REGULAR ELEMENTS IN AN ORDERED SEMIGROUP

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Introduction. In the abstract theory of semigroups, the begriff of regular element was first introduced by Thierrin [11] as a generalization in the semigroup theory of the begriff of inverse element in the group theory. And this begriff of regular element has been effectively used in the ideal theory of semigroups, for example, in Miller and Clifford [5]. But the structure of regular semigroups, that is, semigroups in which all the elements are regular, is complicated and until now very little was known about it. Inverse semigroup is an important sort of regular semigroups, whose structure was completely determined (Preston [7], [8]).

Ordered semigroups have been studied by several authors, for example, Alimov [1] and Clifford [2]. However, as far as we know, none discussed systematically ordered semigroups in our general sense (cf. § 1). In the previous paper [10], we characterized ordered idempotent semigroups, that is, ordered semigroups in which all the elements are idempotent. In the continuation of our investigation of ordered semigroups, in this note, we concern essentially with ordered regular semigroups.

Main purpose of this note is to give a catalog of all possible types of subsemigroups generated by regular pairs of ordered semigroups. The subsemigroups of an ordered semigroup S generated by regular pairs are the analogs of the cyclic subgroups of a group, in fact reduce to exactly these when S is a group. It therefore should be useful in the study of ordered regular semigroups to have catalog of them available. A list of 39 ordered semigroups, each generated by a (non-idempotent) regular pair, is given in this note, and it is shown that every such ordered semigroup is (order-and-product) isomorphic with one of these. Theorems 3, 4 and 5 serve as an index to this catalog.

Moreover, this note contains the following by-products which seem to be interesting:

(a) the set of idempotents of an ordered semigroup S is a subsemigroup of S (Corollary of Lemma 1);

(b) any regular conjugate of an idempotent of an ordered semigroup S is idempotent (Theorem 1);

(c) the set of regular elements of an ordered semigroup S is a subsemigroup of S (Corollary 2 of Lemma 5);

(d) a regular element of finite order of an ordered semigroup S

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can have order only 1 or 2 Theorem 2.

Finally we remark that, even though the subsemigroup of a semigroup S generated by a regular pair need not be regular in general, it is regular if S is ordered.

In § 1, we give some definitions and some elementary results in preparation of the following discussion. In §§ 2–5, we discuss the case when a regular pair is of finite order, while in §§ 6–9, we discuss the case when it is of infinite order. In the final § 10, we remark some applications in special ordered semigroups.

1. Preliminaries. We denote by S an *ordered semigroup*, that is, a semigroup S with a simple order $<$ which satisfies the following condition:

$$(1) \qquad \text{for } x, y, z \in S,\ x < y \text{ implies } xz \leqq yz \text{ and } zx \leqq zy\ .$$

If two elements x and y of S generate the same principal left ideal, then we write $x \equiv y(L)$, while if x and y generate the same principal right ideal, then we write $x \equiv y(R)$. We write $x \equiv y(D)$ if there exists an element z of S such that $x \equiv z(L)$ and $z \equiv y(R)$. As is well-known, these relations are equivalence relation (Green [3]). An element x of S is called *regular* if there exists an element y of S such that

$$(2) \qquad xyx = x \quad \text{and} \quad yxy = y$$

(Miller and Clifford [5]). When a pair (x, y) of elements of S satisfy (2), (x, y) is called a *regular pair* and y is called a *regular conjugate* of x. As is easily seen by (2), for every regular pair (x, y), both xy and yx are idempotents. An element x of S is called *positive* if $x^2 > x$, while x is called *negative* if $x^2 < x$. For an element x of S, the number of distinct natural powers of x is called the *order* of x (Clifford [2]). If x is an element of finite order n, then n is the minimal natural number such that $x^n = x^{n+1}$. Evidently x is of order 1 if and only if x is idempotent. The set of all idempotents of S is denoted by E. For an ordered semigroup S, we call the *multiplicative dual* or, simply, *dual* of S the ordered semigroup constructed from S by interchanging the order of multiplication but by preserving the order of S. An element z of S is said to *lie between x and y*, if either $x \leqq z \leqq y$ *or* $y \leqq z \leqq x$, while z is said to *lie between x and y in the strict sense*, if either $x < z < y$ or $y < z < x$.

LEMMA 1. *If x and y are nonnegative, then, xy is nonnegative. If x and y are non-positive, then xy is non-positive.*

Proof. For nonnegative x and y, if $x \leqq y$, then $xy \leqq x^3y \leqq (xy)^2$, and, if $y \leqq x$, then $xy \leqq xy^3 \leqq (xy)^2$. The second assertion can be proved similarly.

COROLLARY. *The set E of all idempotents of S, if it is nonvoid, is a subsemigroup of S.*

LEMMA 2. *If x is nonnegative, y is non-positive and $x \leqq y$, then both xy and yx are idempotents which lie between x and y.*

Proof.

$$xy \leqq x^3y \leqq (xy)^2 \leqq xy^3 \leqq xy \ ,$$

and so xy is idempotent. Moreover

$$x \leqq x^2 \leqq xy \leqq y^2 \leqq y \ .$$

With respect to the order in S, the subsemigroup E is clearly an ordered semigroup, which plays an important role in the following discussion. As is easily seen, for $g, h \in E$,

(3) $\quad g \equiv h(L)$ in S if and only if $gh = g$ and $hg = h$,

(4) $\quad g \equiv h(R)$ in S if and only if $gh = h$ and $hg = g$.

Hence

(5) $\quad g \equiv h(L)$ and $g \equiv h(R)$ in S if and only if $g = h$.

By (3) and (4), for elements of E, L-equivalence and R-equivalence in E coincide with L-equivalence and R-equivalence in S, respectively. However, for D-equivalence, such a situation does not occur. Of course, for $g, h \in E$, $g \equiv h(D)$ in E implies $g \equiv h(D)$ in S, but the converse is not always true. (The semigroup J in § 8 will offer a counter-example.) The D-equivalence in E is denoted by D_E-equivalence.

LEMMA 3. *If $g, h \in E$ and $g \leqq h$, then the following conditions are equivalent to each other*:

(a) $gh \leqq hg$, (b) $ghg = gh$, (c) $hgh = hg$, (d) $gh \equiv hg(L)$.

Proof. (a) implies (b), for

$$gh = g(gh) \leqq ghg \leqq (gh)h = gh \ .$$

(b) implies (c), for $hg = (hg)(hg) = hgh$. Similarly (c) implies (b). Hence if (c) holds, then both (b) and (c) hold, and so we obtain (d). Finally (d) implies (a), for, by (3), $gh = (gh)(hg) = ghg$, and so by

Lemma 2, $g \leqq gh = ghg \leqq hg \leqq h$.

LEMMA 3'. *If $g, h \in E$ and $g \leqq h$, then the following conditions are eqvivalent to each other*:

$$\text{(a) } hg \leqq gh\ , \quad \text{(b) } hgh = gh\ , \quad \text{(c) } ghg = hg\ , \quad \text{(d) } gh \equiv hg(R)\ .$$

COROLLARY. $gh \equiv hg(D_E)$ *for every* $g, h \in E$.

Ordered idempotent semigroups are studied in our previous paper [10], from which we mention one more lemma without proof.

LEMMA 4. *Each D_E-equivalence class in E consists of either only one L-equivalence class in E or only one R-equivalence class in E.*

A D_E-equivalence class, if it consists of only one L-equivalence class in E, is called *L-typed*, while, if it consists of only one R-equivalence class, it is called *R-typed*. A regular pair (x, y) of S is called *L-typed*, if the D_E-equivalence class which contains $(xy)(yx)$ is L-typed. By Corollary of Lemma 3, a regular pair (x, y) is L-typed if and only if the D_E-equivalence class which contains $(yx)(xy)$ is L-typed. An *R-typed* regular pair is defined similarly. A regular pair (x, y) is said to be of *order n*, if both x and y are elements of order n. A regular pair of order 1 is also called an *idempotent regular pair*.

2. Idempotent regular pair. In this section, we give a theorem which characterizes idempotent regular pairs.

THEOREM 1. (a) *For a regular pair (x, y) of S, x is idempotent if and only if y is idempotent.*

(b) *For $g, h \in E$, (g, h) is a regular pair if and only if $g \equiv h(D_E)$.*

Proof. (a) Suppose that (x, y) is a regular pair and that x is an idempotent. Then $y = yxy = (yx)(xy)$ is an idempotent, by Corollary of Lemma 1. (b) First suppose that (g, h) is an idempotent regular pair and that $g \leqq h$. By (2),

$$g \equiv hg(L)\ , \quad h \equiv gh(L)\ , \quad g \equiv gh(R)\ , \quad h \equiv hg(R)\ .$$

If $gh \leqq hg$, then, by Lemma 3, $hg \equiv gh(L)$, and so $g \equiv h(L)$. If $hg \leqq gh$, then we obtain $g \equiv h(R)$ similarly. Next suppose that $g \equiv h(D_E)$. Then, by Lemma 4, either $g \equiv h(L)$ or $g \equiv h(R)$. If $g \equiv h(L)$, then

$$ghg = (gh)g = g^2 = g \ , \qquad hgh = (hg)h = h^2 = h \ ,$$

and so (g, h) is a regular pair. In the case when $g \equiv h(R)$, we obtain the same result by a similar way.

3. Regular pair of finite order. In this section, we study regular pairs of finite order. But, first of all, we give some lemmas about general regular pairs which are necessary also for coming sections.

LEMMA 5. *For two regular pairs (x, y) and (z, w), (xz, wy) is a regular pair.*

Proof. By Corollary of Lemma 1, both $yxzw$ and $zwyx$ are idempotent. Hence

$$(xz)(wy)(xz) = (xyx)zwyx(zwz) = x(yxzw)z = xz \ ,$$
$$(wy)(xz)(wy) = (wzw)yxzw(yxy) = w(zwyx)y = wy \ .$$

COROLLARY 1. *If (x, y) is a regular pair, then, for every natural number n, (x^n, y^n) is a regular pair.*

COROLLARY 2. *The set of all regular elements of S, if it is nonvoid, is a subsemigroup of S.*

LEMMA 6. *If (p, q) is a regular pair such that $q \leqq p$, then q is non-positive and p is nonnegative.*

Proof. $q^3 \leqq qpq = q$ and $p = pqp \leqq p^3$, from which the lemma follows immediately.

LEMMA 7. *Let (p, q) be a regular pair such that $q \leqq p$ and $qp \leqq pq$. Then the following six conditions are equivalent to each other:*

$$\text{(a)}\ \ pq^2 = pq^2p \ , \qquad \text{(b)}\ \ q^2p = q^2 \ , \qquad \text{(c)}\ \ q^2 = q^3 \ ,$$
$$\text{(d)}\ \ qp^2 = qp^2q \ , \qquad \text{(e)}\ \ p^2q = p^2 \ , \qquad \text{(f)}\ \ p^2 = p^3 \ .$$

Moreover, these conditions imply

$$\text{(g)}\ \ (qp)(pq) \equiv q^2 \equiv p^2 \equiv (pq)(qp)(L) \ .$$

Proof. (a) implies (b), for $q^2p = qpq^2p = qpq^2 = q^2$. (b) implies (c), for $q^2p^2 = q^2p = q^2$ and so q^2 is an idempotent, by Corollary 1 of Lemma 5. (c) implies (a), for

$$pq^2 = pq^3 \leqq pq^2p = pq^2(qp) \leqq pq^2(pq) = pq^2 .$$

Similarly the conditions (d), (e) and (f) are equivalent to each other. Now (c) implies (f), for, by Theorem 1 and Corollary 1 of Lemma 5, (p^2, q^2) is an idempotent regular pair. Similarly (f) implies (c). This proves the first half of the lemma. Next suppose that these conditions hold. Then

$$(qp^2q)q^2 = ((qp^2q)q)q = qp^2q , \qquad q^2(qp^2q) = ((q^3p)p)q = q^2 ,$$
$$q^2p^2 = (q^2p)p = q^2 , \qquad p^2q^2 = (p^2q)q = p^2 ,$$
$$p^2(pq^2p) = ((p^3q)q)p = p^2 , \qquad (pq^2p)p^2 = ((pq^2p)p)p = pq^2p .$$

Hence (g) holds.

LEMMA 7′. *Let (p, q) be a regular pair such that $q \leqq p$ and $pq \leqq qp$. Then the following six conditions are equivalent to each other:*

(a) $q^2p = pq^2p$, (b) $pq^2 = q^2$, (c) $q^2 = q^3$,
(d) $p^2q = qp^2q$, (e) $qp^2 = p^2$, (f) $p^2 = p^3$.

Moreover, these conditions imply

(g) $(qp)(pq) \equiv q^2 \equiv p^2 \equiv (pq)(qp)(R)$.

COROLLARY. *For a regular pair (x, y), x is an element of order 2 if and only if y is of order 2.*

THEOREM 2. *If (x, y) is a regular pair such that either x or y is an element of finite order, then (x, y) is a regular pair of order either 2 or 1.*

Proof. By Theorem 1 and Corollary of Lemma 7, it suffices to show that if x is an element of finite order, then x is of order at most 2. Here we prove this assertion only in the case when $x \leqq y$ and $xy \leqq yx$. Then, by Lemma 6, x is non-positive. Now suppose it were true that $x^{n-1} > x^n = x^{n+1}$ for a natural number $n \geqq 3$. Then $xx^{n-2} = x^{n-1} > x^n = xyx^n$ and so $x^{n-2} > yx^n$. Hence $x^{n-1} \geqq yx^{n+1}$. On the other hand,

$$x^{n-1} = xyx^{n-1} \leqq (yx)x^{n-1} = yx^n = yx^{n+1} .$$

Hence $x^{n-1} = yx^{n+1}$. Then we would have

$$x^n = x^{n-1}x = yx^{n+2} = yx^{n+1} = x^{n-1} ,$$

which is a contradiction.

4. Ordered T-semigroups. In this section, we give some examples of ordered semigroups each of which has a regular element of order 2.

EXAMPLE 1. We denote by T_{1L} the system consisting of eight elements ordered by

$$s < q < e < t < v < f < p < u$$

and with the following multiplication table:

(6)

	s	q	e	t	v	f	p	u
s	s	s	s	s	s	s	s	s
q	s	s	s	s	s	q	e	t
e	s	q	e	t	t	t	t	t
t	t	t	t	t	t	t	t	t
v	v	v	v	v	v	v	v	v
f	v	v	v	v	v	f	p	u
p	v	f	p	u	u	u	u	u
u	u	u	u	u	u	u	u	u .

It can be verified that this system T_{1L} is an ordered semigroup.

EXAMPLE 2. We denote by T_{2L} the system arising from T_{1L} by identifying t and v. Clearly this identification is possible, and the constructed system T_{2L} is an ordered semigroup.

EXAMPLE 3. We denote by T_{1R} the ordered semigroup which is multiplicative dual to T_{1L}. Thus the multiplication table of T_{1R} is symmetric in the main diagonal to the table (6).

EXAMPLE 4. We denote by T_{2R} the ordered semigroup which is dual to T_{2L}.

In each of these four ordered semigroups, (p, q) is a regular pair of order 2 with negative q and positive p. In T_{1L} and T_{2L}, (p, q) is L-typed, while, in T_{1R} and T_{2R}, it is R-typed. The ordered semigroups T_{1L}, T_{2L}, T_{1R} and T_{2R} are called *ordered T-semigroups*. Ordered T-semigroups T_{1L} and T_{2L} are called *L-typed*, while T_{1R} and T_{2R} are called *R-typed*.

5. Regular pair of order 2. In this section, we characterize the subsemigroup generated by a regular pair of order 2.

LEMMA 8. *If (x, y) is a non-idempotent regular pair, then both xy and yx lie between x and y in the strict sense.*

Proof. Suppose that $x \leqq y$. Then, by Lemma 6 and Theorem 1, x is negative and y is positive. Now we suppose that $x \geqq xy$ were true. Then, by Lemma 2, $x = (xy)x$ would be an idempotent, which is a contradiction. Hence $x < xy$. The remaining assertions can be proved similarly.

In the rest of this section, we assume that (p, q) is a regular pair of order 2 such that $q \leqq p$, and set

$$(7) \qquad \begin{array}{lll} s = q^2 , & u = p^2 , & e = \min\{pq, qp\} , \\ f = \max\{pq, qp\} , & t = \min\{ef, fe\} , & v = \max\{ef, fe\} . \end{array}$$

First we suppose that $qp \leqq pq$. Then, by Lemma 7, $(qp)(pq) \equiv (pq)(qp)(L)$, and so, by Lemma 3, $(qp)(pq) \leqq (pq)(qp)$. Hence

$$(7') \qquad e = qp , \quad f = pq , \quad t = ef , \quad v = fe.$$

By Lemma 8, $s = q^2 < q < e$ and $f < p < p^2 = u$. Moreover, by Lemma 7, $pt = p^2q = p^2 > p = pe$ and $qv = q^2p = q^2 < q = qf$, and so $t > e$ and $v < f$. Thus

$$(8) \qquad s < q < e < t \leqq v < f < p < u .$$

Now we denote by T^* the set consisting of elements s, q, e, t, v, f, p and u. By Lemma 7, we can verify that the elements of T^* are multiplied together just as in the table (6) of the ordered semigroup T_{1L} in Example 1 in § 4. Especially T^* is a subsemigroup, which is clearly the subsemigroup generated by (p, q). If $t \neq v$, then T^* is isomorphic to T_{1L}, while, if $t = v$, then T^* is isomorphic to T_{2L}.

Similarly, in the case when $pq \leqq qp$, we can show that, if $t \neq v$, then the subsemigroup T^* generated by (p, q) is isomorphic to T_{1R} and, if $t = v$, then T^* is isomorphic to T_{2R}.

THEOREM 3. *Let (p, q) be a regular pair of order 2 such that $q \leqq p$, and let T^* be the subsemigroup of S generated by (p, q).*

(a) *If $qp \leqq pq$ and $qp^2q \neq pq^2p$, then T^* is isomorphic to the L-typed ordered T-semigroup T_{1L};*

(b) *if $qp \leqq pq$ and $qp^2q = pq^2p$, then T^* is isomorphic to the L-typed ordered T-semigroup T_{2L};*

(c) *if $pq \leqq qp$ and $qp^2q \neq pq^2q$, then T^* is isomorphic to the R-typed ordered T-semigroup T_{1R};*

(d) *if $pq \leqq qp$ and $qp^2q = pq^2p$, then T^* is isomorphic to the R-typed ordered T-semigroup T_{2R}.*

6. Ordered I-semigroups. In this section, we give some examples of ordered semigroups each of which has a regular pair of infinite order.

EXAMPLE 1. The set of all integers forms an ordered semigroup with respect to the usual order and the usual addition. We denote this ordered semigroup by I_0. I_0 is even an ordered group.

EXAMPLE 2. Let U be an ordered semigroup consisting of two elements -1 and 1, with the usual order and the left singular multiplication:

$$ab = a \text{ for every } a, b \in U .$$

We consider the lexicographically ordered direct product of I_0 and U, that is, the system I_{1L} consisting of pairs (i, a) with $i \in I_0$, $a \in U$, in which the order and the multiplication are defined by

$$(i, a) < (j, b) \quad \text{if } i < j \text{ or } i = j, a < b ;$$
$$(i, a)(j, b) = (i + j, ab) = (i + j, a) .$$

It can easily be verified that this system I_{1L} is an ordered semigroup. (Here we remark that lexicographically ordered direct product of two ordered semigroups is not always an ordered semigroup.) In I_{1L}, the subsemigroup, consisting of elements with the second component 1, is isomorphic to the ordered semigroup I_0.

EXAMPLE 3. Let V be a system consisting of six elements with the order

$$e_2 < e_1 < t < v < f_1 < f_2$$

and with the multiplication table:

(9)

	e_2	e_1	t	v	f_1	f_2
e_2	e_2	e_2	e_2	e_2	e_2	e_2
e_1	e_2	e_1	t	t	t	t
t	t	t	t	t	t	t
v	v	v	v	v	v	v
f_1	v	v	v	v	f_1	f_2
f_2	f_2	f_2	f_2	f_2	f_2	f_2 .

It can be verified that V is an ordered idempotent semigroup. Now we define two mappings φ and ψ of V into itself:

$$\varphi(e_2) = \varphi(e_1) = \varphi(t) = \varphi(v) = e_2\,, \quad \varphi(f_1) = e_1\,, \quad \varphi(f_2) = t\,;$$
$$\psi(e_2) = v\,, \quad \psi(e_1) = f_1\,, \quad \psi(t) = \psi(v) = \psi(f_1) = \psi(f_2) = f_2\,.$$

As is easily seen, these mappings have the following properties:

(a) *both* φ *and* ψ *are monotone*:
$g \leqq h$ *implies* $\varphi(g) \leqq \varphi(h)$, $\psi(g) \leqq \psi(h)$;

(b) *both* φ *and* ψ *are semigroup-homomorphisms*:
$\varphi(gh) = \varphi(g)\varphi(h)$, $\psi(gh) = \psi(g)\psi(h)$ *for every* $g, h \in V$;

(c) $\varphi(\varphi(g)) = e_2$, $\psi(\psi(g)) = f_2$ *for every* $g \in V$;

(d) $\varphi(g)e_1 = e_1\varphi(g) = \varphi(g)$, $\psi(g)f_1 = f_1\psi(g) = \psi(g)$ *for every* $g \in V$;

(e) $\varphi(\psi(g)) = e_1 g$, $\psi(\varphi(g)) = f_1 g$ *for every* $g \in V$.

We consider the system K, consisting of pairs (i, g) with $i \in I_0$, $g \in V$, in which the order is defined lexicographically and the multiplication is defined by

$$(10) \qquad (i, g)(j, h) = \begin{cases} (i + j, ge_2) & \text{if } i \leqq -2\,, \\ (i + j, g\varphi(h)) & \text{if } i = -1\,, \\ (i + j, gh) & \text{if } i = 0\,, \\ (i + j, g\psi(h)) & \text{if } i = 1\,, \\ (i + j, gf_2) & \text{if } i \geqq 2\,. \end{cases}$$

Using the properties (a)–(e) of φ and ψ, we can prove that K is an ordered semigroup. Finally we consider the subset I_{2L} of K, consisting of elements with

$$(11) \qquad \begin{aligned} & i \leqq -2,\ g \neq e_1, f_1, \text{ or} \\ & i = -1,\ g \neq f_1, \text{ or} \\ & i = 0,\ g \text{ arbitrary, or} \\ & i = 1,\ g \neq e_1 \text{ or} \\ & i \geqq 2,\ g \neq e_1, f_1\,. \end{aligned}$$

It can also be proved that I_{2L} is closed with respect to the multiplication, and so forms an ordered semigroup. In I_{2L}, the subsemigroup consisting of elements with the second component e_2 or f_2, is isomorphic to I_{1L}, and the subsemigroup consisting of elements with the second component e_2 is isomorphic to I_0.

EXAMPLE 4. Let V, φ and ψ be the same as in the preceding Example 3. We consider the system K', consisting of pairs (i, g) with $i \in I_0$, $g \in V$, in which the order is defined lexicographically and the multiplication is defined by

$$(10') \qquad (i, g)(j, h) = \begin{cases} (i+j, gf_2) & \text{if } i \leqq -2\,, \\ (i+j, g\psi(h)) & \text{if } i = -1\,, \\ (i+j, gh) & \text{if } i = 0\,, \\ (i+j, g\varphi(h)) & \text{if } i = 1\,, \\ (i+j, ge_2) & \text{if } i \geqq 2\,. \end{cases}$$

In a similar way as in Example 3, we can prove that K' is an ordered semigroup, in which the subset I_{3L}, consisting of elements (i, g) with

$$(11') \qquad \begin{aligned} &i \leqq -2,\ g \neq e_1, f_1, \text{ or} \\ &i = -1,\ g \neq e_1, \text{ or} \\ &i = 0,\ g \text{ arbitrary, or} \\ &i = 1,\ g \neq f_1, \text{ or} \\ &i \geqq 2,\ g = e_1, f_1\,, \end{aligned}$$

forms an ordered semigroup. In I_{3L}, the subsemigroup consisting of elements with the second component e_2 or f_2, is isomorphic to I_{1L}, and the subsemigroup consisting of elements with the second component e_2 is isomorphic to I_0.

EXAMPLE 5. We denote by I_{4L} the ordered semigroup constructed from I_{2L} by identifying (i, t) and (i, v) for every $i \in I_0$. It can be seen that this identification is possible.

EXAMPLE 6. The ordered semigroup I_{5L} constructed from I_{3L} by identifying (i, t) and (i, v) for every $i \in I_0$.

EXAMPLE 7. The ordered semigroup I_{1R} which is dual to I_{1L}.

EXAMPLE 8. The ordered semigroup I_{2R} which is dual to I_{2L}.

EXAMPLE 9. The ordered semigroup I_{3R} which is dual to I_{3L}.

EXAMPLE 10. The ordered semigroup I_{4R} which is dual to I_{4L}.

EXAMPLE 11. The ordered semigroup I_{5R} which is dual to I_{5L}.

These eleven ordered semigroups $I_0, I_{1L}, I_{2L}, \cdots, I_{5R}$ are called *ordered I-semigroups*, in which I_0 is called the *fundamental ordered I-semigroup*. Every ordered I-semigroup contains a subsemigroup which is isomorphic to the fundamental ordered I-semigroup I_0.

7. Regular pair of infinite order (1). In this section, we characterize the subsemigroup generated by a regular pair of infinite order under some conditions. For brevity, in this section, always we denote by (p, q) a regular pair of infinite order such that $q \leqq p$ and set

$$(12) \qquad \begin{aligned} e_n &= \min\{q^n p^n, p^n q^n\}, \quad f_n = \max\{q^n p^n, p^n q^n\} \quad (n = 1, 2, 3, \cdots), \\ t &= \min\{e_1 f_1, f_1 e_1\}, \qquad v = \max\{e_1 f_1, f_1 e_1\}. \end{aligned}$$

LEMMA 9. $\cdots < q^3 < q^2 < q < \cdots \leqq e_3 \leqq e_2 \leqq e_1 \leqq t \leqq v \leqq f_1 \leqq f_2 \leqq f_3 \leqq \cdots < p < p^2 < p^3 < \cdots$.

Proof. By Lemma 2, $e_1 \leqq t \leqq v \leqq f_1$. By Lemma 6, q is negative and p is positive, and so $q^{n+1} < q^n$, $p^n < p^{n+1}$ for every natural number n. First we suppose that $qp \leqq pq$. Then

$$q^{n+1}p^{n+1} = q^n(qp)p^n \leqq q^n(pq)p^n = q^n p^n,$$

and similarly $p^{n+1}q^{n+1} \geqq p^n q^n$. Hence we obtain, for every natural number n,

$$\begin{aligned} e_{n+1} = q^{n+1}p^{n+1} \leqq e_n = q^n p^n \leqq e_1 = qp \leqq f_1 = pq \\ \leqq f_n = p^n q^n \leqq f_{n+1} = p^{n+1}q^{n+1}. \end{aligned}$$

Finally, by Corollary 1 of Lemma 5,

$$qq^n = q^{n+1} < q^n = q^n p^n q^n = e_n q^n, \quad f_n p^n = p^n q^n p^n = p^n < p^{n+1} = pp^n$$

and so $q < e_n, f_n < p$. In the case when $pq \leqq qp$, we can prove this theorem in a similar way.

COROLLARY. *For every natural number n, both e_n and f_n are idempotent. If $qp \leqq pq$, then $e_n = q^n p^n, f_n = p^n q^n$. If $pq \leqq qp$, then $e_n = p^n q^n$, $f_n = q^n p^n$.*

LEMMA 10. *For two natural numbers m and n such that $m < n$,*

$$e_m e_n = e_n e_m = e_n, \qquad f_m f_n = f_n f_m = f_n.$$

Proof. We prove only the first assertion in the case when $qp \leqq pq$. By the preceding Corollary and Corollary 1 of Lemma 5,

$$\begin{aligned} e_m e_n &= q^m p^m q^n p^n = (q^m p^m q^m) q^{n-m} p^n = q^m q^{n-m} p^n = e_n, \\ e_n e_m &= q^n p^n q^m p^m = q^n p^{n-m}(p^m q^m p^m) = q^n p^{n-m} p^m = e_n. \end{aligned}$$

Now we remark that two relations

$$(A) \qquad e_2 f_1 \geqq e_1, \qquad e_2 f_1 = e_1 f_1$$

are equivalent to each other. In fact, if $e_2 f_1 \geqq e_1$, then

$$e_2 f_1 = (e_2 f_1) f_1 \geqq e_1 f_1 \geqq e_2 f_1,$$

and so $e_2f_1 = e_1f_1$. If $e_2f_1 = e_1f_1$, then $e_2f_1 = e_1f_1 \geqq e_1^2 = e_1$.

Similarly we can prove that each of the following sets of relations consists of equivalent relations:

(B) $$f_2e_1 \leqq f_1\ , \qquad f_2e_1 = f_1e_1\ ;$$

(A′) $$f_1e_2 \geqq e_1\ , \qquad f_1e_2 = f_1e_1\ ,$$

(B′) $$e_1f_2 \leqq f_1\ , \qquad e_1f_2 = e_1f_1\ .$$

Also the three relations

(C) $$f_2e_1 \geqq f_1\ , \qquad f_2e_1 = f_2\ , \qquad f_2e_2 = f_2$$

are equivalent to each other. In fact, if $f_2e_1 \geqq f_1$, then

$$f_2 \geqq f_2e_1 = f_2(f_2e_1) \geqq f_2f_1 = f_2\ ,$$

and so $f_2e_1 = f_2$. If $f_2e_1 = f_2$, then, without loss of generality by assuming that $qp \leqq pq$, we have $q^2 = q^2f_2 = q^2f_2e_1 = q^3p$, and so $q^2 = q^4p^2$. Therefore $f_2 = p^2q^2 = p^2q^4p^2 = f_2e_2$. Finally, if $f_2e_2 = f_2$, then $f_2e_1 \geqq f_2e_2 = f_2 \geqq f_1$.

Similarly we can prove that each of the following sets of relations consists of equivalent relations:

(D) $$e_2f_1 \leqq e_1\ , \qquad e_2f_1 = e_2\ , \qquad e_2f_2 = e_2\ ;$$

(C′) $$e_1f_2 \geqq f_1\ , \qquad e_1f_2 = f_2\ , \qquad e_2f_2 = f_2\ ;$$

(D′) $$f_1e_2 \leqq e_1\ , \qquad f_1e_2 = e_2\ , \qquad f_2e_2 = e_2\ .$$

In what follows, we refer to the above-mentioned sets of equivalent relations as (A), (B), $\cdots$, (D′), as shown at the left end of each line.

LEMMA 11. *If either* (C) *or* (C′) *holds, then* $e_2 = e_3 = \cdots$. *If either* (D) *or* (D′) *holds, then* $f_2 = f_3 = \cdots$.

Proof. We prove only that (C) implies $e_2 = e_3 = \cdots$ in the case when $qp \leqq pq$. In this case, as is shown in the proof of equivalence of relations in (C), we have $q^2 = q^3p$, and so

$$e_2 = q^2p^2 = e_3 = q^3p^3 = e_4 = q^4p^4 = \cdots\ .$$

LEMMA 12. *If* (p, q) *is L-typed, then*

(a) (C) *is equivalent to* (A′), *and* (D) *is equivalent to* (B′);

(b) (A) *implies* (A′), *and* (B) *implies* (B′).

Proof. (a) We prove only that (C) is equivalent to (A′) in the case when $qp \leqq pq$. If (C) holds, then, as is shown in the proof of equivalence of relations in (C), we have $q^2 = q^3p$, and so

$$f_1e_2 = pq^3p^2 = pq^2p = f_1e_1 .$$

Thus (A′) holds. If (A′) holds, then, by Lemma 3, $f_1e_1f_1 = f_1e_1$, and so

$$\begin{aligned} f_2 = p^2q^2 = pf_1e_1q = pf_1e_2q = p^2q^3p^2q = f_2e_1f_1 \\ = (f_2f_1)e_1f_1 = f_2(f_1e_1) = f_2e_1 . \end{aligned}$$

Thus (C) holds. (b) We prove only (A) implies (A′). By Lemma 3 and (A),

$$\begin{aligned} (f_1e_1)(f_1e_2) = (f_1e_1f_1)e_2 = f_1e_1e_2 = f_1e_2 , \\ (f_1e_2)(f_1e_1) = f_1(e_2f_1)e_1 = f_1e_1f_1e_1 = f_1e_1 . \end{aligned}$$

Hence $f_1e_1 \equiv f_1e_2(R)$. Therefore, by the assumption of being L-typed of (p, q), we obtain $f_1e_1 = f_1e_2$.

LEMMA 12′. *If (p, q) is R-typed, then*
(a) (C′) *is equivalent to* (A), *and* (D′) *is equivalent to* (B);
(b) (A′) *implies* (A), *and* (B′) *implies* (B).

We divide the investigation of a regular pair of infinite order into two cases:

Case 1. *the case when* $e_2 \equiv e_1f_1(D_E)$;

Case 2. *the case when* $e_2 \not\equiv e_1f_1(D_E)$.

In the rest of this section we study *Case* 1, and *Case* 2 will be studied in § 9.

Case 1 is divided into two subcases:

Case 1*L*. *the subcase of Case* 1 *when* (p, q) *is L-typed;*

Case 1*R*. *the subcase of Case* 1 *when* (p, q) *is R-typed.*

Now we consider *Case* 1*L*, that is, suppose that (p, q) is an L-typed regular pair of infinite order such that $q \leqq p$ and $e_2 \equiv e_1f_1(D_E)$. Then $e_2 \equiv e_1f_1 \equiv f_1e_1(L)$, and so $e_2 = e_2(e_1f_1) = e_2f_1$, $f_1e_1 = (f_1e_1)e_2 = f_1e_2$. Hence (D) and (A′) hold. Then, by Lemma 12, also (B′) and (C) hold. Moreover, by Lemma 11, we have

$$e_2 = e_3 = \cdots , \qquad f_2 = f_3 = \cdots . \tag{13}$$

Furthermore, by Lemma 3,

(14) $$t = e_1 f_1 = e_1 f_1 e_1\ , \qquad v = f_1 e_1 = f_1 e_1 f_1\ .$$

We denote by E the set consisting of e_2, e_1, t, v, f_1 and f_2. Then, by (A′), (B′), (C), (D), (13) and (14), we can verify that elements of E are multiplied together just as in the table (9) in Example 3 in § 6.

We divide *Case* 1*L* into two subcases:

Case 1*L*1. *the subcase of Case* 1*L when* $qp \leqq pq$;

Case 1*L*2. *the subcase of Case* 1*L when* $pq \leqq qp$.

First we consider *Case* 1*L*1. In this case, by Corollary of Lemma 9, we have $e_n = q^n p^n$ and $f_n = p^n q^n$. Therefore, by $f_2 e_1 = f_2$ and $e_2 f_1 = e_2$, we obtain

(15) $$p^2 = p^3 q\ , \qquad q^2 = q^3 p\ ,$$

and, by $e_1 f_1 = e_1 f_1 e_1$ and $f_1 e_1 = f_1 e_1 f_1$,

(16) $$p^2 q = p^2 q^2 p\ , \qquad q^2 p = q^2 p^2 q\ .$$

Now we consider the mappings φ and ψ of E into itself which have been defined in Example 3 in § 6, that is,

$$\varphi(e_2) = \varphi(e_1) = \varphi(t) = \varphi(v) = e_2\ , \quad \varphi(f_1) = e_1\ , \quad \varphi(f_2) = t\ ;$$
$$\psi(e_2) = v\ , \quad \psi(e_1) = f_1\ , \quad \psi(t) = \psi(v) = \psi(f_1) = \psi(f_2) = f_2\ .$$

Then, by (15) and (16), it is easily verified that

(17) $$qg = \varphi(g)q\ , \quad pg = \psi(g)p \text{ for every } g \in E\ .$$

Especially we have $qe_2 = e_2 q$, $pf_2 = f_2 p$, and so

(18) $$q^n e_2 = e_2 q^n\ , \quad p^n f_2 = f_2 p^n \text{ for every natural number } n.$$

Moreover by (17)

$$q^2 g = q\varphi(g)q = \varphi(\varphi(g))q^2 = e_2 q^2 = q^2\ ,$$
$$p^2 g = p\psi(g)p = \psi(\psi(g))p^2 = f_2 p^2 = p^2\ ,$$

and so

(19) $$q^n g = e_2 q^n = q^n\ , \quad p^n g = f_2 p^n = p^n \text{ for every } g \in E,\ n \geqq 2\ .$$

By (18) $e_2 q^{m+1} p^{n+1} = e_2 q^m e_1 p^n = q^m e_2 e_1 p^n = q^m e_2 p^n = e_2 q^m p^n$ and $f_2 p^{m+1} q^{n+1} = f_2 p^m q^n$ in a similar way. Hence

(20) $$e_2 q^m p^n = \begin{cases} e_2 q^{m-n} & \text{if } m > n\ , \\ e_2 & \text{if } m = n\ , \\ e_2 p^{n-m} & \text{if } m < n\ , \end{cases}$$

$$f_2p^mq^n = \begin{cases} f_2p^{m-n} & \text{if } m > n , \\ f_2 & \text{if } m = n , \\ f_2q^{n-m} & \text{if } m < n . \end{cases} \tag{21}$$

By (19) and (20),

$$\text{if } m \geqq 2 , \begin{cases} gq^mhq^n = ge_2q^{m+n} , \\ gq^mh = ge_2q^m , \\ gq^mhp^n = \begin{cases} ge_2q^{m-n} & \text{when } m > n , \\ ge_2 & \text{when } m = n , \\ ge_2p^{n-m} & \text{when } m < n . \end{cases} \end{cases} \tag{22}$$

Similarly by (19) and (21),

$$\text{if } m \geqq 2 , \begin{cases} gp^mhp^n = gf_2p^{m+n} , \\ gp^mh = gf_2p^m , \\ gp^mhq^n = \begin{cases} gf_2p^{m-n} & \text{when } m > n , \\ gf_2 & \text{when } m = n , \\ gf_2q^{n-m} & \text{when } m < n . \end{cases} \end{cases} \tag{23}$$

We have mentioned in § 6 that $\varphi(g)e_1 = \varphi(g)$ and $\psi(g)f_1 = \psi(g)$. Hence by (17),

$$\begin{cases} gqhq^n = g\varphi(h)q^{n+1} , \\ gqh = g\varphi(h)q , \\ gqhp = g\varphi(h) , \\ gqhp^n = g\varphi(h)p^{n-1} & \text{if } n \geqq 2 ; \end{cases} \tag{24}$$

$$\begin{cases} gphp^n = g\psi(h)p^{n+1} , \\ gph = g\psi(h)p , \\ gphq = g\psi(h) , \\ gphq^n = g\psi(h)q^{n-1} & \text{if } n \geqq 2 . \end{cases} \tag{25}$$

We denote by I^* the set consisting of elements of the forms gq^n or gp^n or g with $g \in E$ and natural number n. By (22)-(25), we see that I^* is a subsemigroup, which is then clearly the subsemigroup generated by the regular pair (p, q).

Since $q = e_1q$ and $p = f_1p$, we have

$$vq = f_1q , \qquad e_1p = tp . \tag{26}$$

By (26) and relations $e_2q^2 = q^2 = e_1q^2$, $f_2p^2 = p^2 = f_1p^2$,

$$e_2q^n = e_1q^n , \quad vq^n = f_1q^n , \quad e_1p^n = tp^n , \quad f_1p^n = f_2p^n \qquad \text{for } n \geqq 2 . \tag{27}$$

By (18), $q^2(f_2q^{n+1}) = q^{n+3} < q^{n+2} = e_2q^{n+2} = q^2(e_2q^n)$ and $q^2(f_2q) = q^3 < q^2 = e_2q^2 = q^2e_2$, and so

$$f_2q^{n+1} < e_2q^n , \qquad f_2q < e_2 . \tag{28}$$

Similarly

$$f_2 < e_2p , \qquad f_2p^n < e_2p^{n+1} . \tag{29}$$

Thus the elements of I^* are ordered by

$$\begin{aligned} \cdots &< e_2q^2 \leqq tq^2 \leqq vq^2 \leqq f_2q^2 < e_2q \leqq e_1q \leqq tq \leqq vq \leqq f_2q \\ &< e_2 \leqq e_1 \leqq t \leqq v \leqq f_1 \leqq f_2 < e_2p \leqq tp \leqq vp \leqq f_1p \leqq f_2p \\ &< e_2p^2 \leqq tp^2 \leqq vp^2 \leqq f_2p^2 < \cdots . \end{aligned} \tag{30}$$

LEMMA 13. *In this Case* 1*L*1, *all the following relations are equivalent to each other*:

$$\begin{gathered} \cdots,\ e_2q^2 = tq^2,\ vq^2 = f_2q^2,\ e_2q = e_1q,\ e_1q = tq,\ vq = f_2q, \\ e_2 = e_1,\ e_1 = t,\ v = f_1,\ f_1 = f_2,\ e_2p = tp,\ vp = f_1p, \\ f_1p = f_2p,\ e_2p^2 = tp^2,\ vp^2 = f_2p^2,\ \cdots . \end{gathered}$$

Proof. First we prove that the four relations $e_2 = e_1$, $e_1 = t$, $v = f_1$, $f_1 = f_2$ are equivalent to each other. In fact, if $e_2 = e_1$, then

$$e_1 = e_2 = e_2f_1 = e_1f_1 = t .$$

If $e_1 = t$, then

$$f_1 = pe_1q = ptq = f_2 .$$

It can similarly be proved that $f_1 = f_2$ implies $v = f_1$ and that $v = f_1$ implies $e_2 = e_1$. Next we prove that $vq^n = f_2q^n$ are equivalent to $v = f_1$. In fact, if $vq^n = f_2q^n$, then, by taking account of the table (9), $v = vq^np^n = f_2q^np^n = f_2$, and so $v = f_1$. If $v = f_1$, then, by the result proved above, $v = f_1 = f_2$, and so $vq^n = f_2q^n$. Similarly we can prove that each of the remaining relations is equivalent to one of the relations $e_2 = e_1$, $e_1 = t$, $v = f_1$, $f_1 = f_2$.

In a similar way, we can prove the following

LEMMA 14. *In this Case* 1*L*1, *all the following relations are equivalent to each other*:

$$\cdots,\ tq^2 = vq^2,\ tq = vq,\ t = v,\ tp = vp,\ tp^2 = vp^2,\ \cdots .$$

Now we study *Case* 1*L*1 by dividing into subcases.

1°. *Subcase of Case* 1*L*1 *when* $e_2 \neq e_1$, $t \neq v$.

In this subcase, by Lemmas 13 and 14, all the elements of I^* written in (30) are different from each other. We consider a mapping of I_{2L} of Example 3 in § 6 into I^*:

$$(31)\qquad (i, g) \to \begin{cases} gp^i & \text{if } i > 0\,, \\ g & \text{if } i = 0 \\ gq^{-i} & \text{if } i < 0\,. \end{cases}$$

By (11) in § 6, this mapping is one-to-one and onto. Moreover, it is order-preserving. Furthermore, comparing (10) in § 6 and (22)–(25), we see that this mapping is an isomorphism. Thus I^* is isomorphic to I_{2L}. We remark, by the above isomorphism, $(-1, e_1)$ and $(1, f_1)$ are mapped into q and p, respectively.

2°. *Subcase of Case* 1*L*1 *when* $e_2 \neq e_1$, $t = v$.

In this subcase, by Lemmas 13 and 14 and the consideration of 1°, I^* is isomorphic to I_{4L}.

3°. *Subcase of Case* 1*L*1 *when* $e_2 = e_1$, $t \neq v$.

By Lemmas 13 and 14, I^* consists of elements

$$\cdots < e_2q^2 = q^2 < f_2q^2 < e_2q = q < f_2q < e_2 \\ < f_2 < e_2p < f_2p = p < e_2p^2 < f_2p^2 = p^2 < \cdots\,,$$

and, by (22)–(25), we have

$$(32)\qquad \begin{cases} gq^mhq^n = gq^{m+n}\,, \\ gq^mh = gq^m\,, \\ gq^mhp^n = \begin{cases} gq^{m-n} & \text{if } m > n\,, \\ g & \text{if } m = n\,, \\ gp^{n-m} & \text{if } m < n\,, \end{cases} \end{cases}$$

$$(32')\qquad \begin{cases} gp^mhp^n = gp^{m+n}\,, \\ gp^mh = gp^m\,, \\ gp^mhq^n = \begin{cases} gp^{m-n} & \text{if } m > n\,, \\ g & \text{if } m = n\,, \\ gq^{n-m} & \text{if } m < n\,. \end{cases} \end{cases}$$

Also we have

$$(32'')\qquad ghq^n = gq^n\,,\quad gh = g\,,\quad ghp^n = gp^n\,.$$

Thus, in this subcase, the mapping

$$(i, -1) \to \begin{cases} e_2 p^i & \text{if } i > 0 , \\ e_2 & \text{if } i = 0 , \\ e_2 q^{-i} & \text{if } i < 0 , \end{cases}$$
(33)
$$(i, 1) \to \begin{cases} f_2 p^i & \text{if } i > 0 , \\ f_2 & \text{if } i = 0 , \\ f_2 q^{-i} & \text{if } i < 0 , \end{cases}$$

is an isomorphism of I_{1L} onto I^*. We remark, in this isomorphism, $(-1, -1)$ and $(1, 1)$ are mapped into q and p, respectively.

4°. *Subcase of Case* 1*L*1 *when* $e_2 = e_1$, $t = v$.

In this subcase, by Lemmas 13 and 14, I^* consists of elements

$$\cdots < q^3 < q^2 < q < e_2 = e_1 = t = v = f_1 = f_2 < p < p^2 < p^3 < \cdots ,$$

and, by (22)-(25),

$$q^m p^n = p^n q^m = \begin{cases} q^{m-n} & \text{if } m > n , \\ e_1 & \text{if } m = n , \\ p^{n-m} & \text{if } m < n , \end{cases}$$

$$e_1 q^m = q^m e_1 = q^m , \qquad e_1 p^m = p^m e_1 = p^m .$$

Thus the mapping

$$i \to \begin{cases} p^i & \text{if } i > 0 , \\ e_1 & \text{if } i = 0 , \\ q^{-i} & \text{if } i < 0 , \end{cases}$$

is an isomorphism of I_0 onto I^*.

Next we consider *Case* 1*L*2.

1°. *Subcase of Case* 1*L*2 *when* $e_2 \neq e_1$, $t \neq v$.

We can prove, in a similar way as in the corresponding subcase of *Case* 1*L*1, that the subsemigroup I^* generated by (p, q) consists of elements

$$\begin{aligned} \cdots &< e_2 q^2 < tq^2 < vq^2 < f_2 q^2 < e_2 q < tq < vq < f_1 q = q < f_2 q < e_2 \\ &< e_1 < t < v < f_1 < f_2 < e_2 p < e_1 p = p < tp < vp < f_2 p < e_2 p^2 \\ &< tp^2 < vp^2 < f_2 p^2 < \cdots , \end{aligned}$$

and that the mapping given by the same formula (31) as in *Case* 1*L*1 is an isomorphism of I_{3L} onto I^*. In particular, $(-1, f_1)$ and $(1, e_1)$ are mapped into q and p, respectively.

2°. *Subcase of Case* 1*L*2 *when* $e_2 \neq e_1$, $t = v$.

The subsemigroup I^* generated by (p, q) is isomorphic to I_{5L}.

3°. *Subcase of Case* 1*L*2 *when* $e_2 = e_1$, $t \neq v$.

The subsemigroup I^* generated by (p, q) consists of elements

$$\cdots < e_2q^2 < f_2q^2 = q^2 < e_2q < f_2q = q < e_2$$
$$< f_2 < e_2p = p < f_2p < e_2p^2 = p^2 < f_2p^2 < \cdots,$$

and so consists of the same elements as in the corresponding subcase in *Case* 1*L*1. Also the multiplication in I^* is given by the same formula (32)–(32″) as in *Case* 1*L*1. Hence the mapping (33) is an isomorphism of I_{1L} onto I^*. However, in this subcase, $(-1, 1)$ and $(1, -1)$ are mapped into q and p, respectively.

4°. *Subcase of Case* 1*L*2 *when* $e_2 = e_1$, $t = v$.

The subsemigroup I^* generated by (p, q) is the same, in all respects, as that in the corresponding subcase in *Case* 1*L*1.

In *Case* 1*R*, we can argue in a similar way.

THEOREM 4. *Using the notations given in* (12), *let* (p, q) *be a regular pair of infinite order such that* $q \leqq p$ *and* $e_2 \equiv e_1f_1(D_E)$, *let* I^* *be the subsemigroup generated by* (p, q), *and let* $I_0 - I_{5R}$ *be the ordered I-semigroups given in* § 6.

(a) *If* $e_2 = e_1$ *and* $t = v$, *then* I^* *is isomorphic to* I_0;

(b) *if* (p, q) *is L-typed*, $e_2 = e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* I_{1L};

(c) *if* (p, q) *is L-typed*, $qp \leqq pq$, $e_2 \neq e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* I_{2L};

(d) *if* (p, q) *is L-typed*, $pq \leqq qp$, $e_2 \neq e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* L_{3L};

(e) *if* (p, q) *is L-typed*, $qp \leqq pq$, $e_2 \neq e_1$ *and* $t = v$, *then* I^* *is isomorphic to* I_{4l};

(f) *if* (p, q) *is L-typed*, $pq \leqq qp$, $e_2 \neq e_1$ *and* $t = v$, *then* I^* *is isomorphic to* I_{5L};

(g) *if* (p, q) *is R-typed*, $e_2 = e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* I_{1R};

(h) *if* (p, q) *is R-typed*, $pq \leqq qp$, $e_2 \neq e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* I_{2R};

(i) *if* (p, q) *is R-typed*, $qp \leqq pq$, $e_2 \neq e_1$ *and* $t \neq v$, *then* I^* *is isomorphic to* I_{3R};

(j) *if* (p, q) *is R-typed,* $pq \leqq qp$, $e_2 \neq e_1$ *and* $t = v$, *then* I^* *is isomorphic to* I_{4R};

(k) *if* (p, q) *is R-typed,* $qp \leqq pq$, $e_2 \neq e_1$ *and* $t = v$, *then* I^* *is isomorphic to* I_{5R}.

COROLLARY. *Under the assumptions of Theorem* 4, I^* *contains subsemigroup isomorphic to the ordered additive group* I_0 *of integers.*

8. **Ordered J-semigroups.** In § 6, we gave examples of ordered semigroups each of which has a regular pair of infinite order. In this section, we give examples of another kind of such semigroups.

EXAMPLE 1. Let J be the set of pairs (m, n) of nonnegative integers with the multiplication

$$(k, l)(m, n) = \begin{cases} (k + m - l, n) & \text{if } l \leqq m , \\ (k, n + l - m) & \text{if } m \leqq l . \end{cases}$$

As is well known, J is an abstract semigroup (Lyapin [4] or Saitô [9]). It can be verified that the semigroup J turns out to be an ordered semigroup when we define the order in J by

$$(k, l) < (m, n) \quad \text{if } k + n < m + l \quad \text{or } k + n = m + l,\ k < m .$$

This ordered semigroup is denoted by J_{01}.

EXAMPLE 2. It can be verified that the semigroup J in Example 1 turns out to be an ordered semigroup when we define the order in J by

$$(k, l) < (m, n) \quad \text{if } k + n > m + l \quad \text{or } k + n = m + l,\ k < m .$$

This ordered semigroup is denoted by J_{02}.

EXAMPLE 3. The ordered semigroup J_{03}, which is the semigroup J with the order

$$(k, l) < (m, n) \quad \text{if } k + n < m + l \quad \text{or } k + n = m + l,\ k > m .$$

EXAMPLE 4. The ordered semigroup J_{04}, which is the semigroup J with the order

$$(k, l) < (m, n) \quad \text{if } k + n > m + l \quad \text{or } k + n = m + l,\ k > m .$$

In each of the ordered semigroups J_{01}–J_{04}, $((0, 1), (1, 0))$ is a regular pair of infinite order, which generates the corresponding ordered semigroup. In J_{01} and J_{03}, $(0, 1)$ is negative and $(1, 0)$ is positive, while in J_{02} and J_{04}, $(0, 1)$ is positive and $(1, 0)$ is negative.

EXAMPLE 5. Let W be a system consisting of infinite elements ordered by

$$e_2 < e_1 < t < v < f_1 < f_2 < \cdots ,$$

and with the following multiplication table:

(34)

	e_2	e_1	t	v	f_1	f_2	$f_3 \cdots$
e_2	e_2	e_2	t	t	t	f_2	$f_3 \cdots$
e_1	e_2	e_1	t	t	t	f_2	$f_3 \cdots$
t	t	t	t	t	t	f_2	$f_3 \cdots$
v	v	v	v	v	v	f_2	$f_3 \cdots$
f_1	v	v	v	v	f_1	f_2	$f_3 \cdots$
f_2	f_2	f_2	f_2	f_2	f_2	f_2	$f_3 \cdots$
f_3	f_3	f_3	f_3	f_3	f_3	f_3	$f_3 \cdots$
$\cdot$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

It can be verified that W is an ordered idempotent semigroup. We define two mappings φ and ψ of W into itself by

$$\varphi(e_2) = \varphi(e_1) = \varphi(t) = \varphi(v) = e_2,\ \varphi(f_1) = e_1,$$
$$\varphi(f_2) = t,\ \varphi(f_3) = f_2,\ \cdots ;$$
$$\psi(e_2) = v,\ \psi(e_1) = f_1,\ \psi(t) = \psi(v) = \psi(f_1) = f_2,$$
$$\psi(f_2) = f_3,\ \psi(f_3) = f_4,\ \cdots .$$

As is easily seen, these mappings have the following properties:

(a) *both* φ *and* ψ *are monotone;*

(b) *both* φ *and* ψ *are semigroup-homomorphisms;*

(c) $\varphi(g)e_1 = e_1\varphi(g) = \varphi(g)$, $\psi(g)f_1 = f_1\psi(g) = \psi(g)$ *for every* $g \in W$;

(d) $\varphi(\psi(g)) = e_1 g$, $\psi(\varphi(g)) = f_1 g$ *for every* $g \in W$.

For brevity, we use notations:

$$\varphi^1(g) = \varphi(g),\ \varphi^2(g) = \varphi(\varphi(g)),\ \varphi^3(g) = \varphi(\varphi(\varphi(g))),\ \cdots ;$$
$$\psi^1(g) = \psi(g),\ \psi^2(g) = \psi(\psi(g)),\ \psi^3(g) = \psi(\psi(\psi(g))),\ \cdots .$$

Now we consider the system H, consisting of pairs (i, g) with $i \in I_0$ and $g \in W$, where I_0 is the ordered additive group of integers as is defined in § 6. In H, we define the order lexicographically and define the multiplication by

$$(35) \qquad (i, g)(j, h) = \begin{cases} (i + j, g\varphi^{-i}(h)) & \text{if } i < 0 , \\ (i + j, gh) & \text{if } i = 0 , \\ (i + j, g\psi^{i}(h)) & \text{if } i > 0 . \end{cases}$$

Using the properties (a)–(d) of φ and ψ, we can prove that H is an ordered semigroup. Finally, we consider the subset J_{11L} of H consisting of elements with

$$\begin{aligned} & i \leqq -2,\ g \neq e_1, f_1, \text{ or} \\ & i = -1,\ g \neq f_1, \text{ or} \\ & i = 0,\ g \text{ arbitrary, or} \\ & i = 1,\ g \geqq t, \text{ or} \\ & i \geqq 2,\ g \geqq f_i . \end{aligned} \tag{36}$$

It can be verified that J_{11L} is closed with respect to the multiplication, and so forms an ordered semigroup. Here we remark that the ordered semigroup J_{11L} contains a subsemigroup which is isomorphic to J_{01}. In fact, we can verify that the following mapping of J_{01} into J_{11L} is an isomorphism into J_{11L}:

$$(m, n) \to \begin{cases} (m - n, e_2) & \text{if } m = 0 , \\ (m - n, t) & \text{if } m = 1 , \\ (m - n, f_m) & \text{if } m \geqq 2 . \end{cases}$$

EXAMPLE 6. Let W, φ and ψ be the same as in the preceding Example 5. We can verify that the system J_{12L}, consisting of pairs (i, g) with $i \in I_0$, $g \in W$ which satisfies

$$\begin{aligned} & i \leqq -2,\ g \geqq f_i, \text{ or} \\ & i = -1,\ g \geqq t, \text{ or} \\ & i = 0,\ g \text{ arbitrary, or} \\ & i = 1,\ g \neq f_1, \text{ or} \\ & i \geqq 2,\ g \neq e_1, f_1 , \end{aligned}$$

forms an ordered semigroup, when we define the order lexicographically and define the multiplication by

$$(i, g)(j, h) = \begin{cases} (i + j, g\psi^{-i}(h)) & \text{if } i < 0 , \\ (i + j, gh) & \text{if } i = 0 , \\ (i + j, g\varphi^{i}(h)) & \text{if } i > 0 . \end{cases}$$

Moreover we can verify that the mapping of J_{02} into J_{12L} defined by

$$(m, n) \to \begin{cases} (n - m, e_2) & \text{if } m = 0 , \\ (n - m, t) & \text{if } m = 1 , \\ (n - m, f_m) & \text{if } m \geqq 2 \end{cases}$$

is an isomorphism into J_{12L}, and so J_{12L} contains a subsemigroup which is isomorphic to J_{02}.

EXAMPLE 7. Let W' be the system consisting of infinite elements ordered by

$$\cdots < e_3 < e_2 < e_1 < t < v < f_1 < f_2 ,$$

and with the multiplication table arising from the table (34) by means of replacing $e_2, e_1, t, v, f_1, f_2, f_3, \cdots$ by $f_2, f_1, v, t, e_1, e_2, e_3, \cdots$ respectively. It can be verified that W' is an ordered idempotent semigroup. We define two mappings χ and ω of W' into itself by

$$\begin{aligned} &\chi(f_2) = \chi(f_1) = \chi(v) = \chi(t) = f_2,\ \chi(e_1) = f_1, \\ &\qquad\qquad \chi(e_2) = v,\ \chi(e_3) = e_2, \cdots ; \\ &\omega(f_2) = t,\ \omega(f_1) = e_1,\ \omega(v) = \omega(t) = \omega(e_1) = e_2, \\ &\qquad\qquad \omega(e_2) = e_3,\ \omega(e_3) = e_4, \cdots . \end{aligned}$$

In a similar way as in Example 5, the system J_{13L}, consisting of pairs (i, g) with $i \in I_0$, $g \in W'$ which satisfies

$$\begin{aligned} &i \leqq -2,\ g \neq e_1, f_1,\ \text{or} \\ &i = -1,\ g \neq e_1,\ \text{or} \\ &i = 0,\ g\ \text{arbitrary, or} \\ &i = 1,\ g \leqq v,\ \text{or} \\ &i \geqq 2,\ g \leqq e_i \end{aligned}$$

forms an ordered semigroup, when we order it lexicographically and define the multiplication by

$$(i, g)(j, h) = \begin{cases} (i + j, g\chi^{-i}(h)) & \text{if } i < 0 , \\ (i + j, gh) & \text{if } i = 0 , \\ (i + j, g\omega^{i}(h)) & \text{if } i > 0 . \end{cases}$$

The mapping of J_{03} into J_{13L} defined by

$$(m, n) \rightarrow \begin{cases} (m - n, f_2) & \text{if } m = 0 , \\ (m - n, v) & \text{if } m = 1 , \\ (m - n, e_m) & \text{if } m \geqq 2 \end{cases}$$

is an isomorphism into J_{13L}, and so J_{13L} contains a subsemigroup which is isomorphic to J_{03}.

EXAMPLE 8. The ordered semigroup J_{14L} consists of pairs (i, g) with $i \in I_0$, $g \in W'$ which satisfies

$$\begin{aligned} &i \leqq -2,\ g \leqq e_i,\ \text{or} \\ &i = -1,\ g \leqq v,\ \text{or} \\ &i = 0,\ g\ \text{arbitrary, or} \end{aligned}$$

$$i = 1, \ g \neq e_1, \text{ or}$$
$$i \geqq 2, \ g \neq e_1, f_1 .$$

It is ordered lexicographically and has the multiplication

$$(i, g)(j, h) = \begin{cases} (i + j, g\omega^{-i}(h)) & \text{if } i < 0 , \\ (i + j, gh) & \text{if } i = 0 , \\ (i + j, g\chi^i(h)) & \text{if } i > 0 . \end{cases}$$

The mapping of J_{04} into J_{14L} defined by

$$(m, n) \rightarrow \begin{cases} (n - m, f_2) & \text{if } m = 0 , \\ (n - m, v) & \text{if } m = 1 , \\ (n - m, e_m) & \text{if } m \geqq 2 \end{cases}$$

is an isomorphism into J_{14L}.

EXAMPLE 9. The ordered semigroup J_{21L} is constructed from J_{11L} by identifying elements contained in each of the following four pairs:

$$(-1, e_2), \ (-1, e_1); \ (0, e_2), \ (0, e_1); \ (0, v), \ (0, f_1); \ (1, v), \ (1, f_1) .$$

J_{21L} contains a subsemigroup which is isomorphic to J_{01}.

EXAMPLE 10. The ordered semigroup J_{22L} is constructed from J_{12L} by identifying elements contained in each of the following four pairs:

$$(-1, v), \ (-1, f_1); \ (0, e_2), \ (0, e_1); \ (0, v), \ (0, f_1); \ (1, e_2), \ (1, e_1) .$$

J_{22L} contains a subsemigroup which is isomorphic to J_{02}.

EXAMPLE 11. The ordered semigroup J_{23L} is constructed from J_{13L} by identifying elements contained in each of the following four pairs:

$$(-1, f_1), \ (-1, f_2); \ (0, e_1), \ (0, t); \ (0, f_1), \ (0, f_2); \ (1, e_1), \ (1, t) .$$

J_{23L} contains a subsemigroup which is isomorphic to J_{03}.

EXAMPLE 12. The ordered semigroup J_{24L} is constructed from J_{14L} by identifying elements contained in each of the following four pairs:

$$(-1, e_1), \ (-1, t); \ (0, e_1), \ (0, t); \ (0, f_1), \ (0, f_2); \ (1, f_1), \ (1, f_2) .$$

J_{24L} contains a subsemigroup which is isomorphic to J_{04}.

EXAMPLE 13. The ordered semigroup J_{31} is constructed from J_{11L} by identifying (i, t) and (i, v) for each $i \leqq 1$. J_{31} contains a subsemigroup which is isomorphic to J_{01}.

EXAMPLE 14. The ordered semigroup J_{32} is constructed from J_{12L} by identifying (i, t) and (i, v) for each $i \geqq -1$. J_{32} contains a subsemigroup which is isomorphic to J_{02}. Here we remark that the ordered semigroup J_{32} is isomorphic to the dual of ordered semigroup J_{31}. In fact, we can verify that the mapping of the dual of ordered semigroup J_{31} into J_{32} defined by

$$(i, g) \rightarrow \begin{cases} (i, \psi^{-i}(g)) & \text{if } i < 0 , \\ (i, g) & \text{if } i = 0 , \\ (i, \varphi^{i}(g)) & \text{if } i > 0 \end{cases}$$

is an isomorphism onto J_{32}.

EXAMPLE 15. The ordered semigroup J_{33} is constructed from J_{13L} by identifying (i, t) and (i, v) for each $i \leqq 1$. J_{33} contains a subsemigroup which is isomorphic to J_{03}.

EXAMPLE 16. The ordered semigroup J_{34} is constructed from J_{14L} by identifying (i, t) and (i, v) for each $i \geqq -1$. J_{34} contains a subsemigroup which is isomorphic to J_{04}. We remark that J_{34} is isomorphic to the dual of ordered semigroup J_{33}.

EXAMPLES 17–24. The ordered semigroups $J_{11R}, \cdots, J_{24R}$ are multiplicative dual to $J_{11L}, \cdots, J_{24L}$, respectively.

These 24 ordered semigroups given above are called *ordered J-semigroups*. Ordered J-semigroups J_{01}, J_{02}, J_{03} and J_{04} are called *fundamental ordered J-semigroups.*

9. Regular pair of infinite order (2). In this section, the notations of elements e_n, f_n, t, v and the notations of conditions (A)–(D′) are used just as is defined in § 7. In § 7, we divided the investigation of a regular pair of infinite order into two cases, and *Case* 1 was studied in that section. Now we study *Case* 2. Thus, in this section, we suppose that *(p, q) is a regular pair of infinite order such that $q \leqq p$ and $e_2 \not\equiv e_1 f_1 (D_E)$.*

Case 2 is divided into two subcases:

Case 2*L*: *the subcase of Case* 2 *when (p, q) is L-typed;*

Case 2*R*: *the subcase of Case* 2 *when (p, q) is R-typed,*
and moreover *Case* 2*L* is divided into two subcases:

Case 2*L*1: *the subcase of Case* 2*L when the condition* (C) *holds;*

Case 2*L*2: *the subcase of Case* 2*L when* (C) *does not hold.*

Now we consider *Case* 2*L*1. Thus we assume that (p, q) is *L-typed and satisfies* (C). Then, by Lemma 12, the condition (A′) holds. Hence $(e_1f_1)e_2 = e_1(f_1e_1) = e_1f_1$, and so, since $e_2 \not\equiv e_1f_1(D_E)$, we obtain $e_2f_1 = e_2(e_1f_1) \neq e_2$. Hence (D) does not hold, and so, by Lemma 12, (B′) does not hold. Hence (A) and (C′) hold. Moreover, by Lemma 11,

$$e_2 = e_3 = e_4 = \cdots . \tag{37}$$

Since (p, q) is L-typed, we have, by Lemma 3,

$$t = e_1f_1,\ v = f_1e_1,\ e_1f_1e_1 = e_1f_1,\ f_1e_1f_1 = f_1e_1 .$$

Now we denote by E the set consisting of elements $e_2, e_1, t, v, f_1, f_2, f_3, \cdots$.

We have $e_1 < t$. In fact, otherwise, we would have $e_1 = e_1f_1$ and so, by Lemma 10, $e_2 = e_2f_1$, which is a relation in (D). We have $f_1 < f_2$. In fact, otherwise, we would have $f_1 = f_2$ and so $e_1f_1 = e_1f_2$, which is a relation in (B′). We have $f_n < f_{n+1}$ for every $n \geqq 2$. In fact, otherwise, we would have $f_n = f_{n+1}$. Without loss of generality, we assume that $qp \leqq pq$. Then, by Corollary of Lemma 9, $e_n = q^np^n$ and $f_n = p^nq^n$. Hence, by (37), (A) and (A′), we would have

$$\begin{aligned} e_2 = e_n &= q^np^nq^np^n = q^nf_np^n = q^nf_{n+1}p^n = e_nf_1e_n \\ &= e_2f_1e_2 = (e_2f_1)(f_1e_2) = (e_1f_1)(f_1e_1) = e_1f_1 = t , \end{aligned}$$

which contradicts that $e_2 \leqq e_1 < t$. Thus the elements of E are ordered by

$$e_2 \leqq e_1 < t \leqq v \leqq f_1 < f_2 < f_3 < \cdots . \tag{38}$$

Using Lemma 10 and conditions (A), (A′), (C), (C′), we can verify that the elements of E are multiplied together just as in the table (34) in Example 5 in § 8.

Case 2*L*1 is divided into two subcases:

Case 2*L*11: *the subcase of Case* 2*L*1 *when* $qp \leqq pq$;

Case 2*L*12: *the subcase of Case* 2*L*1 *when* $pq \leqq qp$.

Now we consider *Case* 2*L*11. Then $e_n = q^np^n$, $f_n = p^nq^n$, and so, by (C) and (C′),

$$qp^3 = p^2 , \qquad q^3p = q^2 . \tag{39}$$

Moreover, by $e_1f_1 = e_1f_1e_1$ and $f_1e_1 = f_1e_1f_1$, we obtain

$$p^2q = p^2q^2p , \qquad q^2p = q^2p^2q . \tag{40}$$

Now we consider the mappings φ and ψ of E into itself which have been defined in § 8, that is,

$$\varphi(e_2) = \varphi(e_1) = \varphi(t) = \varphi(v) = e_2,\ \varphi(f_1) = e_1,\ \varphi(f_2) = t,\ \varphi(f_3) = f_2, \cdots ;$$
$$\psi(e_2) = v,\ \psi(e_1) = f_1,\ \psi(t) = \psi(v) = \psi(f_1) = f_2,\ \psi(f_2) = f_3,\ \psi(f_3) = f_4, \cdots .$$

Then, by (39) and (40), we can verify that

(41) $$qg = \varphi(g)q\ ,\quad pg = \psi(g)p \text{ for every } g \in E\ .$$

Moreover, as is shown in § 8, φ and ψ satisfy the conditions: (a)–(d) given there. Hence, if $l \geqq m+1$, then

$$\begin{aligned}\varphi^l(g)q^{m+1}p^{n+1} &= \varphi^l(g)q^m e_1 p^n = \varphi^l(g)\varphi^m(e_1)q^m p^n \\ &= \varphi^m(\varphi^{l-m}(g)e_1)q^m p^n = \varphi^m(\varphi^{l-m}(g))q^m p^n = \varphi^l(g)q^m p^n\ .\end{aligned}$$

Similarly, if $l \geqq m+1$, then we have

$$\psi^l(g)p^{m+1}q^{n+1} = \psi^l(g)p^m q^n\ .$$

Using these relations, we can verify that

(42) $$\begin{cases} gq^m hq^n = g\varphi^m(h)q^{m+n}\ , \\ gq^m h = g\varphi^m(h)q^m\ , \\ gq^m hp^n = \begin{cases} g\varphi^m(h)q^{m-n} & \text{if } m > n\ , \\ g\varphi^m(h) & \text{if } m = n\ , \\ g\varphi^m(h)p^{n-m} & \text{if } m < n\ . \end{cases} \end{cases}$$

(43) $$\begin{cases} gp^m hp^n = g\psi^m(h)p^{m+n}\ , \\ gp^m h = g\psi^m(h)p^m\ , \\ gp^m hq^n = \begin{cases} g\psi^m(h)p^{m-n} & \text{if } m > n\ , \\ g\psi^m(h) & \text{if } m = n\ , \\ g\psi^m(h)q^{n-m} & \text{if } m < n\ . \end{cases} \end{cases}$$

Now we denote by J^* the set consisting of elements of the forms gq^n, gp^n or g with $g \in E$ and natural number n. By (42) and (43), we see that J^* is a subsemigroup, which is then clearly the subsemigroup generated by regular pair (p, q).

Since $q = e_1 q$, we have

(44) $$vq = f_1 q\ .$$

By (44) and $e_2 q^2 = q^2 = e_1 q^2$, we have

(45) $$e_2 q^n = e_1 q^n\ ,\quad vq^n = f_1 q^n \text{ for } n \geqq 2\ .$$

Since $tp = qp^2qp = qp^2 = q^2p^3 = e_2p \leqq e_1p \leqq tp$, we have

$$e_2p = e_1p = tp \ . \tag{46}$$

If $n \geqq 2$, then $f_np^n = p^nq^np^n = p^n = qp^{n+1} = q^2p^{n+2} = e_2p^n$, and so,

$$e_2p^n = e_1p^n = tp^n = vp^n = f_1p^n = \cdots = f_np^n \qquad \text{for } n \geqq 2 \ . \tag{47}$$

We have $f_nq^{m+1} < e_2q^m$, for, otherwise, we would have

$$q^{m+n+1} = q^nf_nq^{m+1} \geqq q^ne_2q^m = \varphi^n(e_2)q^{m+n} = q^{m+n} \ ,$$

which contradicts Lemma 9. Therefore we also have $f_nq < e_2$. We have $e_1q^n < tq^n$, for, otherwise, we would have

$$e_n = e_1q^np^n \geqq tq^np^n = te_n = t \ ,$$

which contradicts (38). Similarly we obtain $f_nq^m < f_{n+1}q^m$. We have $f_np^m < e_2p^{m+1}$, for, otherwise, we would have

$$f_nf_mq = f_np^mq^{m+1} \geqq e_2p^{m+1}q^{m+1} = e_2f_{m+1} \geqq e_2 \ ,$$

which contradicts $f_lq < e_2$. If $m \geqq n$, then we have $f_mp^n < f_{m+1}p^n$, for, otherwise, we would have

$$f_m = f_mp^nq^n \geqq f_{m+1}p^nq^n = f_{m+1} \ ,$$

which contradicts (38). Thus the elements of J^* are ordered by

$$\begin{aligned} &\cdots < e_2q^2 < tq^2 \leqq vq^2 < f_2q^2 < f_3q^2 < \cdots < e_2q \leqq e_1q < tq \\ &\leqq vq < f_2q < f_3q < \cdots < e_2 \leqq e_1 < t \leqq v \leqq f_1 < f_2 < f_3 < \cdots \\ &< tp \leqq vp \leqq f_1p < f_2p < f_3p < \cdots < f_2p^2 < f_3p^2 < \cdots \ . \end{aligned} \tag{48}$$

LEMMA 15. *In this Case 2L11, the following four conditions are equivalent to each other*:

$$vq = f_1p,\ v = f_1,\ e_2 = e_1,\ e_2q = e_1q \ .$$

Proof. If $vp = f_1p$, then $v = vpq = f_1pq = f_1$. If $v = f_1$, then $e_2 = qvp = qf_1p = e_1$. If $e_2 = e_1$, then clearly $e_2q = e_1q$. If $e_2q = e_1q$, then $vp = pe_2qp = pe_1qp = p = f_1p$.

LEMMA 16. *In this Case 2L11, all the following conditions are equivalent to each other*:

$$\cdots,\ tq^3 = vq^3,\ tq^2 = vq^2,\ tq = vq,\ t = v,\ tp = vp \ .$$

Proof. If $t = v$, then clearly $tq^n = vq^n$ and $tp = vp$. If $tp = vp$, then $t = tpq = vpq = v$. Similarly, if $tq^n = vq^n$ for some n, then $t = v$.

Now we investigate *Case* 2L11 by dividing into several subcases.

1°. *Subcase of Case* 2*L*11 *when* $e_2 \neq e_1$, $t \neq v$.

By Lemmas 15 and 16, all the elements of J^* written in (48) are different. We consider mapping of J_{11L} of Example 5 in § 8 into I^*:

$$(n, g) \rightarrow \begin{cases} gp^n & \text{if } n > 0 \,, \\ g & \text{if } n = 0 \,, \\ gq^{-n} & \text{if } n < 0 \,. \end{cases} \tag{49}$$

By (36) in § 8, this mapping is one-to-one and onto. Moreover it is order-preserving. Furthermore, comparing (35) in § 8 to (42) and (43), we see that this mapping is an isomorphism. Thus J^* is isomorphic to J_{11L}. We remark, by the above isomorphism, $(-1, e_1)$ and $(1, f_1)$ are mapped into q and p, respectively.

2°. *Subcase of Case* 2*L*11 *when* $e_2 = e_1$, $t \neq v$.

By Lemmas 15 and 16, J^* is isomorphic to J_{21L} of Example 9 in § 8.

3°. *Subcase of Case* 2*L*11 *when* $e_2 \neq e_1$, $t = v$.

By Lemmas 15 and 16, J^* is isomorphic to J_{31} of Example 13 in § 8.

4°. *Subcase of Case* 2*L*11 *when* $e_2 = e_1$, $t = v$.

By Lemmas 15 and 16, J^* consists of elements

$$\begin{aligned} &\cdots < e_2q^2 < tq^2 < f_2q^2 < f_3q^2 < \cdots < e_2q < tq < f_2q < f_3q < \cdots \\ &\cdots < e_2 < t < f_2 < f_3 < \cdots < tp < f_2p < f_3p < \cdots < f_2p^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad < f_3p^2 < \cdots \,. \end{aligned}$$

We can verify that the mapping of J_{01} into J^* defined by

$$(m, n) \rightarrow \begin{cases} e_2q^n & \text{if } m = 0,\ n > 0 \,, \\ e_2 & \text{if } m = n = 0 \,, \\ tq^{n-1} & \text{if } m = 1,\ n > 1 \,, \\ t & \text{if } m = n = 1 \,, \\ tp & \text{if } m = 1,\ n = 0 \,, \\ f_mq^{n-m} & \text{if } m \geqq 2,\ m < n \,, \\ f_m & \text{if } m \geqq 2,\ m = n \,, \\ f_mp^{m-n} & \text{if } m \geqq 2,\ m > n \geqq 0 \end{cases}$$

is an isomorphism onto J^*.

Similarly we can discuss *Case* $2L12$.

Now we consider *Case* $2L2$. In this case, (C) does not hold, and so, by Lemma 12, also (A′) does not hold. Hence (B) and (D′) hold, and so, by Lemma 12, also (B′) and (D) hold. Thus we can argue in a similar way as in *Case* $2L1$.

We can discuss *Case* $2R$ in a similar way.

THEOREM 5. *Using the notations given in* (12), *let* (p, q) *be a regular pair of infinite order such that* $q \leqq p$ *and* $e_2 \not\equiv e_1 f_1(D_E)$, *let* J^* *be the subsemigroups generated by* (p, q), *and let* $J_{01} - J_{24R}$ *be the ordered J-semigroups given in* § 8.

(a) *If* $qp \leqq pq$, $e_2 = e_1$, $t = v$, *then* J^* *is isomorphic to* J_{01};

(b) *if* $pq \leqq qp$, $e_2 = e_1$, $t = v$, *then* J^* *is isomorphic to* J_{02};

(c) *if* $pq \leqq qp$, $f_1 = f_2$, $t = v$, *then* J^* *is isomorphic to* J_{03};

(d) *if* $qp \leqq pq$, $f_1 = f_2$, $t = v$, *then* J^* *is isomorphic to* J_{04};

(e) *if* (p, q) *is L-typed,* $qp \leqq pq$, $f_1 \leqq f_2 e_1$, $e_2 \neq e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{11L};

(f) *if* (p, q) *is L-typed,* $pq \leqq qp$, $f_1 \leqq f_2 e_1$, $e_2 \neq e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{12L};

(g) *if* (p, q) *is L-typed,* $pq \leqq qp$, $f_2 e_1 < f_1$, $f_1 \neq f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{13L};

(h) *if* (p, q) *is L-typed,* $qp \leqq pq$, $f_2 e_1 < f_1$, $f_1 \neq f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{14L};

(i) *if* (p, q) *is L-typed,* $qp \leqq pq$, $f_1 \leqq f_2 e_1$, $e_2 = e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{21L};

(j) *if* (p, q) *is L-typed,* $pq \leqq qp$, $f_1 \leqq f_2 e_1$, $e_2 = e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{22L};

(k) *if* (p, q) *is L-typed,* $pq \leqq qp$, $f_2 e_1 < f_1$, $f_1 = f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{23L};

(l) *if* (p, q) *is L-typed,* $qp \leqq pq$, $f_2 e_1 < f_1$ $f_1 = f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{24L};

(m) *if* $qp \leqq pq$, $e_2 \neq e_1$, $t = v$, *either* (p, q) *is L-typed and* $f_1 \leqq f_2 e_1$ *or* (p, q) *is R-typed and* $f_1 \leqq e_1 f_2$, *then* J^* *is isomorphic to* J_{31};

(n) *if* $pq \leqq qp$, $e_2 \neq e_1$, $t = v$, *either* (p, q) *is L-typed and* $f_1 \leqq f_2 e_1$ *or* (p, q) *is R-typed and* $f_1 \leqq e_1 f_2$, *then* J^* *is isomorphic to* J_{32};

(o) *if* $pq \leqq qp$, $f_1 \neq f_2$, $t = v$, *either* (p, q) *is L-typed and* $f_2 e_1 < f_1$ *or* (p, q) *is R-typed and* $e_1 f_2 < f_1$, *then* J^* *is isomorphic to* J_{33};

(p) *if* $qp \leqq pq$, $f_1 \neq f_2$, $t = v$, *either* (p, q) *is L-typed and* $f_2 e_1 < f_1$ *or* (p, q) *is R-typed and* $e_1 f_2 < f_1$, *then* J^* *is isomorphic to* J_{34};

(q) *if* (p, q) *is R-typed,* $pq \leqq qp$, $f_1 \leqq e_1 f_2$, $e_2 \neq e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{11R};

(r) *if* (p, q) *is R-typed,* $qp \leqq pq$, $f_1 \leqq e_1 f_2$, $e_2 \neq e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{12R};

(s) *if* (p, q) *is R-typed,* $qp \leqq pq$, $e_1f_2 < f_1$, $f_1 \neq f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{13R};

(t) *if* (p, q) *is R-typed,* $pq \leqq qp$, $e_1f_2 < f_1$, $f_1 \neq f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{14R};

(u) *if* (p, q) *is R-typed,* $pq \leqq qp$, $f_1 \leqq e_1f_2$, $e_2 = e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{21R};

(v) *if* (p, q) *is R-typed,* $qp \leqq pq$, $f_1 \leqq e_1f_2$, $e_2 = e_1$, $t \neq v$, *then* J^* *is isomorphic to* J_{22R};

(w) *if* (p, q) *is R-typed,* $qp \leqq pq$, $e_1f_2 < f_1$, $f_1 = f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{23R};

(x) *if* (p, q) *is R-typed,* $pq \leqq qp$, $e_1f_2 < f_1$, $f_1 = f_2$, $t \neq v$, *then* J^* *is isomorphic to* J_{24R}.

COROLLARY. *Under the assumptions of Theorem* 5, J^* *contains a subsemigroup which is isomorphic to one of the fundamental ordered J-semigroups.*

§ 10. Applications. A semigroup S is called an *inverse semigroup* if every element of S is regular and each pair of idempotents of S commute (Munn and Penrose [6]). It can be seen that every subsemigroup of an inverse semigroup S in which every element is regular is an inverse subsemigroup. Hence, by Corollary 2 of Lemma 5, for a regular pair (p, q) of S, the subsemigroup generated by (p, q) is an inverse subsemigroup. Now we see that, except I_0, J_{01}, J_{02}, J_{03} and J_{04}, all ordered semigroups given in examples in §§ 4, 6 and 8 are not inverse semigroups. Hence we have

THEOREM 6. *Let* (p, q) *be a non-idempotent regular pair of an ordered inverse semigroup. Then the subsemigroup generated by* (p, q) *is isomorphic to either the additive ordered group* I_0 *of integers or one of the fundamental ordered J-semigroups.*

Evidently fundamental ordered J-semigroups are not commutative. Hence we have

THEOREM 7. *Let* (p, q) *be a non-idempotent regular pair of an ordered commutative semigroup. Then the subsemigroup generated by* (p, q) *is isomorphic to the additive ordered group* I_0 *of integers.*

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