POLYNOMIAL INTERPOLATION IN POINTS EQUIDISTRIBUTED ON THE UNIT CIRCLE

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1. Introduction. Let distinct points $S_n = \{z_{n1}, z_{n2}, \cdots, z_{nn}\}$ be given on the unit circle $|z| = 1$ in the complex z-plane, let a function f also be given on $|z| = 1$, and let $L_n = L_n(f; z)$ denote the polynomial of degree at most $n - 1$ found by interpolation to f at the points S_n. Consider an infinite sequence of such point sets, $S_1, S_2, \cdots, S_n, \cdots$, and the corresponding sequence $L_1, L_2, \cdots, L_n, \cdots$. If the union of the sets S_n is everywhere dense on $|z| = 1$, does $\lim_{n\to\infty} L_n(f; z)$ exist for $|z| < 1$, and if so, what is it?

Walsh [14, pp. 178–180] proved that if the points S_n are equally spaced for each n, and if f is Riemann integrable, then

$$\lim_{n\to\infty} L_n(f; z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)dt}{t - z} \tag{1.1}$$

uniformly on any closed point set on the region $|z| < 1$. The present author [1] [2] generalized Walsh's result to the case of interpolation on a more or less arbitrary Jordan curve. The problem for equally spaced interpolation points has a pedigree of some length which is described in Walsh's book [14] and in a recent survey given by the author [3].

When the points S_n are not equally spaced, very little is known about the behavior of L_n unless f is analytic on $|z| \leqq 1$. For the analytic case Fejér [4] proved that if the points S_n are equidistributed on an arbitrary Jordan curve C in a sense to be described below in § 2 and if f is analytic on the closed region $\bar{D}$ bounded by C then $L_n \to f$ uniformly on $\bar{D}$. No result of this sort involving equidistribution is at present known for nonanalytic functions f even when C is the unit circle.[1] It is the purpose of this paper to try to shed some light on the situation for nonequally spaced points by means of a probabilistic treatment. We shall let the points of the sequence $S_1, S_2, \cdots$ be random variables defined on a probability space with a structure such that almost certainly a sample sequence is equidistributed. (We use the word "equidistributed" here in connection with sample sequences rather than the more usual words "uniformly distributed" to avoid confusion with the concept of a uniform distribution in the probability sense.) The

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[1] Zygmund [**16**, vol. II, pp. 3-4] points out that a similar gap exists in the theory of trigonometric interpolation.

mean value of L_n formed in the random points S_n is calculated in § 2. The result is consistent with (1.1). But in § 4, in discussing a particular class of equidistributed sample sequences, we shall show how only a slight modification of equal spacing upsets Walsh's deterministic result.

2. A stochastic treatment. In this section we shall first use a stochastic model which is appropriate to the case in which for each $n > 1$ the first $n - 1$ points of S_n are the points of S_{n-1}—or in other words, the first subscripts on the points z_{nk} are superfluous.[2] A slight extension given in the next section will provide the structure for the situation described in the first paragraph of the Introduction, in which S_n may consist entirely of new points not used in S_{n-1}.

Let $\theta_1, \theta_2, \cdots$, be an infinite sequence of mutually independent random variables each with the uniform (or "rectangular") marginal probability distribution on the closed interval $[0, 2\pi]$. Let $z_k = e^{i\theta_k}$, $k = 1, 2, \cdots$. Let the function f be given everywhere on $|z| = 1$, and let

$$L_n(f; z) = L_n(f; z \mid z_1, z_2, \cdots, z_n) = \sum_{k=1}^{n} f(z_k) \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)}, \tag{2.1}$$

where $\omega_n(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$. For each sample sequence of the stochastic process $z_1, z_2, \cdots$, for which the values of $z_1, z_2, \cdots, z_n$ are distinct, the formula (2.1) gives the unique Lagrange polynomial of degree at most $n - 1$ found by interpolation to f in this value. The locus of those points in the n-dimensional interval $0 \leqq \theta_j \leqq 2\pi$, $j = 1, \cdots, n$, for which (2.1) is formally undefined is the union of hyperplanes

$$\bigcup_{j<k}^{n} \{\theta_j, \theta_k \mid \theta_j \equiv \theta_k \bmod 2\pi\} .$$

The probability measure attached to each such hyperplane is zero, so it follows that (2.1) defines a Lagrange polynomial with probability one.

By the Glivenko-Cantelli Theorem [9, pp. 20–21], given any sample sequence of the process $\theta_1, \theta_2, \cdots$, if $N_n(\theta)$ denotes the number of values of the first n terms falling into $[0, \theta]$, then with probability one

$$\lim_{n\to\infty} \frac{N_n(\theta)}{n} = \frac{\theta}{2\pi} \tag{2.2}$$

uniformly in θ. The condition (2.2) is the classical definition of equidistribution or deterministic uniform distribution [15], [11, vol. 1, pp.

[2] This is the only model considered in [4] where Theorem 1 below is announced without proof. It might be noted that Carl Runge, who in 1904 published a proof of the result (1.1) for functions analytic on $|z| \leqq 1$, considered only a sequence of sets of equally spaced points in which each set contained all the previous sets. That is, he interpolated in the nth, $2n$th, $4n$th, etc., roots of unity. See [12, pp. 136–137].

70 ff.], for real numbers. In the work referred to in the Introduction, Fejér translated it to points on an arbitrary Jordan curve C by parametrizing C through the schlicht analytic function $z = \phi(w)$ which gives a conformal map of $|w| > 1$ onto the exterior of C so that the points at infinity correspond. The function can be extended in a continuous and one-to-one manner (Osgood-Taylor-Carathéodory Thoerem) onto $|w| = |e^{i\theta}| = 1$.

The theorem about to be stated and proved deals with the mean value of L_n over the marginal distribution of the vector random variable $(\theta_1, \theta_2, \cdots, \theta_n)$. Given any finite subsequence consisting of k members of the random sequence $\theta_1, \theta_2, \cdots$, and a function g from the interval $I_k = [0, 2\pi] \times [0, 2\pi) \times \cdots \times [0, 2\pi]$ (k factors) to the complex plane integrable in the sense of Lebesgue on I_k, we shall use the symbol $E_k g$ to denote the mean value

$$E_k g = \left(\frac{1}{2\pi}\right)^k \int_0^{2\pi} \cdots \int_0^{2\pi} g(\alpha_1, \alpha_2, \cdots, \alpha_k) d\alpha_1 \cdots d\alpha_k \, .$$

THEOREM 1. *If f is continuous on $|z| = 1$ and possesses there an $(n-2)$th order derivative satisfying a Lipschitz condition with exponent one, then for all z*

$$E_n L_n(f; z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \, , \qquad n \geqq 2 \, , \tag{2.3}$$

where $\sum_0^n a_k z^k$ is the Taylor expansion of the analytic function

$$F(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{f(t)}{t - z} dt \, , \qquad |z| < 1 \, . \tag{2.4}$$

Thus if f is infinitely differentiable on $|z| = 1$, $\lim_{n\to\infty} E_n L_n(f; z)$ exists and equals $F(z)$ for $|z| \leqq 1$. If f is analytic on $|z| \leqq 1$, then this limit exists uniformly and equals $f(z)$ on each closed disk with center at the origin which does not contain a singularity of f.

The statements in the theorem following (2.4) are standard results in function theory relating to the Taylor expansion of F and to the possibility of extending F continuously onto $|z| = 1$. See for example [14, pp. 141 ff.].

The equation (2.3) is trivially true for $n = 1$ if f is merely integrable on $|z| = 1$, because then $L_1(f; z_1) = f(z_1)$ and

$$E_1 L_1(f; z_1) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta_1}) d\theta_1 = \frac{1}{2\pi i} \int_{|t|=1} \frac{f(t)}{t} dt = F(0) = a_0 \, .$$

The derivatives of f referred to in the theorem may be taken with respect to arc length (here identical with θ in the parametrization $z =$

$e^{i\theta}$ of the circle $|z| = 1$), or alternatively they may be taken in a chordal sense:

$$f'(z_1) = \lim_{z\to z_1} \frac{f(z) - f(z_1)}{z - z_1}, \qquad |z| = |z_1| = 1,$$

and so on for higher derivatives. To fix ideas we shall use the latter interpretation.

The Lipschitz condition referred to in the theorem means that for some $l > 0$,

$$|f^{(n-2)}(z_1) - f^{(n-2)}(z_2)| \leqq l\,|z_1 - z_2|,$$

all $|z_1| = |z_2| = 1$. It implies of course that $f^{(k)}$ with $0 \leqq k < n-2$, satisfies a similar condition because $f^{(k)}$ has a continuous derivative on $|z| = 1$.

An alternative expression for the right side of (2.3) is given by

$$\sum_0^{n-1} a_k z^k = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t - z}\left(1 - \frac{z^n}{t^n}\right)dt, \tag{2.5}$$

which follows with no hypotheses on f other than integrability from the validity of

$$F(z) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t}\left(\frac{1}{1 - t/z}\right)dt = \sum_0^\infty \frac{z^k}{2\pi i}\int_{|t|=1} \frac{f(t)}{t^{k+1}}\,dt,\ |z| < 1,$$

and from the uniqueness of Taylor series. Our proof will establish the equivalence of the left side of (2.3) with the right side of (2.5). We need some preliminary results before passing to the main proof.

LEMMA 1.

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-in\theta} - e^{i(m+1)\theta}}{1 - e^{i\theta}}\,d\theta = \begin{cases} 1, & m \geqq 0, n \geqq 0 \\ -1, & m < 0, n < 0 \\ 0, & \text{otherwise,} \end{cases}$$

where n and m are integers.

The integrand consists of the sum of a finite number of positive and negative integral powers of $e^{-i\theta}$. With $m \geqq 0$ and $n \geqq 0$, the coefficient of the zeroth power is one; with $m < 0$, $n < 0$ it is -1; and otherwise there is no zeroth power in the sum at all.

LEMMA 2. *Let $g(\theta)$ have the period 2π and be such that*

$$\int_0^{2\pi}\int_0^{2\pi} \left|\frac{g(\alpha) - g(\beta)}{e^{i\alpha} - e^{i\beta}}\right| d\alpha\, d\beta$$

exists. Then

$$J = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \frac{g(\alpha) - g(\beta)}{e^{i\alpha} - e^{i\beta}} e^{-i(m\alpha+n\beta)} d\alpha d\beta$$

$$= \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{-i(m+n+1)\theta} g(\theta) d\theta\,, & n \geqq 0, m \geqq 0 \\ -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{-i(m+n+1)\theta} g(\theta) d\theta\,, & n < 0, m < 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

For the proof, we make the change of variables $u = \beta - \alpha$, $\alpha = \alpha$, and using periodicity arrive at

$$J = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \frac{e^{-i(m+n+1)\alpha} e^{-inu}(g(\alpha) - g(\alpha+u))}{1 - e^{iu}} d\alpha du\,.$$

Fubini's theorem, applicable because of the integrability hypothesis, allows us to integrate with respect to α first; by doing this and again using the periodicity of g, we get

$$J = \frac{b}{2\pi}\int_0^{2\pi} e^{-inu}\left(\frac{1 - e^{(m+n+1)u}}{1 - e^{iu}}\right) du$$

where

$$b = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(m+n+1)\alpha} g(\alpha) d\alpha\,.$$

An application of Lemma 1 now completes the proof.

The theorem will be proved by expressing L_n in terms of the divided differences of a certain function q related to f and formed in the points $z_1, z_2, \cdots, z_n$. We define these formally as follows:

$$d_1 = d(q \mid z_1, z_2) = \frac{q(z_1) - q(z_2)}{z_1 - z_2}$$

$$d_2 = d(q \mid z_1, z_2, z_3) = \frac{d(q \mid z_1, z_2) - d(q \mid z_3, z_2)}{z_1 - z_3}$$

$$\vdots$$

$$d_{n-1} = d(q \mid z_1, z_2, \cdots, z_n) = \frac{d(q \mid z_1, z_2, \cdots, z_{n-1}) - d(q \mid z_n, z_2, \cdots, z_{n-1})}{z_1 - z_n}\,.$$

The subscripts on the d's refer to the "order" of the divided differences. In our stochastic model, these expressions as they stand are indeterminate with probability zero.

By induction it can be shown [13, p. 15] that

$$d_{n-1} = \sum_1^n \frac{q(z_k)}{\omega_n'(z_k)}\,, \tag{2.6}$$

which incidentally proves that the divided differences are symmetric functions of the z_k's. From (2.1) and (2.6), it is clear that if $q = q_w(z) = f(z)/(w - z)$, $|w| \neq 1$, then

$$L_n(f; w) = \omega_n(w) d(q \mid z_1, \cdots, z_n) . \tag{2.7}$$

If f is such that its first $n - 1$ derivatives satisfy Lipschitz conditions on $|z| = 1$, then it is easily shown that the same must be true for $q_w(z) = f(z)/(w - z)$, $|w| \neq 1$. We omit the details.

We need another lemma which will insure that $E_k L_k(f; w)$ exists for $k = 2, \cdots, n$ and can be calculated by interated itegration.

LEMMA 3. *If a function f given on $|z| = 1$ possesses an $(n - 2)$th derivative satisfying a Lipschitz condition on $|z|=1$, and if $d_1, d_2, \cdots, d_{n-1}$ are respectively the divided differences of f formed successively in the points $z_1, z_2, \cdots, z_n$ on $|z| = 1$, then $|d_1|, |d_2|, \cdots, |d_{n-1}|$ are uniformly bounded for all $z_1, z_2, \cdots, z_n$ for which these divided differences are defined.*

The proof of this lemma is rather long, and is available elsewhere [5]. With proper completion of the definitions of the divided differences by continuity, coincident points z_k are allowable, but that is of no interest for present purposes.

Suppose now that $Q(z)$ is any function satisfying the hypotheses of Lemma 2. Let z and t be any two members of the family of random variables $z_1, z_2, \cdots$ and let w be any fixed complex number. Then by Lemma 2, for any $k \geqq 0$ with $z = e^{i\alpha}$, $t = e^{i\beta}$,

$$\begin{aligned} &E_2(w - t)(w^{k+1}z^{-k} - z) d(Q \mid z, t) \\ &= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \frac{Q(e^{i\alpha}) - Q(e^{i\beta})}{e^{i\alpha} - e^{i\beta}} (w^{k+2}e^{-ik\alpha} - we^{i\alpha} - w^{k+1}e^{-i(k\alpha-\beta)} \\ &\quad + e^{i(\alpha+\beta)}) d\alpha d\beta \\ &= \frac{1}{2\pi}\int_0^{2\pi} Q(e^{i\alpha})(w^{k+2}e^{-i(k+1)\alpha} - 0 - 0 - e^{i\alpha}) d\alpha \\ &= E_1(w^{k+2}z^{-(k+1)} - z) Q(z) . \end{aligned} \tag{2.8}$$

We now use (2.7) and invoke Fubini's Theorem and Lemma 3 as authority for calculating the multiple integral

$$E_n L_n(f; w) = E_n \omega_n(w) d(q \mid z_1, \cdots, z_n)$$

by integration in any convenient order. In what follows, the operator E_1 inside the square brackets refers to integration with respect to z_1 and E_2 inside the square brackets refers to double integration with respect to z_1 and the z_k of the largest subscript. By using (2.8) repeatedly, we obtain:

$$
\begin{aligned}
E_nL_n &= E_{n-2}\Big[\prod_{k=2}^{n-1}(w-z_k)E_2(w-z_1)(w-z_n)d(q\mid z_1,\cdots,z_n)\Big] \\
&= E_{n-2}\Big[\prod_{k=2}^{n-1}(w-z_k)\,E_1(w^2z_1^{-1}-z_1)d_{n-2}\Big] \\
&= E_{n-3}\Big[\prod_{k=2}^{n-2}(w-z_k)E_2(w-z_{n-1})(w^2z_1^{-1}-z_1)d_{n-2}\Big] \\
&= E_{n-3}\Big[\prod_{k=2}^{n-2}(w-z_k)\,E_1(w^3z_1^{-2}-z_1)d_{n-3}\Big] \\
&= \cdots \\
&= E_1[(w-z_2)E_2(w-z_3)(w^{n-2}z_1^{-(n-3)}-z_1)d_2] \\
&= E_1[(w-z_2)E_1(w^{n-1}z_1^{-(n-2)}-z_1)d_1] \\
&= E_2(w-z_2)(w^{n-1}z_1^{-(n-2)}-z_1)d_1 \\
&= E_1(w^nz_1^{-(n-1)}-z_1)q_w(z_1) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(z_1)}{w-z_1}\Big(\frac{w^n}{z_1^{n-1}}-z_1\Big)d\theta \\
&= \frac{1}{2\pi i}\int_{|z_1|=1}\frac{f(z_1)}{z_1-w}\Big(1-\frac{w^n}{z_1^n}\Big)dz_1\,.
\end{aligned}
$$

The last expression is the right member of (2.5), and the proof is now complete for $|w|\neq 1$.

Because of the singularity of $q_w(z)=f(z)/(w-z)$ with $|w|=1$, $|z|=1$, the above argument needs further elaboration to establish that (2.3) holds on the unit circle. However we can prove this by a different approach. It is well known [13, pp. 24–25] that $L_n(f, w)$ is identical with Newton's interpolation formula:

$$L_n(f; w)\equiv f(z_1)+\sum_{k=1}^{n-1}(w-z_1)\cdots(w-z_k)d(f\mid z_1,\cdots,z_{k+1})\,. \tag{2.9}$$

Our hypotheses on f insure that the expected value of each term of this formula exists for all w, and the expected value of the sum (which of course is the sum of the expected values of the terms) is clearly some polynomial in w defined for all w including $|w|=1$. It is equal to the right member of (2.3) for $|w|\neq 1$, and so therefore on $|w|=1$. This establishes (2.3) for all values of w.

An alternative proof of Theorem 1 can be based on (2.9). A consequence of our method of proof of Theorem 1 is this:

THEOREM 2. *Let f given on $|z|=1$ be such that $E_n|d(f|z_1,\cdots,z_n)|$ exists. Then*

$$E_nd(f\mid z_1,\cdots,z_n)=\frac{1}{2\pi}\int_0^{2\pi}e^{-i(n-1)\theta}f(e^{i\theta})d\theta \tag{2.10}$$

$$= \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t^n} dt \ .$$

In Theorem 1 the Lipschitz condition on $f^{(n-2)}$ is used only to insure that $|d(q\,|\,z_1, \cdots, z_n)|$ is integrable. The hypotheses on f could be replaced by this condition, as we did in Theorem 2, and the restriction on f would be lighter.

A generalization of Theorem 1 to the case in which the unit circle is replaced by an arbitrary Jordan curve C and f is analytic on and interior to C is discussed in [7]. The probability distribution of the points z_k on C is defined by the condition that the image points $w_k = e^{i\theta_k}$ under the mapping function $z = \phi(w)$ used by Fejér have uniformly and independently distributed angles θ_k. The generalization seems unsatisfactory because convergence of E_nL_n does not take place unless the singularities of f are all at least a certain distance (characteristic of C) removed from C.

As a result of passing interest here we note that if the points z_k are so distributed, and if f is analytic on and inside a rectifiable Jordan curve C' containing C in its interior, then

$$d(f\,|\,z_1, \cdots, z_n) = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{\omega_n(t)} dt \ .$$

An easy calculation shows that

$$E_n d(f\,|\,z_1, \cdots, z_n) = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{[w_t\phi'(w_t)]^n} dt \ ,$$

where $t = \phi(w_t)$. The Koebe distortion theorem [6, pp. 279–281], suitably modified for exterior-to-exterior mapping functions, yields the inequality

$$\frac{R(R-1)^3}{(R+1)^5} \leqq \frac{r}{R\phi'(Re^{i\theta})} \leqq \frac{R(R+1)^3}{(R-1)^5},\ R > 1,\ 0 \leqq \theta < 2\pi \ ,$$

where r is the capacity of C. The right member is a decreasing function of R, so there exists some value or R such that for all functions f analytic on and inside the level curve C': $\{z\,|\,z = \phi(Re^{i\theta}),\ R \text{ fixed},\ 0 \leqq \theta < 2\pi\}$, the relation $\lim_{n\to\infty} E_n d_{n-1} = 0$ holds.

We conclude this section by noting a consequence of Theorem 1 which is obtained by combining equation (2.3) with a result due to Walsh [14, pp. 153–154]:

THEOREM 3. *Let f be analytic for $|z| < R > 1$ but have a singularity on $|z| = R$. Let $P_n(z)$ be the polynomial of degree at most $n-1$ found by interpolation to f in the* nth *roots of unity. Then*:

$$\lim_{n\to\infty} [P_n(z) - E_nL_n(f; z)] = 0$$

for $|z| < R^2$, *uniformly for* $|z| \leqq R' < R^2$.

3. Equidistributions and uniform probability distributions. In the standard deterministic treatment of polynomial interpolation in the real or complex domain, the interpolation points are presented in a triangular matrix

$$\begin{array}{llll} S_1\colon & z_{11} & & \\ S_2\colon & z_{21} & z_{22} & \\ S_3\colon & z_{31} & z_{32} & z_{33} \\ & \vdots & & \end{array} \tag{3.1}$$

with the implication that more than one—perhaps all—of the points of the nth row may not have appeared previously. The sequence $L_1, L_2, \cdots, L_n, \cdots$ of interpolating polynomials is found by making L_n interpolate to a given function in the points S_n of the nth row. This is the set-up needed to cover, for example, interpolation in successive sets of equally spaced points. The stochastic model of the preceding section provides a probabilistic theory for a deterministic interpolation process of this sort if we think of (3.1) as a sample sequence of the stochastic process $z_1, z_2 \cdots$ with z_{11} a determination of z_1; z_{21} and z_{22} determinations of z_2 and z_3; and so forth. This implies that z_{nn} is a determination of $z_{n(n+1)/2}$. It is convenient now to relabel the random variables $z_1, z_2, \cdots$ so as to correspond with (3.1). We do this by superscripts, denoting the stochastic process now by $z^{11}; z^{21}, z^{22}; z^{31}, z^{32}, z^{23}; \cdots$.

We assume once again that the arguments (angles) of the terms z^{nk} are mutually independent and each is uniformly distributed on $[0, 2\pi]$. The Glivenko-Cantelli theorem states that given any sample sequence of this process, if $N_k(\theta)$ denotes the number of arguments in the first k terms of the sample sequence which do not exceed θ, then with probability one,

$$\lim_{k\to\infty} \frac{N_k(\theta)}{k} = \frac{\theta}{2\pi}, \tag{3.2}$$

for each value of θ. But in the standard deterministic interpolation theory, a stronger equidistribution property is used [14, pp. 164–166]: Let $N_n^*(\theta)$ denote the number of points in the nth set of n points, $S_n = \{z_{n1}, z_{n2}, \cdots, z_{nn}\}$, with arguments not exceeding θ; the required condition is that

$$\lim_{n\to\infty} \frac{N_n^*(\theta)}{n} = \frac{\theta}{2\pi} \tag{3.3}$$

for each θ. We shall call this the strong equidistribution property for a sequence such as (3.1). The roots of unity are so distributed. By elementary methods it is easily shown that if a sequence satisfies (3.3) it satisfies (3.2), but not necessarily conversely. For example if the points z_{nk} in each of a suitably sparse but infinite set of rows of (3.1) were all equal to a constant α, (3.2) might be true but (3.3) certainly could not be true.[3] It is of interest to ask whether almost every sample sequence of our stochastic process has not only property (3.2), but also the strong property (3.3).

The answer is in the affirmative. We here sketch the argument. Let $Pr(A)$ denote the probability of any event A. Consider an infinite sequence of random variables $X^{11}; X^{21}, X^{22}; X^{31}, X^{32}, X^{33}; \cdots$ in which $Pr(X^{nk} = 1) = p$, $Pr(X^{nk} = 0) = q$, $q + p = 1$, $k = 1, \cdots, n$; $n = 1, 2, \cdots$. In this sequence let the random variables in the nth group of n variables be mutually independent for $n = 1, 2, \cdots$. However, successive groups of n need not be independent. Let $\sigma_n = \sum_{k=1}^{n} X^{nk}/n$. In a proof of the Strong Law of Large Numbers for the Bernoulli case given by Feller [8, pp. 190–191], it is shown that for any $\varepsilon > 0$ there exists a number $M > 0$ constant with respect to n such that

$$Pr\{A_n : |\sigma_n - p| > \varepsilon\} < \frac{M}{n^2} .$$

This means that for any infinite subsequence of the sequence of events $A_1, A_2, \cdots$, say A_{n_j}, $j = 1, 2, \cdots$, the series $\sum_j Pr(A_{n_j})$ converges. It follows from the Borel-Cantelli lemma [9, p. 18] [8, p. 188] that the probability is zero that an infinite number of the events A_{n_j} occur. This is the same as saying that $Pr(\lim_{n\to\infty} \sigma_n = p) = 1$.

The author is indebted to Professor Kai-Lai Chung for corroborating the truth of this result in a letter. Professor Chung fiirst refers to a result concerning the standard Strong Law of Large Numbers given in his Columbia University Lecture Notes, 1950–51, in which it is shown that if $X_1, X_2, \cdots$ is a sequence of independent identically distributed random variables with $E(X_1) = 0$ and $E(X_1^4)$ finite, then

$$\sum_{n=1}^{\infty} Pr\left\{B_n : \left|\frac{X_1 + X_2 + \cdots + X_n}{n}\right| > \varepsilon\right\}$$

converges. He then in the Bernoulli case remarks that the marginal probability $Pr(B_n) = Pr(A_n)$ individually for each n, regardless of the joint probabilities for corresponding collections of the events B_n and A_n, and this is the essential link between the classical Strong Law and the present version.

To show that (3.3) holds with probability one for each θ, we simply

[3] The distinction between (3.2) and (3.3) is related to the concept of "well-distributed" points introduced by Petersen [10].

identify the event $X^{nk} = 1$ with the event arg $z^{nk} \leqq \theta$ in the stochastic process $z^{11}; z^{21}, z^{22}; \cdots$. However, the Glivenko-Cantelli Theorem says something more with regard to (3.2); namely that (3.2) holds with probability one uniformly in θ [9, p. 20]. Inspection of the proof reveals that this is true for (3.3) also.

Thus with probability one a sample sequence of the process $z^{11}; z^{21}, z^{22}; \cdots$ has the strong equidistribution property uniformly in θ_1, and so this process may properly be considered the stochastic analogue of the equidistributed sequences used in interpolation theory. We note in conclusion that if $L_n(f; z) = L_n(f; z \mid z_1, z_2, \cdots, z_n)$ in § 2 (see (2.1)) is replaced by $L_n(f; z \mid z^{n1}, z^{n2}, \cdots, z^{nn})$, then Theorem 1 and Theorem 3 still are valid, and so is Theorem 2 with z^{nk} replacing z_k, $k = 1, \cdots, n$, because these theorems depend only on the joint probability distributions of the n visible random variables.

4. Interpolation in certain strong equidistributions. A number of years ago, the late Professor Aurel Wintner asked the author whether it might be possible to extend Walsh's result (1.1) to the case of interpolation in equidistributed points on the unit circle, at least for functions analytic interior to the circle and satisfying smoothness conditions short of analyticity on the closed unit disk. Professor Wintner particularly had in mind interpolation in the points S_n: $\xi, \xi^2, \cdots, \xi^n$, where $|\xi| = 1$ and ξ is no root of unity. It is well known [11, pp. 70–71] that the sequence $S_1, S_2, \cdots$ is equidistributed in the classical sense (3.2) on $|z| = 1$. Moreover it is easy to prove by Weyl's criterion [11, p. 70] that this sequence is strongly equidistributed also if by S_n is meant the nth set of n points in the sequence.

We cannot give an answer here to Wintner's question as to interpolation in the particular sequence S_n defined above. Theorem 1 above seems to be relevant to the more general problem. In the present section by combining the roots of unity with a point ξ we shall construct a strongly equidistributed sequence of interpolation points which demonstrates some of the limitations inherent in interpolation in nonequally spaced points.

In fact, let the nth row of (3.1) consist of $z_{nk} = e^{2\pi ik/(n-1)}$, $k=1,\cdots,n-1$, and $z_{nn} = \xi$, where as above $|\xi| = 1$ and ξ is not a root of unity. This sequence, which we shall denote by $S_1^*, S_2^*, \cdots$, is strongly equidistributed. Let $\omega_n(z) = (z - z_{n1})(z - z_{n2}) \cdots (z - z_{nn})$. It is readily verified that if $f(z)$ is the function $1/z$, then for any interpolation points at all,

$$L_n(f; z) = \frac{1}{z}\left(1 - \frac{\omega_n(z)}{\omega_n(0)}\right) .$$

Here this becomes

$$L_n(f; z) = \frac{1}{z}\left(1 - \frac{(z^{n-1} - 1)(z - \xi)}{(0 - 1)(0 - \xi)}\right).$$

For $|z| < 1$,

$$\lim_{n\to\infty} L_n = \frac{1}{z}\left(1 - \frac{(-1)(z - \xi)}{-\xi}\right) = \frac{1}{\xi}. \tag{4.1}$$

But the value of the right member of (1.1)—that is, the F which appears in (2.4)—is zero for this function. The perturbation caused by the adjoining of a single extra point to the nth roots of unity transmuted Walsh's result. Note that with ξ replaced by ξ^n in S_n there would have been no limit at all.

For a general convergence theory for this set of interpolation points, we write L_n in the following form:

$$L_n(f; z) = f(z_{nn}) + \sum_{k=1}^{n-1} \frac{d(f \mid z_{nk}, z_{nn})\omega_n(z)(z_{nk} - z_{nn})}{\omega_n'(z_{nk})(z - z_{nk})},$$

where $d(f \mid z, t)$ is the first divided difference of f formed in z and t, as in § 2. In the present case, $\omega_n'(z) = (n-1)z^{n-2}(z - \xi) + (z^{n-1} - 1)$, $\omega_n'(z_{nk}) = (n-1)z_{nk}^{-1}(z_{nk} - \xi)$, $k = 1, \cdots, n-1$, $\omega_n'(z_{nn}) = \xi^{n-1} - 1$, and

$$L_n(f; z) = f(\xi) + \frac{(z^{n-1} - 1)(z - \xi)}{2\pi} \sum_{k=1}^{n-1} \frac{d(f \mid z_{nk}, \xi) z_{nk} 2\pi}{(z - z_{nk})(n-1)}.$$

The redundant factors $(1/2\pi)$ and 2π have been inserted to bring out the fact that the summation is a Riemann sum for the function of θ,

$$d(f \mid e^{i\theta}, \xi) \cdot \frac{e^{i\theta}}{z - e^{i\theta}}$$

formed for a partition of $[0, 2\pi]$ into $n - 1$ equal parts.

We now make the hypothesis that $d(f \mid e^{i\theta}, \xi)$ is Riemann integrable with respect to θ. It then follows from the elementary limit theorems that for $|z| < 1$,

$$\begin{aligned} \lim_{n\to\infty} L_n(f; z) &= f(\xi) + \frac{z - \xi}{2\pi} \int_0^{2\pi} d(f \mid e^{i\theta}, \xi) \frac{e^{i\theta}}{e^{i\theta} - z} d\theta \\ &= f(\xi) + \frac{z - \xi}{2\pi i} \int_{|t|=1} \frac{d(f \mid t, \xi)}{(t - z)} dt \\ &= F(z) - \frac{1}{2\pi i} \int_{|t|=1} d(f \mid t, \xi) dt \end{aligned} \tag{4.2}$$

where $F(z)$ is given by (2.4). The convergence is uniform on any closed subset of $|z| < 1$.

It should be noted that if f is analytic on $|z| < 1$, continuous on $|z| \leqq 1$, and such that $d(f \mid t, \xi)$ is Riemann integrable with respect to

t on $|t| = 1$, then by Cauchy's Integral Theorem the integral in the last member of (4.2) is zero and $F(z) = f(z)$, $|z| \leqq 1$, so $\lim_{n\to\infty} L_n(f; z) = f(z)$, $|z| < 1$, as in the theory for equally spaced points.

If S_n consists of the $(n-2)$th roots of unity plus two distinct "mavericks", ξ_1 and ξ_2, $|\xi_1| = |\xi_2| = 1$ and neither ξ_1 nor ξ_2 a root of unity, then (4.2) becomes

$$\lim_{u\to\infty} L_n(f; z) = F(z) - \frac{1}{2\pi i}\int_{|t|=1} d(f\,|\,t, \xi_2)dt - \frac{z-\xi_2}{2\pi i}\int_{t|=1} d(f\,|\,t, \xi_1, \xi_2)dt\ , \tag{4.3}$$

in the divided difference notation of § 2. It must be assumed that the second difference of f formed in the variable point t and the fixed points ξ_1 and ξ_2 is integrable with respect to t. The pattern for adjoining additional mavericks to the roots of unity is apparent from (4.3). No matter how many mavericks there are, if f is analytic on $|z| < 1$, continuous on $|z| \leqq 1$, and sufficiently smooth in the respective neighborhoods of the mavericks, then $\lim_{n\to\infty} L_n(f; z) = f(z)$, $|z| < 1$.

This suggests that Professor Wintner's question may have an affirmative answer, at least for functions which are analytic on $|z| < 1$, continuous on $|z| \leqq 1$, and infinitely differentiable on $|z| = 1$ but not analytic on $|z| \leqq 1$. The question arises as to whether a necessary condition for an affirmative answer to his question for all strongly equidistributed interpolation sets, or just for all sets of the type $\xi, \xi^2, \xi^3, \cdots$, is that f is analytic on $|z| \leqq 1$. The sufficiency is of course covered by Fejér's result.

Recently the author [3] announced some necessary and sufficient conditions for convergent interpolation to functions given on Jordan curves in the complex domain. Some rapid answers can be given by the sequence $S_1^*, S_2^*, \cdots$, to certain questions which might arise in connection with the rôle of equidistribution in these necessary and sufficient conditions. The part of the theorem here of interest can be stated as follows:

(a) *Let the rectifiable Jordan curve C contain the origin of its interior. A necessary and sufficient condition that*

$$\lim_{n\to\infty} L_n(f; z) = F(z_0) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(t)}{t-z_0}dt \tag{4.4}$$

at a single pre-selected point z_0 of the interior D of C for all continuous f is that the sequence $S_1, S_2, \cdots$ of interpolation points be such that

$$\lim L_n(z^{-k}; z_0) = 0\ , \qquad k = 1, 2, \cdots \tag{4.5}$$

and at the same time

$$\sum_{k=1}^{n}\left|\frac{\omega_n(z_0)}{(z_0 - z_{nk})\omega_n'(z_{nk})}\right| \tag{4.6}$$

is uniformly bounded for all n. If (4.6) *is not satisfied, then there exists a continuous f for which* $\lim_{n\to\infty} |L_n(f; z_0)| = \infty$.

(b) *If f is analytic on D and continuous on $C \cup D$, then* (4.6) *alone is a sufficient condition for* (4.4).

(c) *If* (4.6) *holds at only one point z_0, then the image sequence on the unit circle of $S_1, S_2, \cdots$ under the mapping function used by Fejér* (§ 2) *is strongly equidistributed.*

The first question is whether strong equidistribution with each S_n containing only distinct points might be sufficient for either (4.5) or (4.6) to hold. We have already seen in (4.1) that when C is the unit circle the answer for (4.5) is no. As for (4.6), for the set S_n^* the last term of the summation is

$$\left|\frac{\omega_n(z)}{(z_0 - \xi)\omega_n'(\xi)}\right| = \left|\frac{z_0^n - 1}{\xi^n - 1}\right| .$$

The equidistributed points $\xi, \xi^2, \xi^3, \cdots$ are everywhere dense on $|z| = 1$, so this term is unbounded for each z_0, $|z_0| < 1$. Therefore (4.6) cannot hold for this equidistributed sequence, and furthermore for each z_0, $|z_0|<1$, there is a continuous function f for which $|L_n(f; z_0)|$ formed in S_n^* is unbounded as n increases.

The second question is whether (4.6) might be necessary as well as sufficient for convergent interpolation to all functions f analytic on D and continuous on $C \cup D$. When C is the unit circle, we have shown that the sequence $S_1^*, S_2^*, \cdots$ provides convergent interpolation to all such functions for which $d(f \mid t, \xi)$ is integrable in t, and this sequence does not satisfy (4.6). Thus at least for the Lipschitz subclass of the class of functions under consideration, (4.6) is not a necessary condition. But the question remains open as to whether if (4.6) is not satisfied, a function f analytic on D and merely continuous on $C \cup D$ can always be constructed such that $L_n(f; z_0) \to \infty$ for some z_0, $|z_0| < 1$. This appears to be related to another open question communicated to the author by Professor Philip C. Curtis, Jr.: Given any sequence $S_1, S_2, \cdots$, on C which may or may not satisfy (4.6); can a function f analytic on D and continuous on $C \cup D$ always be constructed such that for some point z_0 on C, $L_n(f; z_0) \to \infty$?

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