HOMOGENEITY OF INFINITE PRODUCTS OF MANIFOLDS WITH BOUNDARY

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1. Introduction. In 1931, O. H. Keller [2] proved that the Hilbert cube Q is homogeneous. V. L. Klee, Jr., proved [3] in 1955 that Q is homogeneous with respect to finite sets, and in 1957 strengthened this result [4] by showing that Q is homogeneous with respect to countable closed sets. Our Theorem 1 extends this latter result to spaces which are the product of a countably infinite number of manifolds with boundary. Our method of proof exploits the notion of category for the space of self-homeomorphisms of the product space, and differs considerably from the methods of Keller and Klee, who made use of convexity properties of linear spaces.

In Theorem 2 we prove that if P is the product of a countably infinite number of manifolds with boundary and U and V are countable dense subsets of P, then there is a homeomorphism h of P onto itself such that $h[U] = V$. This theorem is analogous to a well known theorem about Euclidean spaces (see [1], p. 44). In a corollary to our Theorem 2, we show that if U is a countable subset of the Hilbert cube Q, then there is a contraction h_t, $0 \leqq t \leqq 1$, on Q such that if $0 < t < 1$, then h_t is a homeomorphism and $h_t[Q] \cap U = \phi$.

2. Notation and lemmas. For each positive integer n, we let M_n be a compact manifold with boundary, and we let B_n be the boundary of M_n. We let P be the cartesian product space $M_1 \times M_2 \times M_3 \times \cdots$. The projection mapping of P into M_n is denoted by π_n. If $x \in P$, we denote $\pi_n(x)$ by x_n. An admissible metric d_n for M_n is chosen so that M_n has diameter less than 2^{-n}, and we then define an admissible metric d for P by letting

$$d(x, y) = \sum_{n=1}^{\infty} d_n(x_n, y_n) .$$

If f and g are mappings on a compact metric space X into a metric space Y, we let $\rho(f, g)$ denote the least upper bound of the distances between $f(x)$ and $g(x)$ for x in X.

The set of all homeomorphisms of P onto P is denoted by H. Although the metric space (H, ρ) is not complete, it is topologically complete (i.e. homeomorphic to a complete metric space) and hence is

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a second category space.

The following two lemmas can be proved using standard techniques, and the proofs are merely outlined.

LEMMA 1. *If M is a manifold with boundary B, α is an arc lying in B, u and v are the end points of α, and W is an open subset of M which contains α, then there is a homeomorphism ψ of M onto M such that $\psi(u) = v$ and $\psi(x) = x$ for $x \in M - W$.*

Proof. Let S be the set of all points t of α for which there exists a homeomorphism ψ of M onto M such that $\psi(u) = t$ and $\psi(x) = x$ for $x \in M - W$. It is easy to see that S is both open and closed relative to α.

LEMMA 2. *If M is a manifold with boundary B, the dimension of M is at least 2, C is a countable and compact subset of $M - B$, and φ is a homeomorphism on C into $M - B$, then φ can be extended to a homeomorphism Φ on M onto M.*

Proof. For each positive integer n, we can obtain compact sets J_n and K_n such that:

(i) C is contained in the interior of J_n and $\varphi[C]$ is contained in the interior of K_n;

(ii) each component of J_n and of K_n has diameter less than $1/n$ and is homeomorphic to a spherical ball of dimension equal to that of M;

(iii) for each component D of J_n, $\varphi[D \cap C]$ is contained in a single component of K_n; and

(iv) $J_n \supset J_{n+1}$ and $K_n \supset K_{n+1}$.

Using the sets J_n and K_n, it is possible to construct homeomorphisms Φ_n of M onto M such that:

(i) if D is a component of J_n and E is a component of K_n, then $\Phi_n[D] \subset E$ if and only if $\varphi]D \cap C] \subset E$; and

(ii) $\Phi_{n+1}(x) = \Phi_n(x)$ for all $x \in M - J_n$.

The sequence $\Phi_1, \Phi_2, \Phi_3, \cdots$ converges to the desired homeomorphism.

LEMMA 3. *If $p \in P$, there is a residual subset R of H such that if $h \in R$, then $h(p)_n \in M_n - B_n$ for each n.*

Proof. Let $K_n = \{h | h \in H \text{ and } h(p)_n \in B_n\}$. It is obvious that each K_n is closed. We want to prove that K_n if nowhere dense. Thus, suppose $h \in K_n$ for some n and that $\varepsilon > 0$. We seek $g \in H - K_n$ such that $\rho(g, h) < \varepsilon$.

Choose an integer $m \neq n$ such that M_m has diameter less than ε. We define $M = M_n \times M_m$. M is also a manifold with boundary, and the boundary B of M is the set $(M_n \times B_m) \cup (B_n \times M_m)$. Since $h \in K_n$, the point $(h(p)_n, h(p)_m)$ is a member of $B_n \times M_m$. Let q be a point of B_m

such that $q \neq h(p)_m$. There is an arc β in $B_n \times M_m$ which joins $(h(p)_n, h(p)_m)$ to $(h(p)_n, q)$ and has diameter less than ε (since M_m has diameter less than ε). We may now choose a point $r \in M_n - B_n$ and an arc γ joining (r, q) to $(h(p)_n, q)$ such that $\beta \cup \gamma$ is an arc and has diameter less than ε. We let $\alpha = \beta \cup \gamma$.

We now use Lemma 1 to obtain a homeomorphism ψ of M onto M such that ψ maps the point $(h(p)_n, h(p)_m)$ onto (r, q) and the distance from x to $\psi(x)$ is less than ε for all $x \in M$.

Now, we define $g \in H$ by letting $g(y)_k = h(y)_k$ if $n \neq k \neq m$, and letting

$$(g(y)_n, g(y)_m) = \psi((h(y)_n, h(y)_m)) .$$

Since $g(p)_n = r$ and $r \notin B_n$, $g \in H - K_n$. It is easy to see that $\rho(g, h) < \varepsilon$, and hence we have proved that K_n is nowhere dense. We define $R = H - \bigcup_{n=1}^{\infty} K_n$. R is a residual set and if $h \in R$, then $h(p)_n \notin B_n$ for all n.

LEMMA 4. *If p and q are points of P, then there is a residual subset R of H such that if $h \in R$, then $h(p)_n \neq h(q)_n$ for all n.*

Proof. We define $J_n = \{h \mid h \in H \text{ and } h(p)_n = h(q)_n\}$. Each J_n is closed. We want to prove that J_n is nowhere dense. Suppose $h \in J_n$ and $\varepsilon > 0$. We seek $g \in H - J_n$ such that $\rho(g, h) < \varepsilon$.

It follows from Lemma 3, and the fact that residual subsets of H are dense in H, that there exists $f \in H$ such that $\rho(f, h) < \varepsilon/2$ and for all k, $f(p)_k \notin B_k$ and $f(q)_k \notin B_k$. If $f(p)_n \neq f(q)_n$ we can let $g = f$. Otherwise, we choose $m \neq n$ so that $f(p)_m \neq f(q)_m$ and define $M = M_n \times M_m$. Since $(f(p)_n, f(p)_m)$ and $(f(q)_n, f(q)_m)$ are not equal and neither is on the boundary of M, there is a homeomorphis φ of M onto M such that the distance from x to $\varphi(x)$ is less than $\varepsilon/2$ for all $x \in M$ and such that the points $\varphi((f(p)_n, f(p)_m))$ and $\varphi((f(q)_n, f(q)_m))$ have different first coordinates. We now define $g \in H$ by letting $g(y)_k = f(y)_k$ if $n \neq k \neq m$, and $(g(y)_n, f(y)_m) = \varphi((f(y)_n, f(y)_m))$. It is easy to see that $\rho(g, f) < \varepsilon/2$ and hence $\rho(g, h) < \varepsilon$. Moreover, $g(p)_n \neq g(q)_n$ and hence $g \in H - J_n$.

We obtain the desired residual set R by letting $R = H - \bigcup_{n=1}^{\infty} J_n$.

THEOREM 1. *If A is a closed and countable subset of P and f is a homeomorphism on A into P, then f can be extended to a homeomorphism F on P onto P.*

Proof. There is no loss in generality in assuming that each M_n has dimension at least 2, for otherwise we could define $S_n = M_{2n-1} \times M_{2n}$ and represent P as $S_1 \times S_2 \times S_3 \times \cdots$.

It follows from Lemma 3 and Lemma 4 that there is a homeomorphism

$h \in H$ such that for each n, the projection mapping π_n maps both $h[A]$ and $hf[A]$ in a one-to-one manner into $M_n - B_n$. The mapping $\varphi_n = \pi_n hfh^{-1}\pi_n^{-1}$ is one-to-one on $\pi_n h[A]$ onto $\pi_n hf[A]$ and can be extended by Lemma 2 to a homeomorphism Φ_n on M_n onto M_n. We obtain $\Phi \in H$ by letting $\Phi(x)_n = \Phi_n(x_n)$. The desired extension F of f is obtained by defining $F = h^{-1}\Phi h$.

Let h be a homeomorphism on a compact space X into a compact space Y, and let n be a positive integer. We define

$$\eta(h, n) = 2^{-n} \cdot \inf \{d(h(x), h(y)) \mid x, y \in X \text{ and } d(x, y) \geqq 1/n\} .$$

LEMMA 5. *If $h_1, h_2, h_3, \cdots$ is a sequence of homeomorphisms on X onto Y such that $\rho(h_n, h_{n+1}) < \eta(h_n, n)$, then the sequence converges uniformly to a homeomorphism h on X into Y.*

Proof. It is clear that the sequence converges uniformly to a continuous function h on X into Y. We must prove that h is one-to-one.

Suppose u and v are distinct points of X. We choose $n > 1$ so that $d(u, v) > 1/n$. Then, for $k \geqq n$,

$$\begin{aligned} d(h_{k+1}(u), h_{k+1}(v)) &\geqq d(h_k(u), h_k(v)) - d(h_k(u), h_{k+1}(u)) - d(h_k(v), h_{k+1}(v)) \\ &\geqq d(h_k(u), h_k(v)) - 2\eta(h_k, k) \\ &\geqq d(h_k(u), h_k(v)) - 2 \cdot 2^{-k} d(h_k(u), h_k(v)) \\ &\geqq d(h_k(u), h_k(v)) \cdot (1 - 2^{-k+1}) . \end{aligned}$$

Thus,

$$\begin{aligned} d(h(u), h(v)) &= \lim_{k \to \infty} d(h_k(u), h_k(v)) \\ &\geqq d(h_n(u), h_n(v)) \cdot \prod_{j=n}^{\infty} (1 - 2^{-j+1}) \\ &\geqq d(h_n(u), h_n(v))/4 , \quad (\text{since } n > 1) . \end{aligned}$$

This proves that h is one-to-one and hence a homeomorphism.

THEOREM 2. *If U and V are countable dense subsets of P, then there is a homeomorphism h of P onto P such that $h[U] = V$.*

Proof. As we have remarked in the proof of Theorem 1, there is no loss in generality in assuming that each M_n has dimension at least 2. In view of Lemma 3 and Lemma 4, we may also assume that U and V are so situated in P that each π_n maps both U and V in a one-to-one manner into $M_n - B_n$.

We are going to arrange the points of U and V into sequences $u_1, u_2, u_3, \cdots$ and $v_1, v_2, v_3, \cdots$ and choose homeomorphisms h_{ij} for all positive integers i and j. This is done by a fairly complicated inductive process, the first four steps of which are given below. We let $U_1 = U$, $V_1 = V$, and as soon as $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ are defined, we let $U_{n+1} = U_n - \{u_1, \cdots, u_n\}$, $V_{n+1} = V_n - \{v_1, \cdots, v_n\}$. We assume that U and V are well ordered so as to have the order type of the positive integers. We let H_n be the set of homeomorphisms of M_n onto itself.

Step 1. u_1 is chosen to be the first point of U and v_1 is chosen to be the first point of V. $h_{11} \in H_1$ is chosen so that $h_{11}\pi_1(u_1) = \pi_1(v_1)$. $h_{1j} \in H_j$ is the identity for $j > 1$.

Step 2. v_2 is the first point of V_2. $u_2 \in U_2$ is chosen near enough to v_2 for us to obtain $h_{21} \in H_1$ so that: $\rho(h_{21}, h_{11}) < \eta(h_{11}, 1)$ and $h_{21}\pi_1(u_j) = \pi_1(v_j)$ for $j = 1, 2$. $h_{22} \in H_2$ is chosen so that $h_{22}\pi_2(u_j) = \pi_2(v_j)$ for $j = 1, 2$. $h_{2j} \in H_j$ is the identity for $j > 2$.

Step 3. u_3 is the first point of U_3. $v_3 \in V_3$ is chosen near enough to u_3 for us to obtain $h_{3i} \in H_i$ so that: $\rho(h_{3i}, h_{2i}) < \eta(h_{2i}, 2)$ and $h_{3i}\pi_i(u_j) = \pi_i(v_j)$ for $i = 1, 2$ and $j = 1, 2, 3$. $h_{33} \in H_3$ is chosen so that $h_{33}\pi_3(u_j) = \pi_3(v_j)$ for $j = 1, 2, 3$. $h_{3j} \in H_j$ is the identity for $j > 3$.

Step 4. v_4 is the first point of V_4. $u_4 \in U_4$ is chosen near enough to v_4 for us to obtain $h_{4i} \in H_i$ so that: $\rho(h_{4i}, h_{3i}) < \eta(h_{3i}, 3)$ and $h_{4i}\pi_i(u_j) = \pi_i(v_j)$ for $i = 1, 2, 3$ and $j = 1, \cdots, 4$. $h_{44} \in H_4$ is chosen so that $h_{44}\pi_4(u_j) = \pi_4(v_j)$ for $j = 1, \cdots, 4$. $h_{4j} \in H_j$ is the identity for $j > 4$.

We continue this process. By Lemma 5, the homeomorphisms $h_{j1}, h_{j2}, h_{j3}, \cdots$ converge uniformly to a homeomorphism $g_j \in H_j$. It is easy to see that $g_j\pi_j(u_i) = \pi_j(v_i)$ for all i and j. There is determined uniquely a homeomorphism $h \in H$ for which $\pi_j h = g_j \pi_j$ for all j. Since $h(u_i) = v_i$ for all i, and $U = \{u_1, u_2, \cdots\}$, $V = \{v_1, v_2, \cdots\}$, h is the desired homeomorphism.

COROLLARY. *If C is a countable subset of the Hilbert cube Q, then there is a contraction h_t, $0 \leqq t \leqq 1$, defined on Q such that:*

(i) *h_1 is the identity,*

(ii) *h_0 is a constant mapping, and*

(iii) *if $0 < t < 1$, h_t is a homeomorphism of Q into Q and $h_t[Q] \cap C = \phi$.*

Proof. We let M_n be the closed interval $[-5^{-n}, 5^{-n}]$. The resulting space P may then be thought of as the Hilbert cube Q. (This representation is used since M_n was assumed to have diameter less than 2^{-n}.) We let D be the set of all points x in P such that $\pi_i(x)$ is rational for all i, and $\pi_i(x) = 5^{-i}$ for all but a finite number of values of i:

Both $C \cup D$ and D are countable and dense in P, so by Theorem 2 there is a homeomorphism G of P onto P such that $G[C \cup D] = D$. We define $g_t(x) = tx$ for $0 \leqq t \leqq 1$ and $x \in P$. Finally, we let $h_t = G^{-1}g_tG$. It is easy to see that the desired contraction is h_t, $0 \leqq t \leqq 1$.

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