algebra with multiplication, involution and norm given by* (1.2), (1.3) *and* (1.7) *respectively and addition and scalar multiplication defined pointwise; involution is isometric. It is also an algebra over the ring $C_0(Y)$* (resp. $C_0(X)$, $C_0(G)$) *with scalar multiplication given by* (1.4) (resp. 1.5), 1.6)).

THEOREM 1.5. *There is a one-to-one correspondence between bounded (in $\| \cdot \|_1$) representations φ_0 of $C_0(X \times G)$ and representations (φ, P) of G, X. The representation φ_0 which corresponds to φ, P is given by*

$$\varphi_0(f) = \int_G \int_X f(x, \gamma) dP(x) \varphi(\gamma) d\gamma \ . \tag{1.8}$$

The images of φ_0 and of the corresponding (φ, P) generate the same von Neumann algebra. φ_0 is norm decreasing ($\| \varphi_0(f) \| \leqq \| f \|_1$). A unitary operator implements an equivalence between representations φ, P and φ', P' of G, X if and only if it implements an equivalence between the corresponding φ_0 and φ_0'.

THEOREM 1.6. *There is a "canonical procedure" for extending representations (φ, P) of G, X to representations (φ, R) of G, Z.*

If $z \in Z$, let φ be an irreducible representation of $\mathfrak{R}$ with kernel z. Let $x = \pi(z)$ be the x determined by Lemma 1.1A. If E is a closed subset of X then $\pi^{-1}(E) = \{z : f\mathfrak{R} \subset z \text{ if } f(E) = 0, f \in C_0(X)\}$ and is closed. Thus π is continuous and $\pi^{-1}(E)$ is a Borel set if E is. That R extends P means that $R(\pi^{-1}(E)) = P(E)$ for all Borel subsets E of X.

Proof of Theorem 1.4. Let f and g be in $C_0(X \times G)$. Then

$$f^{**}(x, \gamma) = f^*(\gamma^{-1}x, \gamma^{-1})^- \Delta(\gamma^{-1}) = f(x, \gamma)$$

and

$$\begin{aligned}(f*g)^*(x, \gamma) &= \Delta(\gamma^{-1})\int_G f(\gamma^{-1}x, \beta)^- g(\beta^{-1}\gamma^{-1}x, \beta^{-1}\gamma^{-1})^- d\beta \\ &= \int_G g(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) f(\gamma^{-1}x, \gamma^{-1}\beta)^- \Delta(\gamma^{-1}\beta) d\beta \\ &= \int_G g^*(x, \beta) f^*(\beta^{-1}x, \beta^{-1}\gamma) d\beta = (g^* * f^*)(x, \gamma)\end{aligned}$$

and (1.3) defines an involution. Suppose that $x \to d\mu(x, \gamma)$ is a function from X to the finite measures on G which is w^*-continuous and is such that $\bigcup_{x \in X}$ support $d\mu(x, \gamma)$ is contained a compact set. If $f \in C_0(X \times G)$, define $\mu * f$ by the formula

$$\mu * f(x, \gamma) = \int f(\beta^{-1}x, \beta^{-1}\gamma) d\mu(x, \beta) .$$

Then $\mu * f$ has compact support, and by Lemma 1.1, $\mu * f \in C_0(X \times G)$. Furthermore

$$\begin{aligned}(\mu * (f * g))(x, \gamma) &= \int f * g(\alpha^{-1}x, \alpha^{-1}\gamma) d\mu(x, \alpha) \\ &= \int\int_G f(\alpha^{-1}x, \beta) g(\beta^{-1}\alpha^{-1}x, \beta^{-1}\alpha^{-1}\gamma) d\beta d\mu(x, \alpha) \\ &= \int\int_G f(\alpha^{-1}x, \alpha^{-1}\beta) g(\beta^{-1}x, \beta^{-1}\gamma) d\beta d\mu(x, \alpha) \\ &= \int_G \mu * f(x, \beta) g(\beta^{-1}x, \beta^{-1}\gamma) d\beta = ((\mu * f) * g)(x, \gamma) .\end{aligned}$$

In particular if $d\mu(x, \gamma) = h(x, \gamma)d\gamma$, $h \in C_0(X \times G)$ then this proves that multiplication is associative. If h_1 and h_2 are in $C_0(Y)$, then the case $d\mu(x, \sigma) = h_1(x, \sigma)[\Delta_x(\sigma)/\Delta(\sigma)]^{1/2} d(x, \sigma)$ proves that $h_1 * (f * g) = (h_1 * f) * g$. Let $\omega(x, \sigma) = [\Delta_x(\sigma)/\Delta(\sigma)]^{1/2}$. The formula $h_1 * (h_2 * g) = (h_1 * h_2) * g$ follows from the associative law in the measure algebra of G and the fact that $\omega(h_1 * h_2) = (\omega h_1) * (\omega h_2)$. The remaining algebraic assertions of Theorem 1.4 are easy to verify.

The function $\sup\{| g(x, \gamma) | : x \in X\}$ is a lower semicontinuous function of γ and so is measurable. It is bounded and has compact support and so is integrable. If $f, g \in C_0(X \times G)$

$$\begin{aligned}||f * g||_1 &= \int_G \sup_{x \in X} \left| \int_G f(x, \beta) g(\beta^{-1}x, \beta^{-1}\gamma) d\beta \right| d\gamma \\ &\leqq \int_G \int_G \sup_{x \in X} | f(x, \beta) | \sup_{x \in X} | g(\beta^{-1}x, \beta^{-1}\gamma) | d\beta d\gamma = ||f||_1 \, ||g||_1 .\end{aligned}$$

LEMMA 1.7.[2] *Let $\mathfrak{A}$ be a normed $*$-algebra, let $\mathfrak{B}$ be a $*$-algebra and let θ be a representation of $\mathfrak{B}$ as bounded operators on $\mathfrak{A}$ such that $a_1^*(\theta(b)a_2) = (\theta(b^*)a_1)^*a_2$ for a_1, a_2 in $\mathfrak{A}$ and b in $\mathfrak{B}$. Let φ be a continuous representation of $\mathfrak{A}$. Then there is a unique representation ψ of $\mathfrak{B}$ such that*

$$\psi(b)\varphi(a) = \varphi(\theta(b)a) \tag{1.9}$$

for a in $\mathfrak{A}$ and b in $\mathfrak{B}$. Moreover $\|\psi(b)\| \leqq \|\theta(b^*b)\|^{1/2}$ and $\psi(\mathfrak{B})$ is contained in the weak closure of $\varphi(\mathfrak{A})$.

There is at most one representation ψ satisfying (1.9). If A' commutes with $\varphi(\mathfrak{A})$ then $A'\psi(b)\varphi(a) = \psi(b)\varphi(a)A' = \psi(b)A'\varphi(a)$ and A' commutes with $\psi(\mathfrak{B})$. By the double commutant theorem, $\psi(\mathfrak{B})$ is in the weak closure of $\varphi(\mathfrak{A})$.

To prove the existence of $\psi(b)$ it is sufficient to consider the case where the representation space $\mathfrak{H}$ of φ has a vector x which is cyclic with respect to $\varphi(\mathfrak{A})$. Let a be in $\mathfrak{A}$, b be in $\mathfrak{B}$. Then

$$\begin{aligned}\|\varphi(\theta(b)a)x\| &= (\varphi((\theta(b)a)^*\theta(b)a)x, x)^{1/2}\\ &= (\varphi(a^*\theta(b^*b)a)x, x)^{1/2}\\ &= (\varphi(\theta(b^*b)a)x, \varphi(a)x)^{1/2}\\ &\leqq \|\varphi(\theta(b^*b)a)x\|^{1/2}\|\varphi(a)x\|^{1/2}.\end{aligned}$$

Iterating this inequality, we have

$$\begin{aligned}\|\varphi(\theta(b)a)x\| &\leqq \|\varphi(\theta(b^*b)^{2^{n}-1}a)x\|^{2^{-n}}\|\varphi(a)x\|^{1-2^{-n}}\\ &\leqq \|\varphi\|^{2^{-n}}\|\theta(b^*b)\|^{1/2}\|a\|^{2^{-n}}\|x\|^{2^{-n}}\|\varphi(a)x\|^{1-2^{-n}},\end{aligned}$$

and taking limits, $\|\varphi(\theta(b)a)x\| \leqq \|\theta(b^*b)\|^{1/2}\|\varphi(a)x\|$. Thus (1.9) is an unambiguous definition of $\psi(b)$ on $\varphi(\mathfrak{A})x$, $\psi(b)$ is bounded and has a unique bounded extension, $\psi(b)$, defined on all of $\mathfrak{H}$.

Formula (1.9) shows that ψ is linear and multiplicative. $\psi(b)^* = \psi(b^*)$ since $\varphi(a_1)^*\psi(b)\varphi(a_2) = \varphi(a_1^*\theta(b)a_2)) = \varphi((\theta(b^*)a_1)^*a_2) = (\psi(b^*)\varphi(a_1))^*\varphi(a_2)$. $\psi(\mathfrak{B})\mathfrak{H}$ is dense in $\mathfrak{H}$ since $\theta(\mathfrak{B})\mathfrak{A}$ is dense in $\mathfrak{A}$, since φ is bounded and since $\varphi(\mathfrak{A})\mathfrak{H}$ is dense in $\mathfrak{H}$. Thus ψ is a representation and the proof is complete.

Proof of Theorem 1.5. The integral $\int_X f(x, \gamma)dP(x)$ is the ordinary uniformly convergent spectral integral; it is by definition the uniform limit of approximating sums $\sum_{i=1}^{n} P(E_i)f(x_i, \gamma)$, where X is a disjoint union of the Borel sets $E_1, \cdots, E_n$ and $x_i \in E_i$. Since f is continuous

[2] We are indebted to R. Blattner for this lemma and its proof. This replaced considerably more complicated arguments, some of which were in the spirit of [**13**, §5 and 6] and appeared to be limited to separable situations.

and has compact support, the integral $\int_X f(x,\gamma)dP(x)$ exists and is a continuous function (in the operator norm) of γ with compact support. Thus $\varphi_0(f)$ exists; $\|\varphi_0(f)\| \leqq \|f\|_1$ follows from the fact that

$$\left\|\int_X f(x,\gamma)dP(x)\right\| \leqq \sup\{|f(x,\gamma)| : x \in X\} .$$

To show that φ_0 is a representation, let f and g be in $C_0(X \times G)$ and let p and q be in $\mathfrak{H}(\varphi)$. Then

$$\begin{aligned}
(\varphi_0(f*g)p, q) &= \int_G \Big(\int_X\int_G f(x,\beta)g(\beta^{-1}x, \beta^{-1}\gamma)d\beta dP(x)\varphi(\gamma)p, q\Big)d\gamma \\
&= \int_G \lim_{\{E_1,\cdots,E_n\}} \sum_{i=1}^n \Big(P(E_i)\int_G f(x_i,\beta)g(\beta^{-1}x, \beta^{-1}\gamma)d\beta\varphi(\gamma)p, q\Big)d\gamma \\
&= \int_G\int_G \lim_{\{E_1,\cdots,E_n\}} \sum_{i=1}^n (P(E_i)f(x_i,\beta)g(\beta^{-1}x_i, \beta^{-1}\gamma)\varphi(\gamma)p, q)d\gamma d\beta \\
&= \int_G\int_G \lim_{\{E_1,\cdots,E_n\}} \sum_{i=1}^n \Big(P(E_i)f(x_i,\beta)\varphi(\beta)\sum_{j=1}^n P(\beta^{-1}E_j)g(\beta^{-1}x_j,\gamma)\varphi(\gamma)p, q\Big)d\gamma d\beta \\
&= \int_G\int_G \Big(\int_X f(x,\beta)dP(x)\varphi(\beta)\int_X g(x,\gamma)dP(x)\varphi(\gamma)p, q\Big)d\gamma d\beta \\
&= (\varphi_0(f)\varphi_0(g)p, q)
\end{aligned}$$

and

$$\begin{aligned}
(\varphi_0(f^*)p, q) &= \int_G\Big(\int_X f(\gamma^{-1}x, \gamma^{-1})^{-1}\Delta(\gamma^{-1})dP(x)\varphi(\gamma)p, q\Big)d\gamma \\
&= \int_G\Big(\int_X f(\gamma x, \gamma)^{-}dP(x)\varphi(\gamma^{-1})p, q\Big)d\gamma \\
&= \int_G\Big(p, \varphi(\gamma)\int_X f(\gamma x,\gamma)dP(x)p\Big)d\gamma \\
&= \int_G\Big(p, \int_X f(x,\gamma)dP(x)\varphi(\gamma)q\Big)d\gamma = (p, \varphi_0(f)q)
\end{aligned}$$

since $\varphi(\gamma)\int_X h(\gamma x)dP(x)\varphi(\gamma^{-1}) = \int_X h(x)dP(x)$ for any h in $C_0(X)$, as is seen by considering approximating sums to the spectral integrals. Let h be in $C_0(G)$ with support K, and let h_n be a net in $C_0(X)$ which eventually has the value one on each compact subset of X, and suppose $0 \leqq h_n \leqq 1$. Then $\int_X h_n(x)dP(x)$ converges strongly to I and so

$$\int_X h_n(x)dP(x)\varphi(\gamma)p$$

converges to $\varphi(\gamma)p$ uniformly for all γ in K. Thus

$$\begin{aligned}
&|(\varphi_0(h_nh)p - \varphi(h)p, q)| \\
&\quad = \left|\int_G\Big(\int_X h_n(x)h(\gamma)dP(x)\varphi(\gamma) - h(\gamma)\varphi(\gamma)p, q\Big)d\gamma\right|
\end{aligned}$$

$$\leqq \sup_{\gamma \in K} | h(\gamma) | \sup_{\gamma \in K} \left\| \int_X h_n(x) dP(x) \varphi(\gamma) p - \varphi(\gamma) p \right\| \| q \| \int_K d\gamma$$

and so $\varphi_0(h_n h)p \to \varphi(h)p$ strongly. This proves that the set $\varphi_0(C_0(X \times G))\mathfrak{H}(\varphi)$ is dense in $\mathfrak{H}(\varphi)$ and since φ_0 is linear, it is a representation. Since the integrals with respect to dP and $d\gamma$ are weak limits of approximating sums, $\varphi_0(C_0(X \times G))$ lies in the von Neumann algebra generated by the images of φ and P. We have also proved that $\varphi(C_0(G))$ (and so $\varphi(G)$) lies in the weak closure of $\varphi_0(C_0(X \times G))$.

Suppose we are given a representation ψ_0 of $C_0(X \times G)$ which is continuous in $\| \cdot \|_1$. In Lemma 1.7 let $\mathfrak{B}$ be the algebra $C_0(X)$ (resp. $C_0(G)$) and let θ be the multiplication defined by (1.5) (resp. 1.6)). If $e, f \in C_0(X \times G)$, $g \in C_0(X)$ and $h \in C_0(G)$ then

$$\begin{aligned} e^* * (gf)(x, \gamma) &= \int e(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) g(\beta^{-1}x) f(\beta^{-1}x, \beta^{-1}\gamma) d\beta \\ &= \int (g^- e)(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) f(\beta^{-1}x, \beta^{-1}\gamma) d\beta \\ &= (g^- e)^* * f(x, \gamma) , \end{aligned}$$

and $e^* * (h * f) = (h^* * e)^* * f$. To prove the latter formula one could either compute the integrals in question or, as is easier, observe that the formula is true for h in $C_0(X \times G)$ and then approximate h in $C_0(G)$ by elements of $C_0(X \times G)$. Moreover $\| \theta \| \leqq 1$ in both cases. By Lemma 1.7 there are representations ψ of $C_0(G)$ and ψ_1 of $C_0(X)$ such that $\psi_1(g)\psi_0(f) = \psi_0(gf)$, $\psi(h)\psi_0(f) = \psi_0(h * f)$. Since ψ is continuous it comes from a representation ψ of G, and $\psi(\gamma)\psi(h) = \psi(h(\gamma^{-1} \cdot))$. If we let h run through an approximate identity and use the formula $h(\gamma^{-1} \cdot) * f(x, \alpha) = h * f(\gamma^{-1}x, \gamma^{-1}\alpha)$, we conclude that $\psi(\gamma)\psi_0(f) = \psi_0(f(\gamma^{-1} \cdot, \gamma^{-1} \cdot))$. This implies $\psi(\gamma)\psi_1(g)\psi_0(f) = \psi_1(g(\gamma^{-1} \cdot))\psi_0(f(\gamma^{-1} \cdot, \gamma^{-1} \cdot)) = \psi_1(g(\gamma^{-1} \cdot))\psi(\gamma)\psi_0(f)$ and $\psi(\gamma)\psi_1(g)\psi(\gamma^{-1}) = \psi_1(g(\gamma^{-1} \cdot))$. By standard methods (compare [9, p. 93, Theorem], [7, p. 239, Theorem D], or Theorem 1.9), ψ_1 can be extended uniquely to a regular countably additive projection valued measure P on X. Let K_E be the characteristic function of a Borel set E. Since $K_E(\gamma^{-1} \cdot) = K_{\gamma E}(\cdot)$, $\psi(\gamma)P(E)\psi(\gamma^{-1}) = P(\gamma E)$ and (ψ, P) is a representation of (G, X). It follows from Lemma 1.7 that $\psi(C_0(X))$ is contained in the weak closure of $\psi_0(C_0(X \times G))$ and by monotone limits, this is also true for the range of P.

Let φ_0 be defined by (1.8) (with φ replaced by ψ), let $f \in C_0(X)$, $g \in C_0(G)$, $h \in C_0(X \times G)$. Then $fg \in C_0(X \times G)$ and the finite linear combinations of such elements of $C_0(X \times G)$ are dense in $C_0(X \times G)$. If $q, r \in \varphi_0(C_0(X \times G))\mathfrak{H}(\psi_0)$ then

$$(\varphi_0(fg)\psi_0(h)q, r) = \left(\int_G \int_X f(x) g(\gamma) dP(x) \psi(\gamma) d\gamma \psi_0(h) q, r \right)$$

$$= \int_G (\psi_1(f)g(\gamma)\psi(\gamma)\psi_0(h)q, r)d\gamma$$
$$= \int_G (\psi_0(f(\cdot)g(\gamma)h(\gamma^{-1}\cdot, \gamma^{-1}\cdot))q, r)d\gamma$$
$$= \left(\psi_0\left(\int_G f(\cdot)g(\gamma)h(\gamma^{-1}\cdot, \gamma^{-1}\cdot)d\gamma\right)q, r\right)$$
$$= (\psi_0((fg)*h)q, r) = (\psi_0(fg)\psi_0(h)q, r)$$

and so $\varphi_0 = \psi_0$. Thus the correspondence defined by (1. 8) is onto from representations of G, X to representations of $C_0(X \times G)$; one can also check that it is one-to-one. The statement concerning unitary equivalence is verified by a direct computation.

THEOREM 1.8. *If φ, P is a representation of G, X then the formula*

$$\varphi_1(f)\varphi_0(g) = \varphi_0(f*g) \tag{1.10}$$

where $f \in C_0(Y)$, $g \in C_0(X \times G)$ and φ_0 is defined by Theorem 1.5, defines a representation φ_1 of $\mathfrak{K}$. The image of φ_1 lies in the von Neumann algebra generated by the images of φ and P.

Let the $\mathfrak{A}$ (resp. $\mathfrak{B}$) in Lemma 1.7 be $C_0(X \times G)$ (resp. $C_0(Y)$) and let θ be the multiplication defined by (1.4). Let e, g be in $C_0(X \times G)$ and let f be in $C_0(Y)$. Then

$$e^* * (f*g)(x, \gamma)$$
$$= \int_G \int_{G_{\beta^{-1}x}} e(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) f(\beta^{-1}x, \sigma) \cdot g(\beta^{-1}x, \sigma^{-1}\beta^{-1}\gamma)[\Delta_{\beta^{-1}x}(\sigma)\Delta(\sigma^{-1})]^{1/2} d\sigma d\beta$$
$$= \int_G \int_{G_x} c(x, \beta)e(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) f(\beta^{-1}x, \beta^{-1}\sigma\beta) \cdot g(\beta^{-1}x, \beta^{-1}\sigma^{-1}\gamma)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2} d\sigma d\beta$$
$$= \int_G \int_{G_x} c(x, \beta)e(\beta^{-1}x, \beta^{-1}\sigma)^- \Delta(\beta^{-1}) f(\beta^{-1}x, \beta^{-1}\sigma\beta) \cdot g(\beta^{-1}x, \beta^{-1}\gamma)[\Delta_x(\sigma^{-1})\Delta(\sigma)]^{1/2} d\sigma d\beta$$
$$= \int_G \int_{G_{\beta^{-1}x}} e(\beta^{-1}x, \sigma\beta^{-1})^- \Delta(\beta^{-1}) f(\beta^{-1}x, \sigma) \cdot g(\beta^{-1}x, \beta^{-1}\gamma)[\Delta_{\beta^{-1}x}(\sigma^{-1})\Delta(\sigma)]^{1/2} d\sigma d\beta$$
$$= \int_G \int_{G_{\beta^{-1}x}} f^*(\beta^{-1}x, \sigma^{-1})^- e(\beta^{-1}x, \sigma\beta^{-1})^- \Delta(\beta^{-1}) g(\beta^{-1}x, \beta^{-1}\gamma) \cdot \Delta_{\beta^{-1}x}(\sigma^{-1})^{3/2}\Delta(\sigma)^{1/2} d\sigma d\beta$$
$$= \int_G \int_{G_{\beta^{-1}x}} f^*(\beta^{-1}x, \sigma)^- e(\beta^{-1}x, \sigma^{-1}\beta^{-1})^- \Delta(\beta^{-1}) \cdot g(\beta^{-1}x, \beta^{-1}\gamma)[\Delta_{\beta^{-1}x}(\sigma)\Delta(\sigma^{-1})]^{1/2} d\sigma d\beta$$

$$= \int_G f^* * e(\beta^{-1}x, \beta^{-1})^- \Delta(\beta^{-1}) g(\beta^{-1}x, \beta^{-1}\gamma) d\sigma d\beta$$
$$= (f^* * e)^* * g(x, \gamma) ,$$

and

$$\|f * g\|_1 \leqq \int_G \sup_x \int_{G_x} |f(x, \sigma) g(x, \sigma^{-1}\gamma)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2}| \, d\sigma d\gamma$$
$$= \sup_x \int_G \int_{G_x} |f(x, \sigma) g(x, \sigma^{-1}\gamma)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2}| \, d\sigma d\gamma$$

since the function $\gamma \to \int_{Gx} |f(x, \sigma) g(x, \sigma^{-1}\gamma)| \, d\sigma$ is continuous and has compact support for each x in X. We apply Fubini's theorem, substitute $\gamma \to \sigma\gamma$, and conclude that

$$\|f * g\|_1 \leqq \|f(x, \sigma)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2}\|_1 \|g\|_1 . \tag{1.11}$$

Lemma 1.7 shows that (1.10) defines a representation of $C_0(Y)$ and Lemma 1.1, the bound in 1.11) and Lemma 1.7 show that φ_1 is continuous in the inductive limit topology on $C_0(Y)$. By the definition of $\|\cdot\|$, φ_1 is continuous in $\|\cdot\|$ and defines a representation of $\mathfrak{R}$.

Let $\mathfrak{L}$ be the completion of $C_0(X \times G)$ in the norm $\|f\| = \sup \{\|\varphi(f)\|$: φ is a representation of $C_0(X \times G)$ which is continuous in $\|\cdot\|_1\}$. Then $\mathfrak{L}$ is a C^*-algebra. It follows from Theorem 1.8 that the multiplication defined by (1.4) extends to a multiplication between $\mathfrak{R}$ and $\mathfrak{L}$.

THEOREM 1.9. *Let ψ be a representation of a C^*-algebra $\mathfrak{R}$ and let Z be the structure space of $\mathfrak{R}$. If U is an open Borel subset of Z, let $R(U)$ be the projection onto the closed span of*

$$\{\psi(f)p : f \in \bigcap_{Z \sim U} z, \ p \in \mathfrak{H}(\psi)\} .$$

Then R can be extended uniquely to a countably additive projection valued measure on the Borel subsets of Z. The image of R is contained in the center of the weak closure of $\psi(\mathfrak{R})$.

Let $\mathscr{D}$ be the set of proper differences of open sets and let $\mathscr{R}$ be the set of finite disjoint unions of elements of $\mathscr{D}$. By [7, § 5, exercise (2) and (3)], $\mathscr{R}$ is a ring and by [7, § 6, Theorem B] $\mathscr{B}$ is the smallest class of sets containing $\mathscr{R}$ and closed under sequential monotone limits. Thus R has at most one extension to a projection valued Borel measure on Z. $\mathscr{B}$ is the class of Borel sets.

We extend R to $\mathscr{D}$. Let $D_1 = E_1 \sim F_1$ and $D_2 = E_2 \sim F_2$ be in $\mathscr{D}$ where E_i and F_i are open and $E_i \supset F_i$ and suppose $D_1 \supset D_2$. We assert that $R(E_1) - R(F_1) \geqq R(E_2) - R(F_2)$. If $z \in Z$ and $f \in \mathfrak{R}$, let $f(z)$ be the element $f + z$ in the C^*-algebra $\mathfrak{R}/z$. Then $f \in \bigcap \{z: z \in Z \sim U\}$ if and only if $f(z) = 0$ for all z not in U, and in this case we say that f vanishes off U and we let $\mathfrak{J}(U)$ denote the set of all f in $\mathfrak{R}$ which vanish off U. Let p be in Range $R(F_1)$ and let q be in Range $R(E_2) - R(F_2)$. If $f \in \mathfrak{R}$ and f vanishes off F_2 then $\psi(f)q = 0$ and q (resp. p) can be approximated by vectors of the form $\psi(g)q$ (resp. $\psi(h)p$) where g (resp. h) vanishes off E_2 (resp. F_1). Then (p, q) can be approximated by $(p, \psi(h^*g)q)$ which is zero since $h^*g = 0$ off $E_2 \cap F_1 \subset F_2$. Thus $R(F_1) \perp R(E_2) - R(F_2)$. $\mathfrak{J}(E_1) + \mathfrak{J}(F_2)$ is an ideal contained in $\mathfrak{J}(E_1 \cup F_2)$ and its closure $\mathfrak{J}$ is equal to $\mathfrak{J}(E_1 \cup F_2)$ since otherwise $\mathfrak{J}(E_1 \cup F_2)$ has an irreducible representation φ which annihilates $\mathfrak{J}$, φ can be extended to an irreducible representation φ^1 of $\mathfrak{R}$ which annihilates $\mathfrak{J}$ but not $\mathfrak{J}(E_1 \cup F_2)$ and $z = \text{kernel } \varphi^1 \in E_1 \cup F_2$ but $z \notin E_1$ and $z \notin F_2$. Since $E_1 \cup F_2 \supset E_2$, $\mathfrak{J} = \mathfrak{J}(E_1 \cup F_2) \supset \mathfrak{J}(E_2)$. Thus g can be approximated by elements $f_1 + f_2$ of $\mathfrak{R}$, with f_1 in $\mathfrak{J}(E_1)$ and f_2 in $\mathfrak{J}(F_2)$, and q can be approximated by $\psi(f_1)q + \psi(f_2)q = \psi(f_1)q$. This proves that $q \in$ Range $R(E_1)$, $R(E_1) \geqq R(E_2) - R(F_2)$ and $R(E_1) - R(F_1) \geqq R(E_2) - R(F_2)$. If $D_1 = D_2$ then $R(E_1) - R(F_1) = R(E_2) - R(F_2)$, and $R(D)$ is defined unambiguously by the formula $R(D) = R(E_1) - R(F_1)$.

Let $D_1 = E_1 \sim F_1$ and $D_2 = E_2 \sim F_2$ be in $\mathscr{D}$, where $E_i \supset F_i$ and E_i and F_i are open and suppose $D_1 \cap D_2 = \phi$. Let p be in Range $R(D_1)$ and let q be in Range $R(D_2)$. Then p (resp. q) can be approximated by $\psi(f)p$ (resp. $\psi(g)q$) where f(resp g) vanishes off E_1 (resp. E_2). g^*f vanishes off $E_1 \cap E_2 \subset F_1 \cup F_2$ and so g^*f can be approximated by elements $h_1 + h_2$ of $\mathfrak{R}$ with h_i vanishing off F_i. Thus (p, q) can be approximated by $(\psi(g^*f)p, q)$ and by $(\psi(h_1)p + \psi(h_2)p, q)$, which is zero. This proves that $R(D_1) \perp R(D_2)$.

We prove that R is countably additive on $\mathscr{D}$. Let D and D_i, $i = 1, \cdots, \infty$, be in $\mathscr{D}$, let $D = E \sim F$ and $D_i = E_i \sim F_i$ where $E \supset F$, $E_i \supset F_i$ and E, F, E_i and F_i are open and suppose $D = \bigcup_{i=1}^{\infty} D_i$ and suppose the D_i's are disjoint. Then $R(D) \geqq R(D_i)$ and $R(D) \geqq \sum_{i=1}^{\infty} R(D_i)$. To prove $R(D) = \sum_{i=1}^{\infty} R(D_i)$ we assume the contrary and we suppose without loss of generality that $D_1 = \phi = D_2$, $E_1 = E = F_1$ and $E_2 = F = F_2$. Let λ_1, λ_2 and λ_3 be real continuous functions such that $0 \leqq \lambda_i \leqq 1$, $\lambda_i(0) = 0$, $\lambda_i(1) = 1$, $\lambda_1\lambda_2 = \lambda_2$, $\lambda_2\lambda_3 = \lambda_3$, and $\lambda_i(x) > 0$ if $x \in [1/2, 1]$. If $g \in \mathfrak{R}$, if $0 \leqq g \leqq I$, if $p \in \mathfrak{H}(\psi)$ and if $\| \psi(g)p - p \| \leqq \| p \|/3$ then $\psi(\lambda_3(g))p \neq 0$. In fact if $\psi(\lambda_3(g)p = 0$ and if P is the spectral projection for $\psi(g)$ associated with the interval [1/2, 1] then $Pp = 0$ and $\| \psi(g)p \| \leqq \| p \|/2$ and $\| \psi(g)p - p \| \geqq \| p \|/2$. There is by assumption a nonzero p in Range $R(D) - \sum_{i=1}^{\infty} R(D_i)$. We can choose a g in $\mathfrak{R}$ which vanishes off E_1 so that $p_1 = \psi(\lambda_3(g))p \neq 0$. Let $h_1 = \lambda_1(g)$, let $g_1 = \lambda_2(g)$. Let n be a positive integer and suppose inductively that we have chosen

(a) g_n in $\mathfrak{K}$

(b) nonzero vectors $p_1, \cdots, p_n$ in Range $R(D) - \sum_{i=1}^{\infty} R(D_i)$

(c) h_j in $\mathfrak{J}(F_j)$ whenever $p_j \in$ Range $R(F_j)$

in such a manner that if $j \leqq k \leqq n$ then

(i) $p_j \perp$ Range $R(E_j) \Rightarrow p_k \perp$ Range $R(E_j)$

(ii) $p_j \in$ Range $R(F_j) \Rightarrow p_k \in$ Range $R(F_j)$ and $\psi(h_j)p_k = p_k$

(iii) $p_j \in$ Range $R(F_j)$, $p_k \in$ Range $R(F_k)$, and $j < k \Rightarrow h_j h_k = h_k$

(iv) $0 \leqq h_j \leqq I$; $0 \leqq g_n \leqq I$,

and if i is the largest index for which $p_i \in$ Range $R(F_i)$ and if $i \leqq k \leqq n$ then

(v) $h_i g_n = g_n$ and $\psi(g_n)p_k = p_k$.

If $(I - R(E_{n+1}))p_n \neq 0$, let $p_{n+1} = (I - R(E_{n+1}))p_n$ and let $g_{n+1} = g_n$. For each C in $\mathscr{D}$, Range $R(C)$ is invariant under $\psi(\mathfrak{K})$, and since $\psi(\mathfrak{K})$ is closed under the taking of adjoints, $R(C)$ commutes with $\psi(\mathfrak{K})$. $R(C)$ is also a weak limit point of $\psi(\mathfrak{K})$ and so $R(C)$ is in the center of $\psi(\mathfrak{K})^-$, the weak closure of $\psi(\mathfrak{K})$. Using this, it is easy to see that the inductive assumptions are satisfied for $n + 1$. If $(I - R(E_{n+1}))p_n = 0$ then $0 \neq R(F_{n+1})p_n = \psi(g_n)R(F_{n+1})p_n$. Thus there is a g in $\mathfrak{K}$ which vanishes off F_{n+1} such that $p_{n+1} = \psi(\lambda_3(g_n g g_n))R(F_{n+1})p_n \neq 0$. Let $h_{n+1} = \lambda_1(g_n g g_n)$ and let $g_{n+1} = \lambda_2(g_n g g_n)$. Since $\lambda_k(g_n g g_n)$ is a limit of polynomials in $g_n g g_n$, $h_i h_{n+1} = h_{n+1}$, and the remaining inductive assumptions are easy to verify.

Let $\mathfrak{M}$ be the linear subspace of $\mathfrak{K} + \lambda I$ generated by I and h_j if $p_j \in$ Range $R(F_j)$ and $\mathfrak{J}(E_j)$ if $p_j \perp$ Range $R(E_j)$, $j = 1, 2, \cdots$. Let ρ_0 be the linear functional on $\mathfrak{M}$ defined by $\rho_0(I) = 1$, $\rho_0(h_j) = 1$ if $p_j \in$ Range $R(F_j)$ and $\rho_0(\mathfrak{J}(E_j)) = 0$ if $p_j \perp$ Range $R(E_j)$. This definition is consistant and ρ_0 is a state ($=$ positive linear functional normalized by $\rho_0(I) = 1$) of $\mathfrak{M}$, since $\rho_0 = (\lim_n \omega_{p_n} \circ \psi / \| p_n \|^2) \mid \mathfrak{M}$, where ω_{p_n} is the linear functional $A \rightarrow (Ap_n, p_n)$ defined on operators on $\mathfrak{H}(\psi)$. ρ_0 is an extreme point of the set of states of $\mathfrak{M}$. In fact let $\rho_0 = \alpha\tau_1 + (1 - \alpha)\tau_2$, with $\alpha \in (0, 1]$ and τ_1 and τ_2 states. Since $\mathfrak{J}(E_j)$ is generated by its positive elements [16, Lemma 2.3], $\tau_1(\mathfrak{J}(E_j)) = 0$ if $p_j \perp$ Range $R(E_j)$. If $p_j \in$ Range $R(F_j)$ then $\tau_1(h_j) \leqq 1$ and $1 = \alpha\tau_1(h_j) + (1 - \alpha)\tau_2(h_j) \leqq \alpha + 1 - \alpha = 1$. Thus there is equality throughout and $\tau_1(h_j) = 1$, $\tau_1 = \rho_0$, and ρ_0 is an extreme point. ρ_0 can be extended to a state ρ of $\mathfrak{K} + \lambda I$ by a Hahn-Banach type argument and applying the Krein Milman Theorem to the set of such extensions, it is possible to choose ρ to be a pure state (extreme point of the set of states) of $\mathfrak{K} + \lambda I$. The procedure of [15] yields an irreducible representation φ of $\mathfrak{K}$ for which $z =$ kernel φ is the set $\{f: f \in \mathfrak{K}, \rho(g * fh) = 0$ for all g, h in $\mathfrak{K}\}$. If $p_j \in$ Range $R(F_j)$ then $\varphi(h_j) \neq 0$ and so $z \in F_j$. If $p_j \perp$ Range $R(E_j)$ then $\varphi(\mathfrak{J}(E_j)) = 0$ and so $z \notin E_j$.

In particular $z \in F_1 = E$ and $z \notin E_2 = F$. We have proved $z \in D$ but $z \notin D_j$ for any j. This is a contradiction and so $R(D) = \sum_{i=1}^{\infty} R(D_j)$.

Let $F = \bigcup_{i=1}^{m} D_i = \bigcup_{i=1}^{n} E_i$ be in $\mathscr{R}$, where D_i and E_i are in D and $D_i \cap D_j = \phi = E_i \cap E_j$ if $i \neq j$. Then $D_i \cap E_i \in \mathscr{D}$ and

$$\sum_{i=1}^{m} R(D_i) = \sum_{i,j=1}^{m,n} R(D_i \cap E_j) = \sum_{j=1}^{n} R(E_j) .$$

Thus R can be extended to $\mathscr{R}$ by the definition $R(F) = \sum_{i=1}^{m} R(D_i)$, and the same reasoning shows that R is countably additive on $\mathscr{R}$. For each p and q in $\mathfrak{H}(\psi)$, the function $E \rightarrow (R(E)p, q)$ is a measure on $\mathscr{R}$ and can be extended to a measure μ_{pq} on $\mathscr{B}$. If B is a Borel set then there is a unique operator $R(B)$ such that $(R(B)p, q) = \mu_{pq}(B)$ for all p, q. $R(B)$ is a projection and $B \rightarrow R(B)$ is a projection valued measure. If $E \in \mathscr{D}$ then we have already observed that $R(E)$ is in the center of the weak closure of $\psi(\mathfrak{K})$. By finite sums and monotone limits this is true if E is a Borel set

If $\mathfrak{K}$ is separable and type I and if $\mathfrak{H}(\psi)$ is separable then Theorem 1.9 is essentially known and in this case presumably the range of R is all projections in the center of the weak closure of $\psi(\mathfrak{K})$. If $\mathfrak{K}$ is not type I the range of R might not be this large, and in fact might be $\{0, I\}$ even when the weak closure of $\psi(\mathfrak{K})$ is not a factor and is of type I.

R is regular in the sense that for any open U, $R(U)$ is the supremum of the $R(K)$, as K ranges over the compact Borel sets in U. To see this, let p be in $\mathfrak{H}$ and let $f = f^*$ be in $\mathfrak{K}$ and vanish off U. Then $\psi(f)p$ can be approximated by $\psi(g)p$, where $g = g^*$ and g vanishes off $U_\varepsilon = \{z\colon ||f(z)|| > \varepsilon\} \subseteqq \{z\colon ||f(z)|| \geqq \varepsilon\} = K_\varepsilon$. U_ε is open [8, Lemma 4.2] and $\psi(f)p$ can be approximated by $R(U_\varepsilon)p$ and so by $R(K_\varepsilon)p$. K_ε is compact [8, Lemma 4.3] and is a Borel set since $K_\varepsilon = \bigcap_{0<\delta<\varepsilon} U_\delta$.

Proof of Theorem 1.6. Let φ, P be given as in the statement of 1.6, let φ_0 and φ_1 be defined by Theorem 1.5 and 1.8 respectively, and let R be defined by Theorem 1.9 in the case $\psi = \varphi_1$. If $\gamma \in G$, $f \in C_0(Y)$, $g \in C_0(X \times G)$ and $p \in \mathfrak{H}(\varphi)$ then

$$\varphi(\gamma)\varphi_1(f)\varphi(\gamma^{-1})\varphi_0(g)p = (\varphi_1 \circ \gamma_K)(f)\varphi_0(g)p$$

since

$$\begin{aligned} & f*(g(\gamma\,\cdot, \gamma\,\cdot))(\gamma^{-1}x, \gamma^{-1}\beta) \\ &= \int_{G_{\gamma^{-1}x}} f(\gamma^{-1}x, \sigma)g(x, \gamma\sigma^{-1}\gamma^{-1}\beta)[\Delta_{\gamma^{-1}x}(\sigma)\Delta(\sigma^{-1})]^{1/2}d\sigma \\ &= c(x, \gamma)\int_{G_x} f(\gamma^{-1}x, \gamma^{-1}\sigma\gamma)g(x, \sigma^{-1}\beta)[\Delta_x(\sigma)\Delta(\sigma^{-1})]^{1/2}d\sigma \\ &= (\gamma_K(f)*g)(x, \beta) \end{aligned}$$

and since $\varphi(\gamma)\varphi_0(g) = \varphi_0(g(\gamma^{-1}\cdot, \gamma^{-1}\cdot))$. (See the proof of Theorem 1.5.) Let R_γ be the projection valued measure defined on Z by Theorem 1.9 in the case $\psi = \varphi_1 \circ \gamma_K$. If U is an open subset of Z then

$$\begin{aligned} R_\gamma(U)\mathfrak{H}(\varphi) &= \Big\{\varphi_1 \circ \gamma_K(f)\mathfrak{H}(\varphi) \colon f \in \bigcap_{x \in Z \sim U} x\Big\}^- \\ &= \Big\{\varphi_1(f)\mathfrak{H}(\varphi) \colon \gamma_K^{-1}(f) \in \bigcap_{x \in Z \sim U} x\Big\}^- = \Big\{\varphi_1(f)\mathfrak{H}(\varphi) \colon f \in \bigcap_{x \in Z \sim U} \gamma(x)\Big\}^- \\ &= \Big\{\varphi_1(f)\mathfrak{H}(\varphi) \colon f \in \bigcap_{x \in Z \sim \gamma U} x\Big\}^- = R(\gamma U)\mathfrak{H}(\varphi) , \end{aligned}$$

and

$$\begin{aligned} \varphi(\gamma)R(U)\varphi(\gamma^{-1})\mathfrak{H}(\varphi) &= \Big\{\varphi(\gamma)\varphi_1(f)\varphi(\gamma^{-1})\mathfrak{H}(\varphi) \colon f \in \bigcap_{x \in Z \sim U} x\Big\}^- \\ &= R_\gamma(U)\mathfrak{H}(\varphi) . \end{aligned}$$

Both $E \to \varphi(\gamma)R(E)\varphi(\gamma^{-1})$ and $E \to R(\gamma E)$ are projection valued measures which we have just shown to agree with R_γ on open sets. By the uniqueness part of Theorem 1.9, they both are equal to R_γ and thus to each other. This proves that φ, R is a representation of G, Z.

To show that R extends P, it is enough to show this for closed subsets E of X. The range of $I - P(E)$ is the closure of the set of vectors $\int_X f(x)dP(x)p$ where $p \in \mathfrak{H}(\varphi)$, $f \in C_0(X)$ and $f(E) = 0$. This closure is also the closure of the vectors $\varphi_1(fA)p$ where $A \in \mathfrak{K}$ and f and p as before. To see this, use formula (1.10) and choose a suitable approximate identity for $\mathfrak{K}$ in $C_0(Y)$. The element fA of $\mathfrak{K}$ has the property $(fA)(z) = 0$ for z in $\pi^{-1}(E)$. Let B be a self adjoint element of $\mathfrak{K}$ and suppose $B(z) = 0$ for z in $\pi^{-1}(E)$. Let ε be a positive number. Then the set $K = \{z \colon \| B(z) \| \geqq \varepsilon\}$ is a compact subset of $Z \sim \pi^{-1}(E)$ and $\pi(K)$ is a compact subset of X disjoint from E. If g is a function which is one on $\pi(K)$ and zero on E then $\| gB - B \| < \varepsilon$ provided $0 \leqq g \leqq 1$. Thus the range of $I - P(E)$ is the closure of the vectors $\varphi_1(B)p$ where $p \in \mathfrak{H}(\varphi)$, $B \in \mathfrak{K}$ and $B(z) = 0$ for z in $\pi^{-1}(E)$. This is the range of $I - R(\pi^{-1}(E))$ so $R(\pi^{-1}(E)) = P(E)$ and R extends P.

2. Induced representations. It follows from Mackey's work [11] that certain representations of G, X can be constructed in an explicit fashion from the action of G on X; these representations are called induced representations. In this section we determine the topological structure of the space of all irreducible induced representations. This space is homeomorphic to the orbit space $\hat{\mathfrak{K}}/G$. Thus there is a correspondence between properties of $\hat{\mathfrak{K}}/G$ and properties of the induced representations; a simple example of this is Theorem 2.2.

Each φ in $\hat{\mathfrak{K}}$ determines a z in Z, namely $z = \text{kernel}\,\varphi \in Z$ and this z determines an $x = \pi(z)$ in X. $\pi(z)$ is the unique element of X such that all f in $C_0(Y)$ which vanish on $\{x\} \times G_x \subset Y$ are in z. For any f in $C_0(Y)$, $\varphi(f)$ thus depends only on values of f at $\{x\} \times G_x$ and φ defines an irreducible representation φ^1 of $L_1(G_x)$ and so of G_x. If $\tilde{\psi}$ is an irreducible representation of $L_1(G_x)$ for some x in X, then $f \rightarrow \tilde{\psi}(f \mid \{x\} \times G_x)$, f in $C_0(Y)$, defines an irreducible representation ψ of $\mathfrak{K}$, $\pi(\text{kernel}\,\psi) = x$ and $\tilde{\psi} = \psi^1$. The map $\varphi \rightarrow \varphi^1$ preserves unitary equivalence and so $\hat{\mathfrak{K}}$ is in one-to-one correspondence with the pairs x in X and φ^1 in $\hat{G}_x$. The point x determines a correspondence between G/G_x, the right G_x cosets, and the orbit Gx; $G_x\gamma$ corresponds to $\gamma^{-1}x$. This correspondence is a Borel isomorphism since the map $G_x\gamma \rightarrow \gamma^{-1}x$ is one-to-one and continuous and since the restriction of this map to a compact set is a homeomorphism. The induced representation U^{φ^1}, P^{φ^1}, which is a representation of G and G/G_x (G is transformation group acting on G/G_x), defines by means of the correspondence $G_x\gamma \leftrightarrow \gamma^{-1}x$ a representation U^{φ}, P^{φ} of G, X. By means of Theorem 1.5, U^{φ}, P^{φ} define a representation which we shall call Φ of $C_0(X \times G)$ and so of $\mathfrak{L}$. If φ^1 is irreducible, so is the joint action of U^{φ}, P^{φ} [11, § 6] and so is Φ by Theorem 1.5. The map $\varphi^1 \rightarrow U^{\varphi}$, P^{φ} preserves unitary equivalence [11, Theorem 2] as does the map $U^{\varphi}, P^{\varphi} \rightarrow \Phi$ (Theorem 1.5). Thus the map $\varphi \rightarrow \Phi$ is a well defined map of $\hat{\mathfrak{K}}$ into $\hat{\mathfrak{L}}$. We recall that G acts on $\hat{\mathfrak{K}}$ by the map $(\gamma, \varphi) \rightarrow \varphi \cdot \gamma_K^{-1}$.

THEOREM 2.1. *If φ and ψ are in $\hat{\mathfrak{K}}$ then $\Phi = \Psi$ if and only if φ and ψ lie in the same orbit under G, that is if and only if there is a γ in G such that $\psi = \varphi \circ \gamma_K$. The map $\varphi \rightarrow \Phi$ is continuous and the induced map of the orbit space $\hat{\mathfrak{K}}/G$ is a homeomorphism with its image.*

Proof. A. $\psi = \varphi \circ \gamma_K$. Let $\varphi \in \hat{\mathfrak{K}}$ and let $x = \pi(\text{kernel}\,\varphi)$. The Hilbert space $\mathfrak{H}(U^{\varphi})$ is the set of measurable functions f from G to $\mathfrak{H}(\varphi)$ such that $f(\sigma\beta) = \varphi^1(\sigma)f(\beta)$ for σ in G_x and β in G and such that the integral $\int_{G/G_x} \| f(\gamma) \|^2 d\mu(G_x\gamma)$ is finite, where μ is some finite measure on G/G_x which is quasi invariant. If $\psi = \varphi \circ \gamma_K$ then an f in $C_0(Y)$ is in kernel ψ if $\gamma_K(f)$ vanishes on $\{x\} \times G_x$, which occurs if f vanishes on $\{\gamma^{-1}x\} \times G_{\gamma^{-1}x}$. Thus $\pi(\text{kernel}\,\psi) = \gamma^{-1}x$. Let ν be the measure defined on $G/G_{\gamma^{-1}x}$ by means of the formula

$$\int_{G/G_{\gamma^{-1}x}} h(G_{\gamma^{-1}x}\beta)d\nu(G_{\gamma^{-1}x}\beta) = \int_{G/G_x} h(\gamma^{-1}G_x\beta)d\mu(G_x\beta)$$

where $h \in C_0(G/G_{\gamma^{-1}x})$. This makes sense since $\gamma^{-1}G_x\beta = G_{\gamma^{-1}x}\gamma^{-1}\beta$ is a $G_{\gamma^{-1}x}$ coset, and one can see that ν is quasi invariant.

If $f \in \mathfrak{H}(U^{\varphi})$, let $(Uf)(\beta) = f(\gamma\beta)$. Then Uf is a measurable function from G to $\mathfrak{H}(\varphi) = \mathfrak{H}(\psi)$. If $\sigma \in G_{\gamma^{-1}x}$ then $\gamma\sigma\gamma^{-1} \in G_x$ and $(Uf)(\sigma\beta) = f(\gamma\sigma\beta) = \varphi^1(\gamma\sigma\gamma^{-1})f(\gamma\beta) = \varphi^1(\gamma\sigma\gamma^{-1})(Uf)(\beta) = \psi^1(\sigma)(Uf)(\beta)$. The last equality follows from the fact that for g in $C_0(Y)$ and p in $\mathfrak{H}(\varphi)$,

$$\begin{aligned}\psi^1(\sigma)\psi(g)p &= \psi(g(\cdot, \sigma^{-1}\cdot))p = \varphi(c(\cdot, \gamma)g(\gamma^{-1}\cdot, \sigma^{-1}\gamma^{-1}\cdot\gamma))p \\ &= \varphi^1(\gamma\sigma\gamma^{-1})\varphi(c(\cdot, \gamma)g(\gamma^{-1}\cdot, \gamma^{-1}\cdot\gamma))p = \varphi^1(\gamma\sigma\gamma^{-1})\psi(g)p .\end{aligned}$$

If $f_1 \in \mathfrak{H}(U^\varphi)$ also then

$$\int_{G/G_{\gamma^{-1}x}} ((Uf)(\beta), (Uf_1)(\beta))d\nu(G_{\gamma^{-1}x}\beta) = \int_{G/G_x} (f(\beta), f_1(\beta)d\mu(G_x\beta) \tag{2.1}$$

and since the right member of (2.1) is the inner product in $\mathfrak{H}(U^\varphi)$ and the left member is the inner product in $\mathfrak{H}(U^\psi)$, $Uf \in \mathfrak{H}(U^\psi)$ and U is a unitary transformation of $\mathfrak{H}(U^\varphi)$ onto $\mathfrak{H}(U^\psi)$.

Let E be a Borel subset of X. Then $P^\varphi(E)$ (resp. $P^\psi(E)$) is multiplication by the characteristic function of $\{\beta: \beta^{-1}x \in E\}$ (resp. $\{\beta: \beta^{-1}\gamma^{-1}x \in E\}$) and

$$\begin{aligned}(P^\psi(E)Uf)(\beta) &= \chi_E(\beta^{-1}\gamma^{-1}x)f(\gamma\beta) \\ &= U(\chi_E(\cdot^{-1}x)f)(\beta) = U(P^\varphi(E)f)(\beta) ,\end{aligned}$$

where χ_E is the characteristic function of E. Let α be in G. The definition of $U^\varphi(\alpha)f = U^\varphi(\alpha)f$ is

$$U^\varphi(\alpha)f(\beta) = f(\beta\alpha)(\lambda(G_x\beta, \alpha))^{1/2} ,$$

where $\lambda(\cdot, \alpha)$ is a Radon Nikodym derivative of the measure $E \to \mu(E\alpha)$ with respect to μ. Then $\lambda(\gamma\cdot, \alpha)$ is a Radon Nikodym derivative of the measure $E \to \nu(E\alpha)$ with respect to ν and

$$\begin{aligned}(U^\psi(\alpha)Uf)(\beta) &= f(\gamma\beta\alpha)(\lambda(\gamma G_{\gamma^{-1}x}\beta, \alpha))^{1/2} \\ &= f(\gamma\beta\alpha)(\lambda(G_x\gamma\beta, \alpha))^{1/2} = (UU^\varphi(\alpha)f)(\beta) .\end{aligned}$$

Thus U^φ, P^φ is equivalent to U^ψ, P^ψ and so Φ is equivalent to Ψ.

B. $\Phi = \Psi$. Let φ and ψ be in $\hat{\mathfrak{K}}$ and suppose that Φ is unitarily equivalent to Ψ. Let $x = \pi(\text{kernel}\,\varphi)$ and let $y = \pi(\text{kernel}\,\psi)$. $P^\varphi(Gx)$ is multiplication by the characteristic function of $\{\beta: \beta^{-1}x \in Gx\}$ and so $P^\varphi(Gx) = I$ and likewise $P^\psi(Gy) = I$. (Gx is a Borel set since it is a countable union of compact sets.) Since P^φ and P^ψ are equivalent, $P^\varphi(Gy) = I$, $P^\varphi(Gx \cap Gy) = I$, $Gx \cap Gy \neq \phi$ and $Gx = Gy$. Suppose $y = \gamma x$, $\gamma \in G$, and let $\omega = \psi \circ \gamma_K$. Then Ω is equivalent to Ψ by A, and so is equivalent to Φ. Thus U^{φ^1}, P^{φ^1} is equivalent to U^{ω^1}, P^{ω^1} and by [11, Theorem 2], ω^1 is equivalent to φ^1 and so ω is equivalent to φ. Thus φ and ψ have the same orbits under G.

C. The continuity of $\varphi \to \Phi$. The unitary equivalence class of the induced representation is independent of the choice of the quasi-invariant measure μ on G/G_x. We make the choice $\mu = \mu_x$, where μ_x is defined by the formula

$$\int_G f(\gamma)c(x, \gamma)^{-1}d\gamma = \int_{G/G_x}\int_{G_x} f(\sigma\gamma)\Delta_x(\sigma^{-1})d\sigma d\mu_x(G_x\gamma) , \tag{2.2}$$

and $f \in C_0(G)$. That (2.2) defines such a μ_x follows from Lemma 1.5 of [12] and its proof, and it is also shown there that $\Delta(\gamma)c(\cdot^{-1}x, \gamma)^{-1}$ is a Radon Nikodym derivative of the translated measure $E \to \mu_x(E\gamma)$ with respect to μ_x.

LEMMA. *Let M be a compact symmetric subset of G and let s be a nonnegative element of $C_0(G)$ which is positive on M. Then the function $t(x, \gamma) = s(\gamma)[c(x, \gamma)\int_{G_x} s(\sigma\gamma)\Delta_x(\sigma^{-1})d\sigma]^{-1}$ is defined and continuous on the subset $\{(x, \gamma): \gamma^{-1}x \in Mx\}$ of $X \times G$. If $x \in X$ and g is a bounded Borel function on G/G_x and if support $g \subset G_xM$ then*

$$\int_{G/G_x} g(\gamma^{-1}x)d\mu_x(G_x\gamma) = \int_G t(x, \gamma)g(\gamma^{-1}x)d\gamma . \tag{2.3}$$

It is easy to see that t is defined and continuous. If g is continuous then formula (2.3) follows from (2.2). The general case in which g is a bounded Borel function follows by taking monotone limits.

Let φ^m be a net of irreducible representations of $\mathfrak{R}$ converging to an irreducible representation ψ. Let $x_m = \pi(\text{kernel}\,\varphi^m)$, let $y = \pi(\text{kernel}\,\psi)$. If U is a neighborhood of y and if h is a function in $C_0(X)$ which is zero outside U and is one at y and if $x_m \notin U$ then $h\mathfrak{R} \subset \text{kernel}\,\varphi^m$. The set $\{\varphi: h\mathfrak{R} \not\subset \text{kernel}\,\varphi\}$ is a neighborhood of ψ and so for large m, $h\mathfrak{R} \not\subset \text{kernel}\,\varphi^m$ and $x_m \in U$. Thus $x_m \to y$. The topology of $\hat{\mathfrak{R}}$ can be described in terms of w^* convergence of linear functionals, and in particular there are vectors v_m in $\mathfrak{H}(\varphi^m)$ and a w in $\mathfrak{H}(\psi)$ such that $\|v_m\| = 1 = \|w\|$ and such that the linear functionals $(\varphi^m(\cdot)v_m, v_m)$ converge in the w^* topology to $(\psi(\cdot)w, w)$.

If $f \in C_0(X \times G)$, let $f^0(\gamma)(x, \sigma) = f(x, \sigma^{-1}\gamma)$. Then $f^0(\gamma) \in C_0(Y)$ and $\gamma \to f^0(\gamma)$ is continuous in the norm $\|\cdot\|_1$ and so in the norm $\|\cdot\|$. Let $\varphi^{m\prime}$ be the representation of G_{x_m} determined by φ^m. By [12, Lemma 3.1], if

$$V_m(\gamma) = \varphi^m(f^0(\gamma))v_m = \int_{G_{x_m}} f(x_m, \sigma^{-1}\gamma)\varphi^{m\prime}(\sigma)v_m d\sigma$$

then $V_m \in \mathfrak{H}(U^{\varphi^m})$ and likewise $W = (\gamma \to \psi(f^0(\gamma))w)$ is in $\mathfrak{H}(U^\psi)$. We suppose that $W \not\equiv 0$. This is the case for example if f is nonnegative and has its support near $X \times e$. If β and γ are in G then

$$((U^{\varphi^m}(\gamma)V_m)(\beta), V_m(\beta)) = (V_m(\beta\gamma), V_m(\beta))[\Delta(\gamma)c(\beta^{-1}x_m, \gamma)^{-1}]^{1/2}$$

$$
\begin{aligned}
&= (\varphi^m(f^0(\beta)^* * f^0(\beta\gamma))v_m, v_m)[\Delta(\gamma)c(\beta^{-1}x_m, \gamma)^{-1}]^{1/2} \\
(2.4)\qquad &\to (\psi(f^0(\beta)^* * f^0(\beta\gamma)w, w)[\Delta(\gamma)c(\beta^{-1}y, \gamma)^{-1}]^{1/2} \\
&= (W(\beta\gamma), W(\beta))[\Delta(\gamma)c(\beta^{-1}y, \gamma)^{-1}]^{1/2} = ((U^{\psi}(\gamma)W)(\beta), W(\beta))
\end{aligned}
$$

and the convergence in (2.4) is uniform for β and γ in compact sets.

Let g be in $C_0(X \times G)$, let M be a compact symmetric subset of G such that support $f \subset X \times M$ and let $t(x, \gamma)$ be chosen by the lemma. If $\beta \notin G_{x_m} M$ then $V_m(\beta) = 0$ and we have

$$
\begin{aligned}
(\Phi^m(g)V_m, V_m) &= \int_G \Big(\int_X g(x, \gamma)dP^{\varphi^m}(x)U^{\varphi^m}(\gamma)V_m, V_m \Big) d\gamma \\
&= \int_G \int_{G/G_{x_m}} (g(\beta^{-1}x_m, \gamma)(U^{\varphi^m}(\gamma)V_m)(\beta), V_m(\beta))d\mu_{x_m}(G_{x_m}\beta)d\gamma \\
&= \int_G \int_G t(x_m, \beta)(g(\beta^{-1}x_m, \gamma)(U^{\varphi^m}(\gamma)V_m)(\beta), V_m(\beta))d\beta d\gamma \\
&\to \int_G \int_G t(y, \beta)(g(\beta^{-1}y, \gamma)(U^{\psi}(\gamma)W)(\beta), W(\beta))d\beta d\gamma \\
&= \int_G \int_{G/G_y} (g(\beta^{-1}y, \gamma)(U^{\psi}(\gamma)W)(\beta), W(\beta))d\mu_y(G_y\beta)d\gamma \\
&= (\Psi(g)W, W) .
\end{aligned}
$$

This implies that $\Phi^m \to \Psi$ and proves C.

D. The induced map is a homeomorphism. It follows from what we have proved that the map from $\hat{\mathfrak{R}}/G$ into $\hat{\mathfrak{L}}$ induced by the map $\varphi \to \Phi$ is one-to-one and continuous. Let K be a closed G-invariant subset of $\hat{\mathfrak{R}}$ and let $L = \{\Phi; \varphi \in K\}$. To complete the proof we must show that L is relatively closed in the image of $\hat{\mathfrak{R}}$.

Let ψ be in $\hat{\mathfrak{R}}$, let Ψ be the corresponding element of $\hat{\mathfrak{L}}$, let $\pi(\text{kernel}\,\psi) = y$, let g be in $C_0(Y)$, let h be in $C_0(X \times G)$ and let V and W be in $\mathfrak{H}(U^{\psi})$. Then

$$
\begin{aligned}
&(\Psi(g * h)W, V) \\
&\quad = \int_G \int_{G/G_y} (g * h)(\beta^{-1}y, \gamma)((U^{\psi}(\gamma)W)(\beta), V(\beta))d\mu_y(G_y\beta)d\gamma \\
&\quad = \int_G \int_{G/G_y} \int_{G_{\beta^{-1}y}} g(\beta^{-1}y, \sigma)h(\beta^{-1}y, \sigma^{-1}\gamma)((U^{\psi}(\gamma)W)(\beta), V(\beta)) \\
&\qquad\quad \cdot [\Delta_{\beta^{-1}y}(\sigma)/\Delta(\sigma)]^{1/2} d\sigma d\mu_y(G_y\beta)d\gamma .
\end{aligned}
$$

The above integral is absolutely convergent and so we can interchange orders of integration, placing the integration with respect to γ first. If we substitute $\sigma\gamma$ for γ, place the γ integration last again, and then use the substitution $\sigma \to \beta^{-1}\sigma\beta$ as in (1.1), we obtain

$$(\Psi(g * h)W, V)$$

$$= \int_G \int_{G/G_y} \int_{G_y} \beta_K(g)(y, \sigma)h(\beta^{-1}y, \gamma)((U^{\psi}(\beta^{-1}\sigma\beta\gamma)W)(\beta), V(\beta))$$
$$\cdot [\Delta_y(\sigma)/\Delta(\sigma)]^{1/2} d\sigma d\mu_y(G_y\beta)d\gamma$$
$$= \int_G \int_{G/G_y} \int_{G_y} \beta_K(g)(y, \sigma)h(\beta^{-1}y, \gamma)((\psi(\sigma)U^{\psi}(\gamma)W)(\beta), V(\beta))$$
$$\cdot d\sigma d\mu_y(G_y\beta)d\gamma$$
$$= \int_G \int_{G/G_y} h(\beta^{-1}y, \gamma)((U^{\psi}(\gamma)W)(\beta), \psi \circ \beta_K(g^*)V(\beta))d\mu_y(G_y\beta)d\gamma .$$

Since the function $\beta \to \psi \circ \beta_K(g^*)V(\beta)$ is in $\mathfrak{H}(U^{\psi})$,

$$(\Psi(g * h)W, V) = \int_{G/G_y} ((\Psi(h)W)(\beta), \psi \circ \beta_K(g^*)V(\beta))d\mu_y(G_y\beta)$$
$$= \int_{G/G_y} (\psi \circ \beta_K(g)(\Psi(h)W)(\beta), V(\beta))d\mu_y(G_y\beta) ,$$

and by limits converging in the norm in $\mathfrak{R}$, this is true for g in $\mathfrak{R}$.

Let $\mathfrak{J} = \{g; g \in \mathfrak{R}$ and $\varphi(g) = 0$ for all φ in $K\}$. If $\Psi \in L$ then $\Psi(\mathfrak{J} * \mathfrak{L}) = 0$ by the above calculations. Now suppose Ψ is a limit point of L. Then $\Psi(\mathfrak{J} * \mathfrak{L}) = 0$ also. Since $\Psi(\mathfrak{L})$ contains a norm bounded sequence converging strongly to I, if $g \in \mathfrak{J}$ and $V \in \mathfrak{H}(U^{\psi})$ then $\psi \circ \beta_K(g)V(\beta) = 0$ for a.e. β. If we choose V continuous then $\beta \to \psi \circ \beta_K(g)V(\beta)$ is continuous also; this can be seen directly if $g \in C_0(Y)$ and by taking uniform limits otherwise. For such V, $\psi \circ \beta_K(g)V(\beta) = 0$ for all β. By [12, Lemma 3.2], this implies that $\psi \circ \beta_K(g) = 0$ and in particular that $\psi(\mathfrak{J}) = 0$. By the definition of the hull-kernel topology, $\psi \in K^- = K$, $\Psi \in L$ and L is relatively closed. This completes the proof of Theorem 2.1.

If $x \in X$ let φ_x be the one-dimensional representation $f \to \int_{G_x} f(x, \sigma)d\sigma$, $f \in C_0(Y)$. Then φ_x can be extended to $\mathfrak{R}$, $\varphi_x \in \hat{\mathfrak{R}}$, kernel $\varphi_x \in Z$ and $x \to$ kernel φ_x is a homeomorphism of X with its image in Z. This image is invariant under G and so X/G is countably separated (there are G invariant Borel sets $E_1, E_2, \cdots$ in X which separate points of X/G) if Z/G is. However one might be interested only in representations induced from a subset K of $\hat{\mathfrak{R}}$ or of Z, and it is possible that K/G is countably separated when X is not.

THEOREM 2.2. *Let K be a closed G-invariant subset of $\hat{\mathfrak{R}}$ and let L be the closure of its image in $\hat{\mathfrak{L}}$. Let $\mathfrak{J}(K)$(resp. $\mathfrak{J}(L)$) be the set of g in $\mathfrak{R}$(resp. $\mathfrak{L}$) for which $\psi(g) = 0$ if $\psi \in K$(resp. L). Then the following statements are equivalent*:

(1) *$\mathfrak{L}/\mathfrak{J}(L)$ is type I*

(2) *K/G is countably separated*

(3) *$\mathfrak{R}/\mathfrak{J}(K)$ is type I and every factor representation of $\mathfrak{L}$ which*

annihilates $\mathfrak{J}(L)$ *is induced.*

For a C^*-algebra to be type I means that the weak closure of the image of each representation is type I in the sense of Murray and von Neumann.

Suppose (3) is true and let Φ' be a factor representation of $\mathfrak{L}/\mathfrak{J}(L)$. Then the corresponding representation Φ of $\mathfrak{L}$ is induced from a representation φ of $\mathfrak{K}$. By Theorem 1.5 the commutant $\Phi(\mathfrak{L})'$ of $\Phi(\mathfrak{L})$ is the intersection of the commutants of P^{φ} and U^{φ} and by [13, Theorem 6.6], this is isomorphic to $\varphi(\mathfrak{K})'$. Since $\mathfrak{K}/\mathfrak{J}(K)$ is type I, $\varphi(\mathfrak{K})'$ is type I and so is $\Phi'(\mathfrak{L}/\mathfrak{J}(L))'$. Thus Φ' is type I and so is $\mathfrak{L}/\mathfrak{J}(L)$, and $(3) \Rightarrow (1)$.

Suppose (1) is true. By [5, Theorem 2], L is countably separated and by Theorem 2.1, K/G is homeomorphic to a subspace of L. Thus K/G is countably separated, and $(1) \Rightarrow (2)$.

Suppose (2) is true. If $x \in X$, let $K(x)$ be the set of φ in K such that $\pi(\text{kernel}\,\varphi) = x$. If $\gamma \in G$ and φ and $\varphi \circ \gamma_K$ are both in $K(x)$ then $\gamma \in G_x$ and φ is equivalent to $\varphi \circ \gamma_K$. Thus the restriction to $K(x)$ of the quotient map $K \to K/G$ is one-to-one. Let $E_1, E_2, \cdots$ be G invariant Borel subsets of K which separate the points in K/G and let $U_1, U_2, \cdots$ be open subsets of X which separate points of X. Then $\pi^{-1}(U_1), \pi^{-1}(U_2), \cdots$ separate points of $K(x)$ from points of $K(y)$ for $x \neq y$ and $E_1, E_2, \cdots$ separate points of $K(x)$. Thus K is countably separated and by [5, Theorem 2], $\mathfrak{K}/\mathfrak{J}(K)$ is type I.

Let φ_0 be an irreducible representation of $\mathfrak{L}$ which annihilates $\mathfrak{J}(L)$, let φ and P be the corresponding representations of G and X and let R be the projection valued measure on Z which extends X and is given by Theorem 1.6. We assert that $R(Z \sim K) = 0$. Let ψ_1 be the representation of $\mathfrak{K}$ defined by Theorem 1.8. In view of the definition of R, we must show that $\psi_1(\mathfrak{J}(K)) = 0$. Suppose first that $\varphi_0 = \Psi$ is induced from an irreducible representation ψ of $\mathfrak{K}$ which annihilates $\mathfrak{J}(K)$ and let g be in $\mathfrak{J}(K)$ and W in $\mathfrak{H}(U^{\psi})$. As in the proof of Theorem 2.1, D, $(\psi_1(g)W)(\beta) = \psi \circ \beta_K(g)W(\beta)$ for a.e. β, and so $\psi_1(g) = 0$ and $\psi_1(\mathfrak{J}(K)) = 0$. If we no longer assume that φ_0 is induced, φ_0 is in any case a limit of such induced representations Ψ. Thus if W and $V \in \mathfrak{H}(\varphi_0)$ and $h \in C_0(X \times G)$ the representative function

$$g \to (\psi_1(g)\varphi_0(h)W, V) = (\varphi_0(g*h)W, V)$$

defined on $C_0(Y)$ is a limit of uniformly bounded representative functions defined on $\mathfrak{K}$ and vanishing on $\mathfrak{J}(K)$. This implies that $\psi_1(\mathfrak{J}(K)) = 0$ and $R(Z \sim K) = 0$.

Since the images of φ and R are not simultaneously reducible and since K/G is countably separated, R must be concentrated in an orbit ([11]). Thus P is also concentrated in an orbit and by [11] φ and so

φ_0 are induced. This means that the map of $K/G \to L$ is onto, that L is countably separated and by [5 Theorem 2] that $\mathfrak{L}/\mathfrak{J}(L)$ is type I. We have proved that any irreducible representation of $\mathfrak{L}$ which annihilates $\mathfrak{J}(L)$ is induced and thus this is also true for factor representations. We have proved (2) $\Rightarrow$ (3), and this completes the proof of Theorem 2.2.

Some of the results of this section extend results of [3], and this paper is in part addressed to the problems considered in [3] (cf. The final paragraph of [3]).

We conclude with a proof of the result mentioned in the introduction concerning a manifold structure in orbit spaces. We are indebted to R. Palais for discussions concerning this theorem.

THEOREM 2.3. *Let K be a C^∞ or real analytic separable n-dimensional manifold and let G be an analytic group acting smoothly on K. If the orbit space K/G is countably separated and if the orbits all have dimension m then there is an open dense G invariant subset U of K and a unique C^∞ or real analytic n-m dimensional manifold structure on U/G such that a function f defined on U/G is differentiable ($=C^\infty$ or real analytic) near Gx if and only if the corresponding function $x \to f(Gx)$ defined on U is differentiable near x.*

If K/G is countably separated then Theorem 1 of [6] implies that there is a dense open G invariant subset U_1 of K such that U_1/G is T_2; we can suppose $K = U_1$. If $x \in K$, let $\theta_x(\gamma) = \gamma x$, for γ in G. If $\Gamma \in \mathfrak{g}$, the Lie algebra of G, let $\theta^+(\Gamma)$ be the vector field defined by $\theta^+(\Gamma)_x = d\theta_x(\Gamma)$. Then $\theta^+(\mathfrak{g})$ is an m-dimensional involutive differential system $\mathfrak{M}$ on K, by [14, page 35, Theorem 2]. Necessary and sufficient conditions for coordinate functions $x_1, \cdots, x_n$ to be flat with respect to $\mathfrak{M}$ (we use the terminology of [14]) is that $x_j(\gamma y) = x_j(y)$ for γ near e, y in the domain of the x_k and $j = m+1, \cdots, n$. Suppose this is the case, suppose that the coordinate system is cubical of breadth 2a and domain W_a and let $S = S(c_{m+1}, \cdots, c_n)$ denote the slice $\{x; x_j(x) = c_j, j = m+1, \cdots, n\}$ of W_a. Let x be in S. Since $d\theta_x$ maps $\mathfrak{g}$ onto $\mathfrak{M}_x$, θ_x maps each neighborhood of e onto a neighborhood of x in S. Let T be the leaf containing S. Since each y in T is in some such S, $T \cap Gx$ is an open subset of T in the manifold topology for T as a submanifold of K. Since K/G is T_2, Gx is closed and $T \cap Gx$ is a relatively closed subset of T with the relative topology and so is a closed subset of T in the manifold topology. Since T is connected in the manifold topology, $T \subset Gx$. For some neighborhood N of e, $Nx \subset S$, and then $\{\gamma; \gamma x \in T\}$ can be shown to be an open and closed subset of G and thus all of G. Thus the leaves are the orbits.

Let W be a G invariant open subset of K. We show that W contains a G invariant open subset consisting of regular leaves. This will complete the proof since the union U of all open G invariant subsets of K which consist of regular leaves will then be dense, and [14, Theorem 8, page 19] defines the required manifold on U/G. Let $W_\varepsilon = \{x: |x_i(x)| < \varepsilon\}$. There is an ε in $(0, a)$ and a neighborhood N of e such that

$$N(S(c_{m+1}, \cdots, c_n) \cap W_\varepsilon) \subset S(c_{m+1}, \cdots, c_n)$$

for all $c_{m+1}, \cdots, c_n$. By Theorem 1 of [6] there is a nonempty open subset U_0 of W_ε such that for each m in U_0, $Nm \cap U_0 = Gm \cap U_0$. If $S(c_{m+1}, \cdots, c_n) \cap U_0 \neq \phi$ then

$$\begin{aligned}(GS(c_{m+1}, \cdots, c_n)) \cap U_0 &= (G(S(c_{m+1}, \cdots, c_n) \cap U_0)) \cap U_0 \\ &= (N(S(c_{m+1}, \cdots, c_n) \cap U_0)) \cap U_0 = S(c_{m+1}, \cdots, c_n) \cap U_0\end{aligned}$$

and so each orbit that meets U_0 meets it in a set of the form $S(c_{m+1}, \cdots, c_n) \cap U_0$. It follows that each orbit through U_0 is a regular leaf and that GU_0 is the required open subset of W.

D. Mumford has constructed an algebraic quotient using related hypotheses (Conversation with A. Mattuck).

Appendix

J. M. G. Fell has proved the equivalence stated on the first page of this paper. What follows is his proof.

Let G be a locally compact group with unit e and let $\mathscr{S}$ be the family of all closed subgroups of G. Let us give to $\mathscr{S}$ the topology having as a basis for its open sets the family of all

$$\mathscr{U}(C, \mathscr{F}) = \{K \in \mathscr{S}: K \cap C = \phi, K \cap A \neq \phi \text{ for each } A \text{ in } \mathscr{F}\}$$

(where C runs over the compact subsets of G and $\mathscr{F}$ runs over the finite families of nonvoid open subsets of G). This topology makes $\mathscr{S}$ a compact Hausdorff space [4, Theorem 1]. Let us fix a nonnegative function f_0 in $C_0(G)$ such that $f_0(e) > 0$ and for each K in $\mathscr{S}$ let μ_K be the left Haar measure on K for which

$$\int_K f_0(k) d\mu_K(k) = 1.$$

THEOREM. *For each f in $C_0(G)$, the function*

$$K \to \int_K f(k) d\mu_K(k)$$

is continuous on $\mathscr{S}$.

First, we observe that to each compact subset C of G there is a positive number $a = a(C)$ such that

$$\mu_K(C \cap K) \leqq a \tag{1}$$

for all K in $\mathscr{S}$. In fact if $f_0(z) > \varepsilon > 0$ for all z in a neighborhood U of e and if $x \in C$ then choose a neighborhood U_x of x such that $U_x^{-1}U_x \subset U$. A finite number of these, $U_{x_1}, \cdots, U_{x_n}$, cover C. Let $a = n/\varepsilon$, let $J = \{j;\ U_{x_j} \cap K \neq \phi\}$ and if $j \in J$, let y_j be chosen in $U_{x_j} \cap K$. Then

$$\mu_K(C \cap K) \leqq \varepsilon^{-1}\int \sum_{j \in J} f_0(y_j^{-1}k)d\mu_K(k) \leqq n/\varepsilon = a \ .$$

The essential technique is that of generalized limits. Let K_n be a net in $\mathscr{S}$ converging to K and let K_n be directed by a set N. A generalized limit is a positive linear functional Γ defined on the space B of all bounded real valued functions on N such that if $s \in B$ and $\lim_{n\to\infty} s_n$ exists then $\Gamma(s) = \lim_{n\to\infty} s_n$. If $s \in B$ and $\Gamma(s)$ is the same for all possible generalized limits, then $\lim_{n\to\infty} s_n$ must exist and equal $\Gamma(s)$.

Now let Γ be any generalized limit and let f be in $C_0(G)$. By (1), the function $\int_{K_n} f(k)d\mu_{K_n}(k)$ defined on N is bounded. Let

$$\Phi(f) = \Gamma\Big(\int_{K_n} f(k)d\mu_{K_n}(k)\Big) \ .$$

Φ is a positive linear functional on $C_0(G)$. If $f = 0$ on K, choose f_δ in $C_0(G)$ converging to f uniformly and such that the support of f_δ is contained in $\{x: |f(x)| \geqq \delta\}$. Then $\mathscr{U}(\text{suppt } f_\delta, \phi)$ is a neighborhood of K and if K_n is in this neighborhood then $\int_{K_n} f_\delta(k)d\mu_{K_n}(k) = 0$ and so $\Phi(f_\delta) = 0$ and $\Phi(f) = 0$. Also every g in $C_0(K)$ extends to an f in $C_0(G)$, so the definition

$$\varphi(f \mid K) = \Phi(f) \ , \qquad f \in C_0(G)$$

gives a positive linear functional φ on $C_0(K)$.

If $k_0 \in K$ and if $\varepsilon > 0$ then by (1) we can choose an open neighborhood U of k_0 such that

$$\Big|\int_H f(k_0k)d\mu_H(k) - \int_H f(k_1k)d\mu_H(k)\Big| < \varepsilon$$

for all k_1 in U and H in $\mathscr{S}$. For large n, $K_n \in \mathscr{U}(\phi, U)$ and so there is a k_n in $K_n \cap U$. Hence

$$\begin{aligned}
&|\varphi(f(k_0\,\cdot) \mid K) - \varphi(f \mid K)| \\
&\quad \leqq \limsup_n \Big|\Gamma\Big(\int_{K_n} f(k_0k)d\mu_{K_n}(k) - \int_{K_n} f(k_nk)d\mu_{K_n}(k)\Big)\Big| \\
&\qquad + \limsup_n \Big|\Gamma\Big(\int_{K_n} f(k_nk)d\mu_{K_n}(k)\Big) - \varphi(f \mid k)\Big| \\
&\quad \leqq \varepsilon \,\|\Gamma\| + \limsup_n \Big|\Gamma\Big(\int_{K_n} f(k)d\mu_{K_n}(k)\Big) - \varphi(f \mid k)\Big| = \varepsilon \,\|\Gamma\| \ ,
\end{aligned}$$

so φ is left invariant on K and thus is a left Haar measure. Since

$$\varphi(f_0 \mid K) = \Gamma\left(\int_{K_n} f_0(k) d\mu_{K_n}(k)\right) = \Gamma(1) = 1 ,$$

we must have

$$\Phi(f) = \int_K f(k) d\mu_K(k)$$

for all f in $C_0(G)$. The right member of the previous equation is independent of the choice of Γ and hence so is the left member. Thus

$$\lim_n \int_{K_n} f(k) d\mu_{K_n}(k) = \int_K f(k) d\mu_K(k) ,$$

and the theorem is proved.

If G_x is a continuous function of x and if $\mu_x = \mu_{G_x}$ is chosen as above then $x \to \mu_x$ is a continuous choice of the Haar measures. Conversely suppose we are given a continuous choice $x \to \mu_x$ of Haar measures on the G_x and suppose that $\{x_n : n \in N\}$ is a net in X converging to y and that $\mathscr{U}(K, \mathscr{F})$ is a neighborhood of G_y. If $G_{x_n} \cap K$ is not eventually empty then for all n in a cofinal subset of N, there is a σ_n in $G_{x_n} \cap K$, and if we pass to a suitable subnet, $\sigma_n \to \sigma$. However $\sigma \in K \cap G_y$ which contradicts the fact that $\mathscr{U}(K, \mathscr{F})$ is a neighborhood of G_y. Let $V \in \mathscr{F}$ and let f be a nonnegative nonzero element of $C_0(G)$ with support in V. Then $\int_{G_y} f(\sigma) d_y(\sigma) > 0$ and so $\int_{G_{x_n}} f(\sigma) d_{x_n}(\sigma)$ is eventually greater that zero. Hence $G_{x_n} \cap V$ is eventually not empty, G_{x_n} is eventually in $\mathscr{U}(K, \mathscr{F})$, and G_x is a continuous function of x.

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