SOME EXACT SEQUENCES IN COHOMOLOGY THEORY FOR KÄHLER MANIFOLDS

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1. Introduction. In this note some results announced in [1] concerning exact sequences for Kähler manifolds are proved. The main result is that there exists an exact sequence relating the usual bigraded groups of harmonic forms on a compact Kähler manifold K and an imbedded submanifold L with certain mixed relative cohomology groups of (K, L). These mixed cohomology groups have been introduced by Hodge [6]. Using the results of Hodge the exactness of the given sequence is derived in a straightforward manner. H. Guggenheimer considered in [5] such a sequence too. Finally the exact sequence is applied to deduce some results concerning the imbedding of a complex manifold L in a Kähler manifold K, in particular the following statement is proved: if the imbedding $L \subset K$ is homology faithful, then $b_{r,s}(K) = b_{r,s}(L)$ implies $b_{r-k,s-k}(K) = b_{r-k,s-k}(L)$ for $k = 0, 1, 2, \cdots$ ($b_{r,s}$ = rank of the module of harmonic (r, s)-forms). The paper is organized the following way: § 2 gives the necessary notations and a few known results; § 3 contains the main theorem (Theorem 1) the proof of which is given in § 4; some applications follow in §§ 5 and 6, in particular Theorem 2 which implies the foregoing statement on homology faithful imbeddings.

2. Notations and known results.

(a) We use the following notations:

$F^{r,s}$: group of the complex valued $C^\infty (r + s)$-forms of type (r, s) on a complex manifold $M = M^{2n}$.

$F^k = \sum_{r+s=k} F^{r,s}$.

$d = d' + d''$: exterior differentiation operator on M; $d: F^k \to F^{k+1}$, $d': F^{r,s} \to F^{r+1,s}$, $d'': F^{r,s} \to F^{r,s+1}$ (cf [2], [6]).

$*$: $F^{r,s} \to F^{n-s,n-r}$: Hodge—de Rham duality operator which associates to α its (metrically) dual form $*\alpha$, with the help of a Hermitian metric (of class C^∞) on M (cf. [6] and [3]; $** = (-1)^{k(2n-k)} = (-1)^k$: $F^k \to F^k$).

$\delta = -*d* = \delta' + \delta'' = -(*d''* + *d'*)$ ($\delta' = -*d''*$ resp. $\delta'' = -*d'*$ is an operator of type $(-1, 0)$ resp. $(0, -1)$).

$\nabla = d'd''$, $\nabla^* = \delta''\delta' = (-1)^{k+1}*\nabla*$.

$\Delta = d\delta + \delta d$: Laplace—Beltrami operator.

$\Delta' = d'\delta' + \delta'd'$, $\Delta'' = d''\delta'' + \delta''d''$.

We define $Z_{d'}^{r,s} = \{\alpha \mid \alpha \in F^{r,s}, d'\alpha = 0\}$, similarly $Z_{d''}^{r,s}$, $Z_{\nabla}^{r,s}$, $Z_d^{r,s} = Z_{d',d''}^{r,s}$,

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moreover $R_{d'}^{r,s} = \{\alpha \mid \alpha \in F^{r,s},\ \alpha = d'\beta \text{ for some } \beta\} = d'F^{r-1,s}$, similarly $R_{d''}^{r,s} = d''F^{r,s-1}$, $R_{\nabla}^{r,s} = \nabla F^{r-1,s-1}$, $R_{d',d''}^{r,s} = d'F^{r-1,s} \oplus d''F^{r,s-1}$, $R_{d}^{r,s} = \{\alpha \mid \alpha \in F^{r,s},\ \alpha = d\beta \text{ for some } \beta\}$, and dually $Z_{\delta''}^{r,s}$ etc[1]. Let $H_{\Delta}^{r,s} = Z_{\Delta}^{r,s}$ be the group of harmonic (r, s)-forms, $H_{\Delta'}^{r,s} = Z_{\Delta'}^{r,s}$ and $H_{\Delta''}^{r,s} = Z_{\Delta''}^{r,s}$.

Considering the components of different type in $dd\alpha$, the relation $dd = 0$ implies

$$(1) \qquad d'd' = 0\ , \quad d''d'' = 0\ , \quad d'd'' + d''d' = 0\ ,$$

and therefore the following relations hold:

$$(2) \qquad \begin{cases} R_{d'}^{r,s} \subset Z_{d'}^{r,s},\ R_{d''}^{r,s} \subset Z_{d''}^{r,s}\ , \\ R_{\nabla}^{r,s} \subset R_{d'}^{r,s} \cap R_{d''}^{r,s} \cap R_{d}^{r,s} \subset Z_{d}^{r,s} = Z_{d',d''}^{r,s} = Z_{d'}^{r,s} \cap Z_{d''}^{r,s}\ , \\ R_{d}^{r,s} \subset R_{d',d''}^{r,s} = R_{d'}^{r,s} \oplus R_{d''}^{r,s} \subset Z_{d'}^{r,s} \oplus Z_{d''}^{r,s} \subset Z_{\nabla}^{r,s}\ . \end{cases}$$

(2) enables us to form the cohomology groups $H_{d'}^{r,s} = Z_{d'}^{r,s}/R_{d'}^{r,s}$, $H_{d''}^{r,s} = Z_{d''}^{r,s}/R_{d''}^{r,s}$ (Dolbeault groups), and $H_{d/\nabla}^{r,s} = H_{d',d''/\nabla}^{r,s} = Z_{d}^{r,s}/R_{\nabla}^{r,s}$, $H_{\nabla/d',d''}^{r,s} = Z_{\nabla}^{r,s}/R_{d',d''}^{r,s}$ (mixed cohomology groups). Dually to (1), (2), there are relations $\delta'\delta' = 0$ etc., $R_{\delta'}^{r,s} \subset Z_{\delta'}^{r,s}$ etc., and cohomology groups $H_{\delta'}^{r,s}$ etc.

(b) If M is a Kähler manifold (cf. [2], [6], [4]) one has

$$(3) \qquad \Delta = 2\Delta' = 2\Delta''\ ,$$

$$(4) \qquad d'\delta'' + \delta''d' = 0, \quad d''\delta' + \delta'd'' = 0\ ,$$

such that $H_{\Delta} = H_{\Delta'} = H_{\Delta''}$ and (using the obvious notation) $Z_{d'\delta''} = Z_{\delta''d'}$, $Z_{d''\delta'} = Z_{\delta'd''}$. Moreover, (4) makes it possible to consider the further mixed cohomology groups $H_{d',\delta''/d'\delta''}^{r,s}$, $H_{d'\delta''/d',\delta''}^{r,s}$, $H_{d'',\delta'/d''\delta'}^{r,s}$, $H_{d''\delta'/d'',\delta'}^{r,s}$ (using again the obvious notation).

For a compact Kähler manifold M the following isomorphisms (listed for completeness sake) are well known: $H_{\Delta}^{r,s} \cong H_{\Delta}^{s,r}$ (Eckmann-Guggenheimer), $H_{\Delta}^{r,s} \cong H_{\Delta}^{n-s,n-r}$ (refined Poincaré duality; holds for any compact complex manifold with a Hermitian metric). The de Rham—Hodge theorem

$$(5) \qquad H^p \cong H_d^p \cong H_{\Delta}^p \cong \sum_{r+s=p} H_{\Delta}^{r,s}$$

will be used later on. Here H^p is the usual pth cohomology group of M over the field C of the complex numbers and H_d^p the ordinary pth de Rham cohomology group of M over C. The most important results frequently used in the sequel are the isomorphisms proved by Hodge in [6]:

$$(6) \qquad H_{\Delta}^{r,s} \cong H_{d'}^{r,s} \cong H_{d''}^{r,s}\ ,$$

$$(7) \qquad H_{\Delta}^{r,s} \cong H_{d/\nabla}^{r,s}\ ,$$

1 《etc.》 indicates—here and sometimes in the sequel—the possibility of analogous formulas or expressions using dual and/or complex conjugate operators.

$$H_{\Delta}^{r,s} \cong H_{\nabla/d',d''}^{r,s} \,. \tag{8}$$

Dually $H_{\Delta}^{r,s} \cong H_{\delta''}^{r,s}$ etc. [6] contains also $H_{\Delta}^{r,s} \cong H_{d',\delta''/d'\delta''}^{r,s} \cong H_{d'\delta''/d',\delta''}^{r,s} \cong H_{d'',\delta'/d''\delta'}^{r,s} \cong H_{d''\delta'/d'',\delta'}^{r,s}$. The proofs for these isomorphisms and for (6), (7), (8) are strongly related; one applies the decomposition theorem for the forms on a compact Kähler manifold, deduced with the help of (3) and (4). The considered isomorphisms are induced in a natural way by inclusion ($Z_{\Delta}^{r,s} \subset Z_{d'}^{r,s}$ and so on).[2]

3. The main theorem.

(a) Let $K = K^{2n}$ and $L = L^{2m}$ be compact Kähler manifolds, $m < n$, and let L be imbedded in K, regularly (and complex analytically) such that there exist coordinates $z_1, z_2, \cdots, z_n$ in the neighborhood $U(x) \subset K$ of any point $x \in L$ with $L \cap U(x) = \{p \mid p \in U(x), z_1(p) = \cdots = z_{n-m}(p) = 0\}$. We consider the relative groups $F^{r,s}(K, L) = \{\alpha \mid \alpha \in F^{r,s}(K), \alpha = 0$ along $L\}$, $Z_{d'}^{r,s}(K, L) = \{\alpha \mid \alpha \in F^{r,s}(K, L),\ d'\alpha = 0\} = Z_{d'}^{r,s}(K) \cap F^{r,s}(K, L)$ etc., $R_{d'}^{r,s}(K, L) = d'F^{r-1,s}(K, L)$ etc. ⟪$\alpha = 0$ along L⟫ means: α vanishes tangentially along L. The inclusion $L \xrightarrow{i} K$ induces homomorphisms $F^{r,s}(K) \xrightarrow{i^\#} F^{r,s}(L)$, $F^{r,s}(K, L) \xrightarrow{j^\#} F^{r,s}(K)$, and $i^\#, j^\#$ commute with d', d'', ∇. Clearly the relations (2) hold still in the relative case, so the relative cohomology groups $H_{d'}^{r,s}(K, L)$, $H_{d''}^{r,s}(K, L)$, $H_{d/\nabla}^{r,s}(K, L)$, $H_{\nabla/d',d''}^{r,s}(K, L)$ are well defined: $H_{d'}^{r,s}(K, L) = Z_{d'}^{r,s}(K, L)/R_{d'}^{r,s}(K, L)$ etc. Using (6), (7), (8), the operators $i^\#, j^\#, d', d'', \nabla$ induce the cohomology homomorphisms i^*, j^*, d', d'', ∇ in the following sequences (cf. [5]):

$$\cdots \xrightarrow{i^*} H_{\Delta}^{r-1,s}(L) \xrightarrow{d'} H_{d'}^{r,s}(K, L) \xrightarrow{j^*} H_{\Delta}^{r,s}(K) \xrightarrow{i^*} H_{\Delta}^{r,s}(L) \xrightarrow{d'} \cdots, \tag{9}$$

$$\cdots \xrightarrow{i^*} H_{\Delta}^{r,s-1}(L) \xrightarrow{d''} H_{d''}^{r,s}(K, L) \xrightarrow{j^*} H_{\Delta}^{r,s}(K) \xrightarrow{i^*} H_{\Delta}^{r,s}(L) \xrightarrow{d''} \cdots, \tag{10}$$

$$\cdots \xrightarrow{i^*} H_{\Delta}^{r-1,s-1}(L) \xrightarrow{\nabla} H_{d/\nabla}^{r,s}(K, L) \xrightarrow{j^*} H_{\Delta}^{r,s}(K) \xrightarrow{i^*} H_{\Delta}^{r,s}(L) \xrightarrow{\nabla} \cdots. \tag{11}$$

i^*, j^* are given in the natural way; d', d'', ∇ are explained in (b). In § 4 we prove the

THEOREM 1. *The sequences* (9), (10), (11) *are exact.*

We will treat only (11); the considerations can easily be carried over for to get the analogous results for (9) and (10)[3]. In the sequel, all the occurring sequences of groups and homomorphisms are exact.

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2 It follows from (6) that dim $H_{\Delta}^{r,s}$ is independent of the Kähler metric on M. $-H_{\Delta}^{r,0} \cong H_{d''}^{r,0}$ is an equality $H_{\Delta}^{r,0} = H_{d''}^{r,0} = Z_{d''}^{r,0}$ and says: every holomorphic r-form is a harmonic $(r, 0)$-form and vice versa.

3 The constructions in (9) and (10) and the proof for Theorem 1 in these cases are completely parallel to the corresponding procedures in the case of the de Rham cohomology (using the operator d). Hence we are mostly interested in (11).

(b) To describe the homomorphism ∇ in (11) we follow the usual procedure as it can be found e.g. in [7], p. 192. Every element $\alpha \in F^{r,s}(L)$ is the trace of an element $\tilde{\alpha} \in F^{r,s}(K)$, and one has

$$0 \longrightarrow F^{r,s}(K,L) \xrightarrow{j^\#} F^{r,s}(K) \xrightarrow{i^\#} F^{r,s}(L) \longrightarrow 0 .$$

Applying ∇, we get the commutative diagram

$$(12) \quad \left\{ \begin{array}{ccccccccc} 0 & \longrightarrow & F^{r,s}(K, L) & \xrightarrow{j^\#} & F^{r,s}(K) & \xrightarrow{i^\#} & F^{r,s}(L) & \longrightarrow & 0 \\ & & \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \\ 0 & \longrightarrow & F^{r+1,s+1}(K, L) & \xrightarrow{j^\#} & F^{r+1,s+1}(K) & \xrightarrow{i^\#} & F^{r+1,s+1}(L) & \longrightarrow & 0 . \end{array} \right.$$

Let $\alpha \in Z_\Delta^{r,s}(L)$, and let $\alpha = i^\# \tilde{\alpha}$, $\tilde{\alpha} \in F^{r,s}(K)$. Then $i^\# \nabla \tilde{\alpha} = \nabla i^\# \tilde{\alpha} = \nabla \alpha = 0$, i.e. $\nabla \tilde{\alpha} \in F^{r+1,s+1}(K, L)$. $d'\nabla = d''\nabla = 0$ implies $\nabla \tilde{\alpha} \in Z_d^{r+1,s+1}(K, L)$. Let $\tilde{\beta}$ be a second extension of α: $\alpha = i^\# \tilde{\beta}$, $\tilde{\beta} \in F^{r,s}(K)$. Since $\nabla(\tilde{\alpha} - \tilde{\beta}) = \nabla \tilde{\alpha} - \nabla \tilde{\beta}$, $\tilde{\alpha} - \tilde{\beta} = 0$ along L, $\nabla \tilde{\alpha}$ and $\nabla \tilde{\beta}$ determine the same element in $H_{d/\nabla}^{r+1,s+1}(K, L)$ such that we get a homomorphism $Z_\nabla^{r,s}(L) \xrightarrow{\nabla} H_{d/\nabla}^{r+1,s+1}(K, L)$. Moreover $R_{d',d''}^{r,s}(L) \xrightarrow{\nabla} 0$ since every form $\alpha \in R_{d',d''}^{r,s}(L)$ is trace of a form $\tilde{\alpha} \in R_{d',d''}^{r,s}(K)$ (and because $\nabla d' = \nabla d'' = 0$). Therefore, (12) gives rise to a homomorphism $H_{\nabla/d',d''}^{r,s}(L) \xrightarrow{\nabla} H_{d',d''/\nabla}^{r+1,s+1}(K, L) = H_{d/\nabla}^{r+1,s+1}(K, L)$ and to the sequence

$$(13) \qquad \cdots \xrightarrow{i^*} H_{\nabla/d',d''}^{r-1,s-1}(L) \xrightarrow{\nabla} H_{d/\nabla}^{r,s}(K, L) \xrightarrow{j^*} H_\Delta^{r,s}(K) \xrightarrow{i^*} H_{\nabla/d',d''}^{r,s}(L) \xrightarrow{\nabla} \cdots$$

with $i^* = \bar{i}\, I$ where $H_\Delta^{r,s}(K) \xrightarrow{I} H_{\nabla/d',d''}^{r,s}(K)$ is the canonical isomorphism (8) and $H_{\nabla/d',d''}^{r,s}(K) \xrightarrow{\bar{i}} H_{\nabla/d',d''}^{r,s}(L)$ is the restriction homomorphism belonging to the inclusion i, and with $j^* = J\bar{j}$ where $H_{d/\nabla}^{r,s}(K, L) \xrightarrow{\bar{j}} H_{d/\nabla}^{r,s}(K)$ is the inclusion homomorphism belonging to i and $H_{d/\nabla}^{r,s}(K) \xrightarrow{J} H_\Delta^{r,s}(K)$ is the inverse of the canonical isomorphism (7). Replacement of $H_{\nabla/d',d''}^{r,s}(L)$ by $H_\Delta^{r,s}(L)$ in (13) in virtue of (8) leads to the sequence (11), and in order to prove exactness of (11) it is enough to prove exactness of (13).

4. Proof of the main theorem.

(a) $\quad H_{\nabla/d',d''}^{r-1,s-1}(L) \xrightarrow{\nabla} H_{d/\nabla}^{r,s}(K, L) \xrightarrow{\bar{j}} H_{d/\nabla}^{r,s}(K).$

Proof. $\bar{j}\nabla = 0$ immediate, i.e. image$(\nabla) \subset$ kernel$(\bar{j})$. Let $\bar{\alpha} \in H_{d/\nabla}^{r,s}(K, L)$ with $\bar{j}\bar{\alpha} = 0$, and $\bar{\alpha}$ be represented by $\alpha \in Z_d^{r,s}(K, L)$. $\bar{j}\bar{\alpha} = 0$ implies $\alpha = \nabla \alpha'$, $\alpha' \in F^{r-1,s-1}(K)$. Put $\alpha_1 = i^\# \alpha'$; then $\nabla \alpha_1 = \nabla i^\# \alpha' = i^\# \nabla \alpha' = i^\# \alpha = 0$, therefore $\alpha_1 \in Z_\nabla^{r-1,s-1}(L)$, α_1 represents an element $\bar{\alpha}_1 \in H_{\nabla/d',d''}^{r-1,s-1}(L)$ with

$\nabla\bar{\alpha}_1 = \bar{\alpha}$, i.e. $\operatorname{im}(\nabla) \supset \ker(\bar{j})$.

(b) $H^{r,s}_{d/\nabla}(K, L) \xrightarrow{\bar{j}} H^{r,s}_{d/\nabla}(K) \xrightarrow{\bar{\bar{i}}} H^{r,s}_{d/\nabla}(L)$.

Proof. $\bar{\bar{i}}\bar{j} = 0$ immediate. Let $\bar{\alpha} \in H^{r,s}_{d/\nabla}(K)$ with $\bar{\bar{i}}\bar{\alpha} = 0$, represented by α. Then $i^\#\alpha = \nabla\alpha'$ for an $\alpha' \in F^{r-1,s-1}(L)$. Let $\tilde{\alpha}'$ be an extension of α' $(i^\#\tilde{\alpha}' = \alpha')$ such that $\beta = \alpha - \nabla\tilde{\alpha}' \in Z^{r,s}_d(K, L)$. β represents $\bar{\beta} \in H^{r,s}_{d/\nabla}(K, L)$, and $\bar{j}\bar{\beta} = \bar{\alpha}$. This shows $\operatorname{im}(\bar{j}) \supset \ker(\bar{\bar{i}})$.

(c) $H^{r,s}_{\nabla/d',d''}(K) \xrightarrow{\bar{i}} H^{r,s}_{\nabla/d',d''}(L) \xrightarrow{\nabla} H^{r+1,s+1}_{d/\nabla}(K, L)$.

Proof. $\nabla\bar{i} = 0$ immediate. Let $\bar{\alpha} \in H^{r,s}_{\nabla/d',d''}(L)$ with $\nabla\bar{\alpha} = 0$, represented by α. Let $i^\#\tilde{\alpha} = \alpha$, $\tilde{\alpha} \in F^{r,s}(K)$. $\beta = \nabla\tilde{\alpha}$ represents then the zero element in $H^{r+1,s+1}_{d/\nabla}(K, L)$; therefore $\beta = \nabla\alpha_1$ for an $\alpha_1 \in F^{r,s}(K, L)$. Hence $\gamma = \tilde{\alpha} - \alpha_1 \in Z^{r,s}_\nabla(K)$, γ represents an element $\bar{\gamma} \in H^{r,s}_{\nabla/d',d''}(K)$, and $\bar{i}\bar{\gamma} = \bar{\alpha}$. Therefore $\operatorname{im}(\bar{i}) \supset \ker(\nabla)$.

(d) We have the commutative diagram

$$\begin{array}{ccc} H^{r,s}_{d/\nabla}(K) & \xrightarrow{\bar{\bar{i}}} & H^{r,s}_{d/\nabla}(L) \\ \big\downarrow h_K & & \big\downarrow h_L \\ H^{r,s}_{\nabla/d',d''}(K) & \xrightarrow{\bar{i}} & H^{r,s}_{\nabla/d',d''}(L) \end{array}$$

where h_K, h_L are naturally induced isomorphisms (by [6], cf. (7) and (8)).

(a), (b), (c), (d) and (7), (8) imply exactness of (13) and (11).

5. Some remarks and first applications.

(a) REMARK 1. If K is a Kähler manifold, and if the complex manifold L is complex analytically immersed in K, then L is a Kähler manifold too since the Kähler metric in K induces one in L. So Theorem 1 holds if K is a compact Kähler manifold and L a regularly imbedded complex manifold in K.

REMARK 2. The homomorphism $H^{r,s}_{\nabla/d',d''}(L) \xrightarrow{\nabla} H^{r+1,s+1}_{d/\nabla}(K, L)$ can be defined also in the following situation: K and L complex manifolds of the same dimension, L open subset of K. One makes use of the local character of the operator ∇ (i.e. $T\nabla\alpha \subset T\alpha$ for $T\alpha$ = carrier of α). Inclusion and restriction homomorphisms are defined the usual way, and (a), (b), (c) hold in this case too.

REMARK 3. (9), (10) hold in the form

$$(9') \quad \cdots \xrightarrow{i^*} H^{r-1,s}_{d'}(L) \xrightarrow{d'} H^{r,s}_{d'}(K, L) \xrightarrow{j^*} H^{r,s}_{d'}(K) \xrightarrow{i^*} H^{r,s}_{d'}(L) \xrightarrow{d'} \cdots ,$$

$$(10'') \quad \cdots \xrightarrow{i^*} H_{d''}^{r,s-1}(L) \xrightarrow{d''} H_{d''}^{r,s}(K, L) \xrightarrow{j^*} H_{d''}^{r,s}(K) \xrightarrow{i^*} H_{d''}^{r,s}(L) \xrightarrow{d''} \cdots,$$

for complex manifolds K, L (Kählerian or not) with $\dim L < \dim K$ and L regularly imbedded in K, or with $\dim L = \dim K$ and $L =$ open subset of K.

REMARK 4. An analogous dual construction in the situation $L =$ open subset of K is possible for δ', δ'' where one gets exact sequences dual to (9'), (10''). ∇^* gives rise to the dual statements of (a), (b), (c) (again for $L =$ open subset of K).

(b) We return to our original assumptions as stated in § 3 (a). We put $H_{d'}^p = \sum_{r+s=p} H_{d'}^{r,s}$ etc., and (5), (6), (7), (8) imply

$$H^p(K) \cong H_d^p(K) \cong H_\Delta^p(K) \cong H_{d'}^p(K) \cong H_{d''}^p(K) \cong H_{d/\nabla}^p(K) \cong H_{\nabla/d',d''}^p(K)$$

for a compact Kähler manifold K. Moreover, considering the exact sequences (9), (10) besides those which belong to the de Rham cohomology and to the usual cohomology, we get ([1], p. 293)[4]

$$(14) \qquad H^p(K, L) \cong H_d^p(K, L) \cong H_{d'}^p(K, L) \cong H_{d''}^p(K, L) .$$

It seems harder to express $H_{d/\nabla}^p(K, L)$ (in which we are interested because of (11)) by means of known quantities. A formula similar to (14) is certainly not correct for these groups. We have the commutative diagram

$$(15) \quad \begin{array}{ccccccccccccccc}
\cdots \xrightarrow{i^*} & H_\Delta^{r-1,s-1}(L) & \xrightarrow{d'} & H_{d'}^{r,s-1}(K,L) & \xrightarrow{j^*} & H_\Delta^{r,s-1}(K) & \xrightarrow{i^*} & H_\Delta^{r,s-1}(L) & \xrightarrow{d''} & H_{d''}^{r,s}(K,L) & \xrightarrow{j^*} & H_\Delta^{r,s}(K) & \xrightarrow{i^*} & H_\Delta^{r,s}(L) & \xrightarrow{d'} \cdots \\
& \updownarrow & & & & & & & & & & \updownarrow & & \updownarrow & \\
\cdots \xrightarrow{i^*} & H_\Delta^{r-1,s-1}(L) & & & \xrightarrow{\nabla} & & & H_{d/\nabla}^{r,s}(K,L) & & & \xrightarrow{j^*} & H_\Delta^{r,s}(K) & \xrightarrow{i^*} & H_\Delta^{r,s}(L) & \xrightarrow{\nabla} \cdots
\end{array}$$

which implies $H_{d/\nabla}^{r,s}(K, L) \cong d' H_\Delta^{r-1,s-1}(L) \oplus j^* H_{d''}^{r,s}(K, L)$, i.e.

$$H_{d/\nabla}^{r,s}(K, L) \oplus j^* H_{d'}^{r,s-1}(K, L) \cong H_{d'}^{r,s-1}(K, L) \oplus j^* H_{d''}^{r,s}(K, L) .$$

Moreover, (15) implies together with the diagram obtained from (15) by interchanging d' with d'' and adjusting the dimension indices

$$\nabla H_\Delta^{r,s}(L) \cong d' H_\Delta^{r,s}(L) \cong d'' H_\Delta^{r,s}(L) ,$$

$$j^* H_{d/\nabla}^{r,s}(K, L) \cong j^* H_{d'}^{r,s}(K, L) \cong j^* H_{d''}^{r,s}(K, L) ,$$

such that

4 As always, we are interested only in the additive structure of the different cohomology groups, so the isomorphisms are understood to be additive. The investigation of the multiplicative structures—where such are possible—is interesting in connection with Poincaré duality and has not yet been carried out in great details for the situation considered in this paper.

$$H_{d'}^{r,s}(K, L) \cong d'H_{\Delta}^{r-1,s}(L) \oplus j^* H_{d'}^{r,s}(K, L) \cong d''H_{\Delta}^{r-1,s}(L) \oplus j^* H_{d'}^{r,s}(K, L) ,$$

$$H_{d''}^{r,s}(K, L) \cong d''H_{\Delta}^{r,s-1}(L) \oplus j^* H_{d''}^{r,s}(K, L) \cong d'H_{\Delta}^{r,s-1}(L) \oplus j^* H_{d''}^{r,s}(K, L) ,$$

and

$$H_{d'}^{r,s}(K, L) \oplus d'H_{\Delta}^{r,s-1}(L) \cong H_{d''}^{r,s}(K, L) \oplus d''H_{\Delta}^{r-1,s}(L) .$$

(c) A very simple application of (10″) is the following. We have

$$0 \longrightarrow H_{d''}^{r,0}(K, L) \xrightarrow{j^*} H_{d''}^{r,0}(K) \xrightarrow{i^*} H_{d''}^{r,0}(L) \xrightarrow{d''} \cdots ,$$

and suppose $H_{d''}^{r,0}(L) = 0$, i.e. $H_{\Delta}^{r,0}(L) = 0$ in the case of a compact Kähler manifold L ($b_{r,0}(L) = 0$, where $b_{r,s} = \dim_C H_{\Delta}^{r,s}$). Then $H_{d''}^{r,0}(K, L) \cong H_{d''}^{r,0}(K)$, this isomorphism being induced by inclusion, saying that every holomorphic r-form in K vanishes along L. This also follows immediately from $H_{d''}^{r,0}(L) = 0$. As a consequence, the complex projective space $P^{(m)}$ cannot be regularly imbedded in a complex torus $T^{(m+N)}$ (with a complex parallelizable structure), $N \geqq 0$. The same is true for any complex manifold $L^{(m)}$ instead of $P^{(m)}$ with $H_{d''}^{r,0}(L) = 0$ for some r, $1 \leqq r \leqq m$. Another example: let L be again a complex manifold with $H_{d''}^{r,0}(L) = 0$ for some r, $1 \leqq r \leqq m = \dim_C(L)$, and consider $M = L \times T^{(N)}$, $N \geqq r$, as a fibre bundle with base L and fibre T. Then the only complex analytic cross sections in M are given by $L \times p$, $p \in T^{(N)}$.[5]

6. Further applications to the imbedding $L \subset K$. In a compact Kähler manifold $K = K^{(n)}$ multiplication with the fundamental class ω induces according to [4]

$$(16) \qquad 0 \longrightarrow H_{\Delta}^{r-1,s-1}(K) \xrightarrow{L^*} H_{\Delta}^{r,s}(K) \quad \text{for } r + s \leqq n$$

(ω is the cohomology class belonging to the form induced by the Kähler metric). Considering the commutative diagram (in which we use (16))

$$\begin{array}{ccc} H_{d/\nabla}^{r,s}(K, L) & \xrightarrow{j^*} & H_{\Delta}^{r,s}(K) \\ \uparrow \Lambda^* & & \uparrow L^* \\ H_{d/\nabla}^{r-1,s-1}(K, L) & \xrightarrow{j^*} & H_{\Delta}^{r-1,s-1}(K) \\ & & \uparrow \\ & & 0 \end{array}$$

where Λ^* is also induced by multiplication with ω, the condition

$$(17) \qquad 0 \longrightarrow H_{d/\nabla}^{r-1,s-1}(K, L) \xrightarrow{j^*} H_{\Delta}^{r-1,s-1}(K)$$

yields

5 *Added in proof.* Cf. S. S. Chern, Geometrical Structures on Manifolds, Coll. Lectures Summer Meeting AMS, Michigan 1960, pp. 26-30, for further remarks on the possibilities of holomorphic maps f: $V \to M$.

$$(18)\qquad 0 \longrightarrow H_{d/\nabla}^{r-1,s-1}(K, L) \xrightarrow{\Lambda^*} H_{d/\nabla}^{r,s}(K, L) .$$

In view of (11) the condition (17) is equivalent to

$$(19)\qquad H_\Delta^{r-2,s-2}(K) \xrightarrow{i^*} H_\Delta^{r-2,s-2}(L) \longrightarrow 0 ,$$

and we get the

LEMMA. *If K, L is a pair of compact Kähler manifolds, L regularly imbedded in K, then* (19) *implies* (18) (*for* $r + s \leqq n = \dim_C K$).

If in addition $H_\Delta^{r-1,s-1}(K) \xrightarrow{i^*} H_\Delta^{r-1,s-1}(L) \longrightarrow 0$ and $0 \longrightarrow H_\Delta^{r,s}(K) \xrightarrow{i^*} H_\Delta^{r,s}(L)$, then (by (11)) $H_{d/\nabla}^{r,s}(K, L) = 0$, and the Lemma implies $H_{d/\nabla}^{r-1,s-1} = 0$ such that (again by (11)) $0 \longrightarrow H_\Delta^{r-1,s-1}(K) \xrightarrow{i^*} H_\Delta^{r-1,s-1}(L) \longrightarrow 0$, i.e. we proved the

THEOREM 2. *If K, L is a pair of compact Kähler manifolds, L regularly imbedded in K, and if for some r, s, $r + s \leqq n = \dim_C K$, $0 \longrightarrow H_\Delta^{r,s}(K) \xrightarrow{i^*} H_\Delta^{r,s}(L)$, $H_\Delta^{r-1,s-1}(K) \xrightarrow{i^*} H_\Delta^{r-1,s-1}(L) \longrightarrow 0$, $H_\Delta^{r-2,s-2}(K) \xrightarrow{i^*} H_\Delta^{r-2,s-2}(L) \longrightarrow 0$, then $H_\Delta^{r-1,s-1}(K) \xrightarrow{i^*} H_\Delta^{r-1,s-1}(L)$ is an isomorphism (and therefore $b_{r-1,s-1}(K) = b_{r-1,s-1}(L)$).*

Repeated application of Theorem 2 leads to

COROLLARY 1. *K, L, r, s as in Theorem 2. Let $0 \longrightarrow H_\Delta^{r,s}(K) \xrightarrow{i^*} H_\Delta^{r,s}(L)$ and $H_\Delta^{r-k,s-k}(K) \xrightarrow{i^*} H_\Delta^{r-k,s-k}(L) \longrightarrow 0$ for $k = 1, 2, \cdots$, then these epimorphisms are isomorphisms (and therefore $b_{r-k,s-k}(K) = b_{r-k,s-k}(L)$ for $k = 1, 2, \cdots$).*

COROLLARY 2. *K, L, r, s as above. Let the imbedding $L \subset K$ be homology faithful (i.e. $H_\Delta^{u,v}(K) \xrightarrow{i^*} H_\Delta^{u,v}(L) \longrightarrow 0$ for all u, v). Then $0 \longrightarrow H_\Delta^{r,s}(K) \xrightarrow{i^*} H_\Delta^{r,s}(L)$ implies*

$$0 \longrightarrow H_\Delta^{r-k,s-k}(K) \xrightarrow{i^*} H_\Delta^{r-k,s-k}(L) \longrightarrow 0 \quad \textit{for } k = 0, 1, 2, \cdots$$

(*or*: $b_{r,s}(K) = b_{r,s}(L)$ *implies* $b_{r-k,s-k}(K) = b_{r-k,s-k}(L)$).

EXAMPLE. The Poincaré polynomial of $P^{(m)} \times P^{(m)}$ is

$$\Pi(P^{(m)} \times P^{(m)}) = 1 + 2t^2 + 3t^4 + \cdots + (m + 1)t^{2m} + \cdots + 2t^{4m-1} + t^{4m} .$$

Let $P_*^{(n)}$ ($n \geqq 2m$) be the complex manifold which we get from $P^{(n)}$ by means of σ-modifications at m different points. $P_*^{(n)}$ is a compact Kähler (even algebraic) manifold with

$$\Pi(P_*^{(n)}) = 1 + (m + 1)t^2 + (m + 1)t^4 + \cdots + (m + 1)t^{2n-2} + t^{2n} .$$

Corollary 2 implies: there is no homology faithful regular imbedding $P^{(m)} \times P^{(m)} \subset P_*^{(n)}$. There are regular imbeddings of $P^{(u)} \times P^{(v)}$ in $P^{(uv+u+v)}$, therefore in $P_*^{(n)}$ for $n \geqq uv + u + v$. It follows: these imbeddings are not homology faithful. Similar results hold for $P^{(m)} \times P^{(m)} \subset \tilde{P}^{(n)}$, $\tilde{P}^{(n)}$ obtained from $P^{(n)}$ by σ-modifications at q different points, $1 \leqq q \leqq m$.

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