ON DIRECT SUMS AND PRODUCTS OF MODULES

STEPHEN U. CHASE

A well-known theorem of the theory of abelian groups states that the direct product of an infinite number of infinite cyclic groups is not free ([6], p. 48.) Two generalizations of this result to modules over various rings have been presented in earlier papers of the author ([3], [4].) In this note we exhibit a broader generalization which contains the preceding ones as special cases.

Moreover, it has other applications. For example, it yields an easy proof of a part of a result of Baumslag and Blackburn [2] which gives necessary conditions under which the direct sum of a sequence of abelian groups is a direct summand of their direct product. We also use it to prove the following variant of a result of Baer [1]: If a torsion group T is an epimorphic image of a direct product of a sequence of finitely generated abelian groups, then T is the direct sum of a divisible group and a group of bounded order. Finally, we derive a property of modules over a Dedekind ring which, for the ring Z of rational integers, reduces to the following recent theorem of Rotman [10] and Nunke [9]: If A is an abelian group such that $\mathrm{Ext}_Z(A, T) = 0$ for any torsion group T, then A is slender.

In this note all rings have identities and all modules are unitary.

1. The main theorem. Our discussion will be based on the following technical device.

DEFINITION 1.1. Let $\mathscr{F}$ be a collection of principal right ideals of a ring R. $\mathscr{F}$ will be called a *filter of principal right ideals* if, whenever aR and bR are in $\mathscr{F}$, there exists $c \in aR \cap bR$ such that cR is in $\mathscr{F}$.

We proceed immediately to the principal result of this note.

THEOREM 1.2. *Let $A^{(1)}, A^{(2)}, \cdots$ be a sequence of left modules over a ring R, and set $A = \prod_{i=1}^{\infty} A^{(i)}$, $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$. Let $C = \sum_{\alpha} \oplus C_{\alpha}$, where $\{C_{\alpha}\}$ is a family of left R-modules and α traces an index set I. Let f: $A \to C$ be an R-homomorphism, and denote by f_{α}: $A \to C_{\alpha}$ the composition of f with the projection of C onto C_{α}. Finally, let $\mathscr{F}$ be a filter of principal right ideals of R. Then there exists aR in $\mathscr{F}$ and an integer $n > 0$ such that $f_{\alpha}(aA_n) \subseteqq \bigcap_{bR \in \mathscr{F}} bC_{\alpha}$ for all but a finite number of α in I.*

Proof. Assume that the statement is false. We shall first construct

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inductively sequences $\{x_n\} \subseteqq A$, $\{a_nR\} \subseteqq \mathscr{F}$, and $\{\alpha_n\} \subseteqq I$ such that the following conditions hold:

(i) $a_nR \supseteqq a_{n+1}R$.

(ii) $x_n \in a_nA_n$.

(iii) $f_{\alpha_n}(x_n) \not\equiv 0 \pmod{a_{n+1}C_{\alpha_n}}$.

(iv) $f_{\alpha_n}(x_k) = 0$ for $k < n$.

We proceed as follows. Select any a_1R in $\mathscr{F}$. Then there exists $\alpha_1 \in I$ such that $f_{\alpha_1}(a_1A_1) \not\subset \bigcap_{bR \in \mathscr{F}} bC_{\alpha_1}$, and hence we may select bR in $\mathscr{F}$ such that $f_{\alpha_1}(a_1A_1) \not\subset bC_{\alpha_1}$. Since $\mathscr{F}$ is a filter of principal right ideals, there exists $a_2 \in a_1R \cap bR$ such that $a_2R \in \mathscr{F}$, in which case $f_{\alpha_1}(a_1A_1) \not\subset a_2C_{\alpha_1}$. Hence there exists $x_1 \in a_1A_1$ such that $f_{\alpha_1}(x_1) \not\equiv 0 \pmod{a_2C_{\alpha_1}}$. Then conditions (i)–(iv) above are satisfied for $n = 1$.

Proceed by induction on n; assume that the sequences $\{x_k\}$ and $\{\alpha_k\}$ have been constructed for $k < n$ and the sequence $\{a_kR\}$ has been constructed for $k \leqq n$ such that conditions (i)–(iv) are satisfied. Now, there exist $\beta_1, \cdots, \beta_r \in I$ such that, if $\alpha \neq \beta_1, \cdots, \beta_r$, then $f_\alpha(x_k) = 0$ for all $k<n$. We may then select $\alpha_n \neq \beta_1, \cdots, \beta_r$ such that $f_{\alpha_n}(a_nA_n) \not\subset \bigcap_{bR \in \mathscr{F}} bC_{\alpha_n}$; for, if we could not do so, then the theorem would be true. Hence there exists $bR \in \mathscr{F}$ such that $f_{\alpha_n}(a_nA_n) \not\subset bC_{\alpha_n}$. Since $\mathscr{F}$ is a filter of principal right ideals, there exists $a_{n+1} \in a_nR \cap bR$ such that $a_{n+1}R$ is in $\mathscr{F}$, in which case $f_{\alpha_n}(a_nA_n) \not\subset a_{n+1}C_{\alpha_n}$. Thus we may select $x_n \in a_nA_n$ such that $f_{\alpha_n}(x_n) \not\equiv 0 \pmod{a_{n+1}C_{\alpha_n}}$. It is then clear that the sequences $\{x_k\}$ and $\{\alpha_k\}$ for $k \leqq n$ and $\{a_kR\}$ for $k \leqq n + 1$ satisfy conditions (i)–(iv), and hence the construction of all three sequences is complete.

Now write $x_k = (x_k^{(i)})$, where $x_k^{(i)} \in A^{(i)}$. Since $x_k \in a_kA_k$, $x_k^{(i)} = 0$ for $k > i$, and $x^{(i)} = \sum_{k=1}^{\infty} x_k^{(i)}$ is a well-defined element of $A^{(i)}$. Also, since $a_nR \supseteqq a_{n+1}R \supseteqq \cdots$, it follows that there exists $y_n^{(i)} \in A^{(i)}$ such that $x^{(i)} = x_1^{(i)} + \cdots + x_n^{(i)} + a_{n+1}y_n^{(i)}$. Therefore, setting $x = (x^{(i)})$ and $y_n = (y_n^{(i)})$, we see that $x = x_1 + \cdots + x_n + a_{n+1}y_n$ for all $n \geqq 1$.

It follows immediately from inspection of conditions (iii) and (iv) above that $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence there exists n such that $f_{\alpha_n}(x) = 0$. Writing $x = x_1 + \cdots + x_n + a_{n+1}y_n$ as above, we may then apply f_{α_n} and use condition (iv) to conclude that $f_{\alpha_n}(x_n) = -a_{n+1}f_{\alpha_n}(y_n) \equiv 0 \pmod{a_{n+1}C_{\alpha_n}}$, contradicting condition (iii). The proof of the theorem is hence complete.

In the following discussion we shall use the symbol $|X|$ to denote the cardinality of the set X.

COROLLARY 1.3 ([3], Theorem 3.1, p. 464). *Let R be a ring, and $A = \prod_{\alpha \in J} R^{(\alpha)}$, where $R^{(\alpha)} \approx R$ as a left R-module and $|J| \geqq \aleph_0$. Suppose that A is a pure submodule of $C = \sum_\beta \oplus C_\beta$, where each C_β is a left R-*

module and $|C_\beta| \leqq |J|$.[1] Then R must satisfy the descending chain condition on principal right ideals.

Proof. Since J is an infinite set, it is easy to see that $A \approx \prod_{i=1}^{\infty} A^{(i)}$, where $A^{(i)} \approx A$, and so without further ado we shall identify A with $\prod_{i=1}^{\infty} A^{(i)}$. Let $f: A \to C$ be the inclusion mapping, and $f_\beta: A \to C_\beta$ be the composition of f with the projection of C onto C_β. Finally, set $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$.

Suppose that the statement is false. Then there exists a strictly descending infinite chain $a_1R \supsetneqq a_2R \supsetneqq \cdots$ of principal right ideals of R. These ideals obviously constitute a filter of principal right ideals of R, and so we may apply Theorem 1.2 to conclude that there exists $n \geqq 1$ and $\beta_1, \cdots, \beta_r$ such that $f_\beta(a_nA_n) \subseteqq a_{n+1}C_\beta$ for $\beta \neq \beta_1, \cdots, \beta_r$.

Now let $C' = C_{\beta_1} \oplus \cdots \oplus C_{\beta_r}$; then the projection of C onto C' induces a Z-homomorphism $g: a_nC/a_{n+1}C \to a_nC'/a_{n+1}C'$, where Z is the ring of rational integers. Also, the restriction of f to A_n induces a Z-homomorphism $h: a_nA_n/a_{n+1}A_n \to a_nC/a_{n+1}C$. A_n is a direct summand of A, which is a pure submodule of C, and so A_n is likewise a pure submodule of C. Hence h is a monomorphism. We may then apply the conclusion of the preceding paragraph to obtain that the composition gh is a monomorphism. In particular, $|a_nA_n/a_{n+1}A_n| \leqq |a_nC'/a_{n+1}C'| \leqq |C'|$.

Observe that $|C'| \leqq |J|$, since J is infinite and $|C_\beta| \leqq |J|$ for all β. However, since $a_nR \neq a_{n+1}R$, $a_nR/a_{n+1}R$ contains at least two elements; therefore $|a_nA_n/a_{n+1}A_n| = |a_nA/a_{n+1}A| \geqq 2^{|J|} > |J|$. We have thus reached a contradiction, and the corollary is proved.

2. Applications to integral domains.

Throughout this section R will be an integral domain. If C is an R-module, we shall denote the maximal divisible submodule of C by $d(C)$. In addition, we shall write $R^\omega C = \bigcap aC$, where a traces the nonzero elements of R.

Our principal result concerning modules over integral domains is the following theorem.

THEOREM 2.1. *Let $\{A^{(i)}\}$ be a sequence of R-modules, and set $A = \prod_{i=1}^{\infty} A^{(i)}$, $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$. Let $C = \sum_\alpha \oplus C_\alpha$, where each C_α is an R-module. Let $f: A \to C$ be an R-homomorphism, and $f_\alpha: A \to C_\alpha$ be the composition of f with the projection of C onto C_α. Then there exists an integer $n \geqq 1$ and $a \in R$, $a \neq 0$, such that $af_\alpha(A_n) \subseteqq R^\omega C_\alpha$ for all but finitely many α.*

Proof. Let $\mathscr{F}$ be the set of all nonzero principal ideals of R. Since R is an integral domain, it is clear that $\mathscr{F}$ is a filter of principal ideals. The theorem then follows immediately from Theorem 1.2.

[1] A is a pure submodule of C if $A \cap aC = aA$ for all $a \in R$.

COROLLARY 2.2 (see [4].) Same hypotheses and notation as in Theorem 2.1, with the exception that now each C_α is assumed to be torsion-free. Then there exists an integer $n \geqq 1$ such that $f_\alpha(A_n) \subseteqq d(C_\alpha)$ for all but finitely many α. In particular, if each C_α is reduced (i.e., has no divisible submodules) then $f_\alpha(A_n) = 0$ for all but finitely many α.

Proof. This follows immediately from Theorem 2.1 and the trivial observation that, since each C_α is torsion-free, $R^\omega C_\alpha = d(C_\alpha)$.

Next we present our proof of the afore-mentioned result of Baumslag and Blackburn concerning direct summands of direct products of abelian groups ([2], Theorem 1, p. 403.)

THEOREM 2.3. *Let $\{A^{(i)}\}$ be a sequence of modules over an integral domain R, and set $A = \prod_{i=1}^\infty A^{(i)}$, $C = \sum_{i=1}^\infty \oplus A^{(i)}$ (then C is, in the usual way, a submodule of A.) If C is a direct summand of A, then there exists $n \geqq 1$ and $a \neq 0$ in R such that $aA^{(i)} \subseteqq d(A^{(i)})$ for $i > n$.*

Proof. Assume that C is a direct summand of A, and let $f: A \rightarrow C$ be the projection. Then the composition of f with the projection of C onto $A^{(i)}$ is an epimorphism $f_i: A \rightarrow A^{(i)}$. We then obtain from an easy application of Theorem 2.1 that there exists $n \geqq 1$ and $a \neq 0$ in R such that $af_i(A) \subseteqq R^\omega A^{(i)}$. Since each f_i is an epimorphism, it follows that $aA^{(i)} \subseteqq R^\omega A^{(i)}$ for $i > n$.

Now let $z \in R^\omega A^{(i)}$, where $i > n$. If $b \neq 0$ is in R, then there exists $x \in A^{(i)}$ such that $abx = z$. Hence, setting $y = ax$, we have that $y \in R^\omega A^{(i)}$ and $by = z$. It then follows that $R^\omega A^{(i)}$ is divisible, and so $R^\omega A^{(i)} \subseteqq d(A^{(i)})$. Therefore $aA^{(i)} \subseteqq R^\omega A^{(i)} \subseteqq d(A^{(i)})$ for $i > n$, completing the proof of the theorem.

We end this section with a proposition which will be useful in the proof of some later results.

PROPOSITION 2.4. Let $\{A^{(i)}\}$ be a sequence of finitely generated modules over an integral domain R, and set $A = \prod_{i=1}^\infty A^{(i)}$. Let $C = \sum_\alpha \oplus C_\alpha$, where each C_α is a finitely generated torsion R-module. If $f: A \rightarrow C$ is an R-homomorphism, then there exists $c \in R$ such that $cf(A) = 0$ but $c \neq 0$.

Proof. As before we let $\mathscr{F}$ be the filter of all nonzero principal ideals of R. Clearly $R^\omega C_\alpha = 0$ for all α, and so we may apply Theorem 2.1 to obtain $a \neq 0$ in R and an integer $n > 0$ such that $af_\alpha(A_n) = 0$ for all but finitely many α, where $A_n = \prod_{i=n+1}^\infty A^{(i)}$ and $f_\alpha: A \rightarrow C_\alpha$ is defined as before. Say this condition holds for $\alpha \neq \alpha_1, \cdots, \alpha_r$; then, since each C_α is finitely generated and torsion, there exists $a' \neq 0$ in R such that $a'C_{\alpha_i} = 0$ for $i = 1, \cdots, r$, in which case $aa'f(A_n) = 0$. Since each $A^{(i)}$ is finitely generated and C is a torsion module, there exists $a'' \neq 0$ in R such that $a''f(A^{(i)}) = 0$ for $i \leqq n$. Set $c = aa'a''$; then $c \neq 0$ and, since $A = A^{(1)} \oplus \cdots \oplus A^{(n)} \oplus A_n$, it is clear that $cf(A) = 0$, completing the proof of the proposition.

3. Applications to Abelian groups.

This section is devoted to a discussion of the results of Baer, Rotman, and Nunke mentioned in the introduction.

THEOREM 3.1 (see [1], Lemma 4.1, p. 231). *Let $\{A^{(i)}\}$ be a sequence of finitely generated modules over a principal ideal domain R, and set $A = \prod_{i=1}^{\infty} A^{(i)}$. If C is a torsion R-module which is an epimorphic image of A, then C is the direct sum of a divisible module and a module of bounded order.*

Proof. For each prime p in R, let C_p be the p-primary component of C and C'_p be a basic submodule of C_p (see [5], p. 98;) i.e., C'_p is a direct sum of cyclic modules and is a pure submodule of C_p, and C_p/C'_p is divisible.[2] Set $C' = \sum_p \oplus C'_p$; then, since $C = \sum_p \oplus C_p$, C' is a pure submodule of C and C/C' is divisible. Also, C' is a direct sum of cyclic modules.

We now apply the fundamental result of Szele ([5], Theorem 32.1, p. 106) to conclude that C'_p is an endomorphic image of C_p for each prime p, from which it follows that C' is an endomorphic image of C. Since by hypothesis C is an epimorphic image of A, we then see that there exists an epimorphism $f: A \rightarrow C'$. By Proposition 2.4, there exists $c \neq 0$ in R such that $cC = cf(A) = 0$; i.e., C' has bounded order. Since C' is a pure submodule of C, we may apply Theorem 7 of [6] (p. 18) to conclude that C' is a direct summand of C. Since C/C' is divisible, the proof is complete.

For the case in which R is the ring of rational integers, the assertion of Theorem 3.1 follows from the work of Nunke [9].

In the remainder of this note, R will be a Dedekind ring which is not a field. If A and C are R-modules, we shall write $\operatorname{Ext}(A, C)$ for $\operatorname{Ext}^1_R(A, C)$. The following two lemmas are well-known, but to our knowledge have not appeared explicitly in the literature.

LEMMA 3.2. *Let $a \neq 0$ be a nonunit in R, and let A and C be R-modules. Assume that $aC = 0$, and a operates faithfully on A* (i.e., *$ax = 0$ for $x \in A$ only if $x = 0$.*) *Then* $\operatorname{Ext}(A, C) = 0$.

[2] The definition and properties of basic submodules used here, as well as the theorem of Szele applied in the following paragraph, are in [5] given only for the special case in which R is the ring of rational integers. However, it is well-known that these results can be trivially extended to modules over an arbitrary principal ideal domain.

Proof. Since a operates faithfully on A, we obtain the exact sequence—

$$0 \longrightarrow A \xrightarrow{m_a} A \longrightarrow A/aA \longrightarrow 0$$

where m_a is defined by $m_a(x) = ax$. This gives rise to the exact cohomology sequence—

$$\operatorname{Ext}(A, C) \xrightarrow{m_a^*} \operatorname{Ext}(A, C) \longrightarrow 0$$

where $m_a^*(u) = au$ for u in $\operatorname{Ext}(A, C)$. But, since $aC = 0$, we have that $m_a^* = 0$, and so it follows from exactness that $\operatorname{Ext}(A, C) = 0$, completing the proof.

LEMMA 3.3. *Let $a \neq 0$ be a nonunit in R, and A, C be R-modules. Assume that a operates faithfully on A. Then the following statements are equivalent*:

(a) *a operates faithfully on* $\operatorname{Ext}(A, C)$.

(*b*) *The natural mapping* $\operatorname{Hom}(A, C) \rightarrow \operatorname{Hom}(A, C/aC)$ *is an epimorphism.*

Proof. Consider the exact sequence—

$$0 \longrightarrow C_a \longrightarrow C \xrightarrow{m_a} C \longrightarrow C/aC \longrightarrow 0$$

where $C_a = \{x \in C / ax = 0\}$ and m_a is defined as in Lemma 3.2. This sequence may be broken up into the following short exact sequences:

$$0 \longrightarrow C_a \longrightarrow C \xrightarrow{\mu} aC \longrightarrow 0$$

$$0 \longrightarrow aC \xrightarrow{\nu} C \longrightarrow C/aC \longrightarrow 0$$

where ν is the inclusion mapping and μ differs from m_a only by the obvious contraction of the range. Since $aC_a = 0$ and a operates faithfully on A, we obtain from Lemma 3.2 that $\operatorname{Ext}(A, C_a) = 0$, and so the relevant portions of the resulting cohomology sequences are as follows:

$$0 \longrightarrow \operatorname{Ext}(A, C) \xrightarrow{\mu_*} \operatorname{Ext}(A, aC) \longrightarrow 0$$

$$\operatorname{Hom}(A, C) \longrightarrow \operatorname{Hom}(A, C/aC) \longrightarrow \operatorname{Ext}(A, aC) \xrightarrow{\nu_*} \operatorname{Ext}(A, C) .$$

Since $m_a = \nu\mu$, we have that $m_{a*} = \nu_*\mu_*$, where $m_{a*}\colon \operatorname{Ext}(A, C) \rightarrow \operatorname{Ext}(A, C)$ is defined by $m_{a*}(u) = au$ for u in $\operatorname{Ext}(A, C)$. Hence (a) holds if and only if m_{a*} is a monomorphism. But this is true if and only if ν_* is a monomorphism, since μ_* is an isomorphism. But it is clear from the second exact sequence above that ν_* is a monomorphism if and only if (b) holds. The proof is hence complete.

In the remainder of this section we shall set $\Pi = \prod_{i=1}^{\infty} R^{(i)}$, where $R^{(i)} \approx R$.

THEOREM 3.4. *Let R be a Dedekind ring, and $a \neq 0$ be a nonunit in R. Set $C = \sum_{n=1}^{\infty} \oplus R/a^n R$. Let A be a torsion-free R-module satisfying the following conditions*:

(a) *Every submodule of A of finite rank is projective.*

(b) *a operates faithfully on* $\operatorname{Ext}(A, C)$.

Then, if $f \in \operatorname{Hom}(\Pi, A)$, $f(\Pi)$ has finite rank.

Proof. Assume that the statement is false for some $f \in \operatorname{Hom}(\Pi, A)$. Then $f(\Pi)$ contains a submodule F_0 of countably infinite rank. Let $F = \{x \in A / a^n x \in F_0 \text{ for some } n\}$. Then F likewise has countably infinite rank. We may then apply condition (a) and a result of Nunke ([8], Lemma 8.3, p. 239) to obtain that F is projective, and then a result of Kaplansky ([7], Theorem 2, p. 330) to conclude that F is free. Let $x_1, x_2, \cdots$ be a basis of F. Then there exist nonnegative integers $\nu_1, \nu_2, \cdots$ such that $y_n = a^{\nu_n} x_n$ is in F_0.

Let z_n generate the direct summand of C isomorphic to $R/a^n R$, and let $\bar{z}_n$ be the image of z_n under the natural mapping of C onto $\bar{C} = C/aC$. Define an R-homomorphism $\theta_1: F \to \bar{C}$ by $\theta_1(x_n) = \bar{z}_{n+\nu_n}$. Observe that $\theta_1(aF) = 0$, and so θ_1 induces a homomorphism $\theta_2: F/aF \to \bar{C}$. Now, it follows easily from the construction of F that the sequence $0 \to F/aF \to A/aF \to A/F \to 0$ is exact, and a operates faithfully on A/F. We may then apply Lemma 3.2 to conclude that this sequence splits. It is then clear that θ_2 can be extended to a homomorphism $\theta: A \to \bar{C}$. We emphasize the fact that $\theta(x_n) = \bar{z}_{n+\nu_n}$.

Since a operates faithfully on $\operatorname{Ext}(A, C)$, we may now apply Lemma 3.3 to obtain $\varphi \in \operatorname{Hom}(A, C)$ such that the diagram—

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\theta"'] & C \arrow[d] \\ & \bar{C} \end{tikzcd}$$

is commutative. Observe that, since $\theta(x_n) = \bar{z}_{n+\nu_n}$, $\varphi(x_n) \equiv z_{n+\nu_n} \pmod{aC}$. That is, the coefficient of $z_{n+\nu_n}$ in the expansion of $\varphi(x_n)$ is $1 + at_n$ for some $t_n \in R$. Since $y_n = a^{\nu_n} x_n$, the coefficient of $z_{n+\nu_n}$ in the expansion of $\varphi(y_n)$ is $a^{\nu_n} + a^{\nu_n+1} t_n$.

Set $g = \varphi f$; then $g \in \operatorname{Hom}(\Pi, C)$, and so we may apply Proposition 2.4 to conclude that $cg(\Pi) = 0$ for some $c \neq 0$ in R. Since each y_n is in $f(\Pi)$, and z_n generates a direct summand of C isomorphic to $R/a^n R$, it then follows from the preceding paragraph that $c(a^{\nu_n} + a^{\nu_n+1} t_n)$ is in $a^{n+\nu_n} R$ for all n, in which case $c(1 + at_n)$ is in $a^n R$ for all n. Let P

be any prime ideal in R containing a; then $1 + at_n$ is a unit modulo P^n for all $n > 0$, and so $c \in P^n$ for all n. Therefore $c = 0$, a contradiction. This completes the proof of the theorem.

COROLLARY 3.5. *Let R be a Dedekind ring (not a field,) and let A be an R-module with the property that* $\mathrm{Ext}(A, C) = 0$ *for any torsion module C. Then, if $f \in \mathrm{Hom}(\Pi, A), f(\Pi)$ is a projective module of finite rank.*

Proof. We may apply a result of Nunke ([8], Theorem 8.4, p. 239) to obtain that A is torsion-free and every submodule of A of finite rank is projective. The corollary then follows immediately from Theorem 3.4.

The following special case of Theorm 3.4 was first proved by Rotman ([10], Theorem 3, p. 250) under an additional hypothesis whitch was later removed by Nunke ([9], p. 275.)

COROLLARY 3.6. *Let A be an abelian group such that* $\mathrm{Ext}(A, C) =$ 0 *for any torsion group C. Then A is slender.*[3]

Proof. We need only show that, for any $f \in \mathrm{Hom}(\Pi, A)$, $f(\Pi)$ is slender. By Corollary 3.5, $f(\Pi)$ is free of finite rank. But it is well-known that a free abelian group is slender (see [5], Theorems 47.3 and 47.4, pp. 171–172.) The proof is hence complete.

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PRINCETON UNIVERSITY

[3] For the definition of a slender Abelian group we refer the reader to [9].