# ON THE NÖRLUND SUMMABILITY OF FOURIER SERIES

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1. Let $f(x)$ be a function integrable $-L$ over the interval $(-\pi, \pi)$ and periodic with period $2\pi$, outside this interval.

Let

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s(x)\}\,, \tag{1.1}$$

and

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \operatorname{Sin} nx) \tag{1.2}$$

be the Fourier Series of the function $\phi(t)$.

Nörlund Summability of Fourier Series (1.2) has been considered by Woronoi [6] and later on by Nörlund [4]. These results have been extended by Hille and Tamarkin [2], [3], and later on by Astrachan [1]. Recently, extending a result due to Hille and Tamarkin [3], Varshney [5] has proved the proved the following:

THEOREM. V. *If the sequence* $\{p_n\}$ *satisfies the following conditions*:

$$\frac{n\,|\,pn\,|}{\log n} < c\,|\,P_n\,|\,, \tag{1.3}$$

$$\sum_{k=0}^{n} \frac{k\,|\,p_k - p_{k-1}\,|}{\log(k+1)} < c\,|\,P_n\,| \tag{1.4}$$

*and*

$$\sum_{k=0}^{n} \frac{P_k}{k \log(k+1)} < c\,|\,P_n\,| \tag{1.5}$$

*and also if*

$$\bar{\Phi}_1(t) = \int_0^t |\,\phi(u)\,|\,du = 0\left(t/\log\frac{1}{t}\right) \tag{1.6}$$

*then the Fourier Series* (1.2) *associated with the function* $\phi(t)$ *is summable by Nörlund means i.e. summable* $(N, p_n)$ *to the sum zero at the point* $t = x$.

The object here is to prove the following:

THEOREM. *If the sequence* $\{p_n\}$ *satisfies the following conditions*

Received January 3, 1961.

$$\frac{n\,|\,p_n\,|}{(\log n)^r} < c\,|\,P_n\,| \tag{1.7}$$

*and*

$$\sum_{k=0}^{n} \frac{k\,|\,p_k - p_{k-1}\,|}{\{\log (k+1)\}^r} < c\,|\,P_n\,| \tag{1.8}$$

*and also if*

$$\bar{\Phi}_1(t) = \int_0^t |\,\phi(u)\,|\,du = 0\left\{t\Big/\left(\log\frac{1}{t}\right)^r\right\} \tag{1.9}$$

*and*

$$\frac{1}{P(n)}\int_{\pi/n}^{\delta} \frac{|\,\phi(t+\pi/n) - \phi(t)\,|}{t} P\left(\frac{1}{t}\right)dt = 0(1) \tag{1.10}$$

*then the Fourier Series* (1.2) *associated with the function* $\phi(t)$ *is summable by Nörlund means i.e. summable* $(N, p_n)$ *to the sum zero at the point* $t = x$ *for all* $0 \leqq r \leqq 1$.

2. The following notations will be used in the sequel.

We write $S_n(x)$ as the $n$th partial sum of the series (1.2) and the Nörlund transform of the partial sum of the series (1.2) we denote by $\sigma_n(x)$.

Also we write ; where $P_n \equiv P(n)$ ,

$$N_n(t) = \frac{1}{\pi P_n}\sum_{k=0}^{n} p_{n-k}\frac{\text{Sin}\,(k+1/2)t}{t}\,. \tag{2.1}$$

We recall that the conditions of regularity of the method of summation are

$$\sum_{k=0}^{n} r_k = \sum_{k=0}^{n} |\,p_k\,| < c\,|\,P_n\,| \tag{2.2}$$

and

$$\{p_n/P_n\} \to 0 \quad \text{as} \quad n \to \infty\,. \tag{2.3}$$

3. If we write

$$S_n(x) = \frac{1}{\pi}\int_0^{\pi} \phi(t)\frac{\text{Sin}\,(n+1/2)t}{t}\,dt \tag{3.1}$$

then we have

$$\begin{aligned}\sigma_n(x) &= \frac{1}{\pi P(n)}\int_0^\pi \phi(t)\Big(\sum_{k=0}^{n} p_{n-k}\frac{\mathrm{Sin}\,(k+1/2)t}{t}\Big)dt\\ &= \int_0^\pi \phi(t)N_n(t)dt\\ &= \Big(\int_0^{\pi/n}+\int_{\pi/n}^{\delta}+\int_\delta^\pi\Big)\phi(t)N_n(t)dt\\ &= I_1+I_2+I_3\,,\quad \text{say} \qquad (3.2)\end{aligned}$$

where $\delta$ is fixed.

Hence

$$\begin{aligned}I_2 &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\sum_{k=0}^{n}(p_k\,\mathrm{Sin}\,(n-k+\tfrac{1}{2})t)dt\\ &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\sum_{k=0}^{n}p_k\{\mathrm{Sin}\,(n+\tfrac{1}{2})t\cdot\cos kt-\cos(n+\tfrac{1}{2})t\cdot\mathrm{Sin}\,kt\}dt\\ &= I_{2,1}-I_{2,2}\quad\text{say.} \qquad (3.3)\end{aligned}$$

Now, if we write

$$\begin{aligned}I_{2,1} &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\,\mathrm{Sin}\,(n+\tfrac{1}{2})t\cdot\Big\{\sum_{kt\le 1}+\sum_{kt>1}\Big\}p_k\cos kt\cdot dt\\ &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\,\mathrm{Sin}\,(n+\tfrac{1}{2})t\sum_{kt\le 1}p_k\cos ktdt\\ &\quad+\frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\cdot\mathrm{Sin}\,(n+\tfrac{1}{2})t\sum_{kt>1}p_k\cos kt\cdot dt \qquad (3.4)\\ &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\,\mathrm{Sin}\,(n+\tfrac{1}{2})t\Sigma_1 dt\\ &\quad+\frac{1}{\pi P(n)}\int_{\pi/n}^{\delta}\frac{\phi(t)}{t}\,\mathrm{Sin}\,(n+\tfrac{1}{2})t\Sigma_2 dt\\ &= I_{2,1,1}+I_{2,1,2}\,,\quad\text{say.} \qquad (3.5)\end{aligned}$$

**4.** We shall require the following lemma.

LEMMA. *If we write*

$$|p_n| = r_n\,,\qquad R_n = r_0+r_1+r_2+\cdots+r_n \qquad (4.1)$$

*and*

$$r(u) = r_{[u]}\,,\qquad R(u) = R_{[u]} \qquad (4.2)$$

*where* $[u]$ *denotes the integer (largest)* $\le u$, *and*

$$V_0\equiv 0\,,\qquad V_n\equiv\sum_{k=0}^{n}|p_k-p_{k-1}|\,,\qquad V_u\equiv V_{[u]} \qquad (4.3)$$

*then we have, from* (3.4) ,

$$\Sigma_1 > P_t \cos 1 > \tfrac{1}{2} P(\tfrac{1}{t}) \tag{4.4}$$

*and*

$$|\Sigma_2| = \frac{A}{t}\left\{r\left(\frac{1}{t}\right) + r(n) + V(n) - V\left(\frac{1}{t} - 1\right)\right\} . \tag{4.5}$$

This is known Hille and Tamarkin [3].

5. Now we shall prove the theorem.

*Proof.* Since

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{\pi/n} \phi(t) N_n(t)dt \\ &= \frac{1}{\pi}\int_0^{\pi/n} |\phi(t)| O(n)dt \\ &= O(n)[\bar{\Phi}_1(t)]_0^{\pi/n} \qquad \text{by (1.9)} \\ &= O(n)\left[0\left\{t\Big/\left(\log\frac{1}{t}\right)^r\right\}\right]_0^{\pi/n} \\ &= 0(1) , \qquad \text{as } n \to \infty . \end{aligned} \tag{5.1}$$

From (3.5) and Lemma, above, we have

$$\begin{aligned} I_{2,1,1} &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n + \tfrac{1}{2})tP\left(\frac{1}{t}\right)dt \\ &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt + o(1) , \end{aligned}$$

by the regularity of the method of summation.

$$\begin{aligned} &= -\frac{1}{\pi P(n)}\int_0^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} P\left(\frac{1}{t+\pi/n}\right) \operatorname{Sin} nt\, dt + o(1) \\ &= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt - \frac{1}{2\pi P(n)}\int_0^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \operatorname{Sin} nt \\ &\quad \cdot P\left(\frac{1}{t+\pi/n}\right)dt + o(1) \\ &= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt + \frac{1}{2\pi P(n)}\int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt \\ &\quad \cdot P\left(\frac{1}{t}\right)dt - \frac{1}{2\pi P(n)}\int_0^{\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right)dt \\ &\quad - \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right)dt + o(1) \end{aligned}$$

$$= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \left\{ \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n} P\left(\frac{1}{t+\pi/n}\right) \right\} \operatorname{Sin} nt\, dt$$
$$- \frac{1}{2\pi P(n)} \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right) dt$$
$$+ \frac{1}{2\pi P(n)} \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) \operatorname{Sin} nt\, dt + o(1)$$
$$= \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \left[ \left\{ \frac{\phi(t)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t}\right) \right\} \right.$$
$$+ \left\{ \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t+\pi/n}\right) \right\}$$
$$\left. + \left\{ \frac{\phi(t+\pi/n)}{t} P\left(\frac{1}{t+\pi/n}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n} P\left(\frac{1}{t+\pi/n}\right) \right\} \right] \operatorname{Sin} nt\, dt$$
$$- \frac{1}{2\pi P(n)} \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right) dt$$
$$+ \frac{1}{2\pi P(n)} \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right) dt + o(1)$$
$$= \frac{1}{2\pi P(n)} \left( \int_{\pi/n}^{\delta-\pi/n} \left\{ \frac{[\phi(t) - \phi(t+\pi/n)]}{t} P\left(\frac{1}{t}\right) dt \right. \right.$$
$$+ \left[ P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right) \right] \frac{\phi(t+\pi/n)}{t}$$
$$\left. + P\left(\frac{1}{t+\pi/n}\right) \phi(t+\pi/n) \frac{\pi/n}{t(t+\pi/n)} \right\} \operatorname{Sin} nt\, dt$$
$$- \int_0^{\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right) dt$$
$$\left. + \int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right) dt \right)$$
$$+ o(1)$$

$$(5.2) \quad = (P_1 + P_2 + P_3) + P_4 + P_5\,, \quad \text{say.}$$

By virtue of (1.10) we have

$$(5.3) \qquad P_1 = o(1)\,, \quad \text{as} \quad n \to \infty\,.$$

Also

$$P_2 = \frac{1}{2\pi P(n)} \int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{t} \left\{ P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right) \right\} \operatorname{Sin} nt\, dt\,.$$

Since, for all $0 < 1/\alpha < 1/\beta$, we have

$$P\left(\frac{1}{\beta}\right) - P\left(\frac{1}{\alpha}\right) = \int_{1/\alpha}^{1/\beta} p(\beta) ds + O\left\{ p\left(\frac{1}{\alpha}\right) + p\left(\frac{1}{\beta}\right) \right\}.$$

Hence

$$P_2 = O\Big(\frac{1}{P(n)}\Big)\Big\{\int_{\pi/n}^{\delta} |\phi(t+\pi/n)| \frac{dt}{t}\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}$$

$$+ O\Big(\frac{1}{P(n)}\int_{\pi/n}^{\delta} |\phi(t+\pi/n)|\, r\Big(\frac{1}{t}\Big)\frac{dt}{t}\Big)$$

$$+ O\Big(\frac{1}{P(n)}\int_{\pi/n}^{\delta} |\phi(t+\pi/n)|\, r\Big(\frac{1}{t+\pi/n}\Big)\frac{dt}{t}\Big)$$

$$= P_{2,1} + P_{2,2} + P_{2,3} \quad \text{say.} \tag{5.4}$$

Now

$$P_{2,1} = O\Big(\frac{1}{P(n)}\Big)\Big\{\bar{\Phi}_1(t+\pi/n)\frac{1}{t}\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}_{\pi/n}^{\delta}$$

$$+ O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \bar{\Phi}_1(t+\pi/n)\frac{dt}{t^2}\int_{1/(t+\pi/n)}^{1/t} r(s)ds$$

$$+ O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \bar{\Phi}_1(t+\pi/n)\frac{1}{t}\, r\Big(\frac{1}{t}\Big)\frac{dt}{t^2}$$

$$= P_{2,1,1} + P_{2,1,2} + P_{2,1,3}\,, \quad \text{say.} \tag{5.5}$$

We have

$$P_{2,1,1} = O\Big(\frac{1}{P(n)}\Big)\Big\{o\Big(\frac{1}{(\log 1/t)^r}\Big)\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}_{\pi/n}^{\delta} \quad \text{by (1.9)}$$

$$= o\Big(\frac{1}{P(n)}\Big)\Big\{\frac{1}{(\log n)^r}\int_{n/2\pi}^{n/\pi} r(s)ds\Big\}$$

$$+ o\Big(\frac{1}{P(n)}\Big)\Big\{\frac{1}{(\log 1/\delta)^r}\int_{1/(\delta+\pi/n)}^{1/\delta} r(s)ds\Big\}$$

$$= o(1)\,, \quad \text{as} \quad n\to\infty\,, \quad \text{by (1.8).} \tag{5.6}$$

And

$$P_{2,1,2} = O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} o\Big(\frac{t}{(\log 1/t)^r}\Big)\frac{dt}{t^2}\int_{1/(t+\pi/n)}^{1/t} r(s)ds \quad (1.9)$$

$$= o\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \frac{dt}{t(\log 1/t)^r}\int_{1/(t+\pi/n)}^{1/t} r(s)ds$$

$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} r(s)ds\int_{(1/s)-\pi/n}^{1/s} \frac{dt}{t(\log 1/t)^r} + o(1)\,,$$

by change of order of integration.

$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s)ds}{(\log 1/s)^r}\int_{(1/s)-\pi/n)}^{1/s} \frac{dt}{t} + o(1)$$

$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s)ds}{(\log 1/s)^r}(s\,\pi/n) + o(1)$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n}\frac{r(k)}{(\log(k+1))^r} + o(1)$$

$$= o(1)\,, \quad \text{as} \quad n\to\infty\,, \quad \text{by (1.8).} \tag{5.7}$$

Finally, considering $P_{2,1,3}$, we have

$$P_{2,1,3} = O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right)\frac{dt}{t}\, r\left(\frac{1}{t}\right)\frac{1}{t^2}\,, \qquad \text{by (1.9)}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta}\frac{r(1/t)}{(\log 1/t)^r}\frac{dt}{t^2}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{n/\pi}^{1/\delta}\frac{r(s)}{(\log s)^r}ds$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n} r(k)/\{\log(k+1)\}^r$$

$$= o(1) \quad \text{as} \quad n\to\infty\,, \quad \text{by (1.8).} \tag{5.8}$$

Thus from (5.5), (5.6), (5.7) and (5.8) we see that

$$P_{2,1} = o(1) \quad \text{as} \quad n\to\infty\,. \tag{5.9}$$

Estimating $P_{2,2}$ we find that

$$P_{2,2} = O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta}|\phi(t+\pi/n)|\, r\left(\frac{1}{t}\right)dt$$

$$= O\left(\frac{1}{P(n)}\right)\left\{\left[\bar{\Phi}_1(t) r\left(\frac{1}{t}\right)\frac{1}{t}\right]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta}\bar{\Phi}_1(t) r\left(\frac{1}{t}\right)\frac{dt}{t^2}\right.$$

$$\left. - \int_{\pi/n}^{\delta}\bar{\Phi}_1(t)\frac{1}{t}\, dr\left(\frac{1}{t}\right)\right\} + o(1)\,.$$

Here, the integrated part is $o(1)$, by virtue of (1.7) and the fact that $P(n)\to\infty$ as $n\to\infty$. The second part is

$$o\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{1}{(\log 1/t)^r}\right) r\left(\frac{1}{t}\right)\frac{dt}{t}\,, \qquad \text{by (1.9)}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{n/\pi}^{1/\delta}\frac{r(s)}{s(\log s)^r}ds$$

$$= o(1)\,, \quad \text{by (1.8).}$$

The third term is

$$O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right)\frac{1}{t}\, dr\left(\frac{1}{t}\right)$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n}\frac{|r_k - r_{k-1}|}{(\log(k+1))^r}$$

$$= o(1) \quad \text{as} \quad n\to\infty\,, \quad \text{by (1.8).}$$

Thus we see that

(5.10) $$P_{2,2} = o(1) \quad \text{as} \quad n \to \infty .$$

Similarly, we can show that

(5.11) $$P_{2,3} = o(1) \quad \text{as} \quad n \to \infty .$$

Hence by (5.4), (5.9), (5.10) and (5.11) we get

(5.12) $$P_2 = o(1) \quad \text{as} \quad n \to \infty .$$

Evaluating $P_3$ we have

$$\begin{aligned} P_3 &= O\Big(\frac{\pi}{nP(n)}\Big)\int_{\pi/n}^{\delta} |\phi(t+\pi/n)|\, P\Big(\frac{1}{t+\pi/n}\Big)\frac{dt}{t(t+\pi/n)} \\ &= O\Big(\frac{1}{n}\Big)\int_{\pi/n}^{\delta} |\phi(t+\pi/n)|\frac{dt}{t^2} \\ &= O\Big(\frac{1}{n}\Big)\Big[\Big\{\bar{\Phi}_1(t+\pi/n)\frac{1}{t^2}\Big\}_{\pi/n}^{\delta} + 2\int_{\pi/n}^{\delta} \bar{\Phi}_1(t+\pi/n)\frac{dt}{t^3}\Big] \\ &= o\Big(\frac{1}{n}\Big) + o\Big(\frac{1}{(\log n)^r}\Big) \quad \text{by (1.9)} \end{aligned}$$

(5.13) $$= o(1) , \quad \text{as} \quad n \to \infty .$$

And

$$\begin{aligned} P_4 &= \frac{1}{2\pi P(n)}\int_0^{\pi/n} \phi(t+\pi/n)\, \text{Sin}\, nt\, P\Big(\frac{1}{t+\pi/n}\Big)\frac{dt}{t+\pi/n} \\ &= -\frac{1}{2\pi P(n)}\int_{\pi/n}^{2\pi/n} \phi(t)\, \text{Sin}\, nt\, P\Big(\frac{1}{t}\Big)\frac{dt}{t} \\ &= O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{2\pi/n} |\phi(t)|\, O(nt) P\Big(\frac{1}{t}\Big)\frac{dt}{t} \\ &= O(n)\int_{\pi/n}^{2\pi/n} |\phi(t)|\, dt \\ &= o\Big(\frac{1}{(\log n)^r}\Big), \quad \text{by (1.9)} \end{aligned}$$

(5.14) $$= o(1) \quad \text{as} \quad n \to \infty.$$

Also

$$P_5 = O\Big(\frac{1}{P(n)}\Big)\int_{\delta-\pi/n}^{\delta} |\phi(t)|\, P\Big(\frac{1}{t}\Big)\frac{dt}{t}$$

(5.15) $$= o(1) ,$$

by the regularity of the method of summation and since the interval $(\delta - \pi/n, \delta)$ tends to zero as $n \to \infty$.

Consequently from (5.2), (5.3), (5.12), (5.13), (5.14) and (5.15) we have

$$I_{2,1,1} = o(1) , \quad \text{as} \quad n \to \infty . \tag{5.16}$$

Now

$$\begin{aligned} I_{2,1,2} &= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t\Sigma_2 dt \\ &= O\Big(\frac{1}{P(n)}\Big) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t} \Big\{\frac{r(1/t)}{t} + \frac{r(n)}{t} \\ &\quad + \frac{1}{t}\Big[V(n) - V\Big(\frac{1}{t} - 1\Big)\Big]\Big\} dt , \quad \text{by (4.5)} \\ &= Q_1 + Q_2 + Q_3 , \quad \text{say.} \end{aligned} \tag{5.17}$$

We have

$$\begin{aligned} Q_1 &= O\Big(\frac{1}{P(n)}\Big) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2} r\Big(\frac{1}{t}\Big) dt \\ &= O\Big(\frac{1}{P(n)}\Big)\Big\{\Big[o\Big(\frac{t}{\{\log 1/t\}^r}\Big) r\Big(\frac{1}{t}\Big)\frac{1}{t^2}\Big]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} o\Big(\frac{t}{\{\log 1/t\}^r}\Big) r\Big(\frac{1}{t}\Big)\frac{dt}{t^3} \\ &\quad + \int_{\pi/n}^{\delta} o\Big(\frac{t}{\{\log 1/t\}^r}\Big)\frac{1}{t^2} dr\Big(\frac{1}{t}\Big)\Big\} \quad \text{by (1.9).} \end{aligned}$$

Here the integrated part is $o(1)$, by (1.7) and the fact that $P(n) \to \infty$ as $n \to \infty$. Also the second term is

$$\begin{aligned} & o\Big(\frac{1}{P(n)}\Big) \int_{\pi/n}^{\delta} r\Big(\frac{1}{t}\Big) \frac{dt}{t^2} \frac{1}{(\log 1/t)^r} \\ &= o\Big(\frac{1}{P(n)}\Big) \int_{n/\pi}^{1/\delta} \frac{r(s)}{(\log s)^r} ds \\ &= o\Big(\frac{1}{P(n)}\Big) \sum_{k=0}^{n} \frac{k\,|\,p_k - p_{k-1}\,|}{(\log (k+1))^r} \\ &= o(1) \quad \text{by (1.8).} \end{aligned}$$

The third part is

$$\begin{aligned} & o\Big(\frac{1}{P(n)}\Big) \int_{n/\pi}^{1/\delta} \frac{s dr(s)}{(\log s)^r} \\ &= o\Big(\frac{1}{P(n)}\Big) \sum_{k=0}^{n} \frac{k\,|\,p_k - p_{k-1}\,|}{(\log (k+1))^r} \\ &= o(1) \quad \text{as} \quad n \to \infty , \quad \text{by (1.8).} \end{aligned}$$

Thus we see that

$$Q_1 = o(1) \quad \text{as} \quad n \to \infty . \tag{5.18}$$

Now

$$Q_2 = O\left(\frac{r(n)}{P(n)}\right)\int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2}\,dt$$
$$= O\left(\frac{r(n)}{P(n)}\right)\left\{\left(\frac{\bar{\Phi}_1(t)}{t^2}\right)_{\pi/n}^{\delta} + 2\int_{\pi/n}^{\delta}\bar{\Phi}_1(t)\frac{dt}{t^3}\right\}$$
$$= o\left(\frac{r(n)}{P(n)}\right) + o\left(\frac{nr(n)}{P(n)(\log n)^r}\right)$$
$$+ o\left(\frac{r(n)}{P(n)}\right)\int_{\pi/n}^{\delta}\frac{dt}{t^2(\log 1/t)^r}$$
$$= o\left\{\frac{(\log n)^r}{n}\right\} + o(1) + o\left\{\frac{r(n)n}{P(n)(\log n)^r}\right\}$$
$$= o(1) + o\left(\frac{1}{n}\right)$$
$$= o(1) \quad \text{as} \quad n \to \infty\,. \tag{5.19}$$

Lastly

$$Q_3 = O\left(\frac{1}{P(n)}\int_{\pi/n}^{\delta}\frac{|\phi(t)|}{t^2}\left[V(n) - V\left(\frac{1}{t} - 1\right)\right]dt\right.$$
$$= O\left(\frac{1}{P(n)}\right)\left\{\left[\frac{\bar{\Phi}_1(t)}{t^2}\left(V(n) - V\left(\frac{1}{t} - 1\right)\right)\right]_{\pi/n}^{\delta}\right.$$
$$+ 2\int_{\pi/n}^{\delta}\frac{\bar{\Phi}_1(t)}{t^3}\left[V(n) - V\left(\frac{1}{t} - 1\right)\right]dt$$
$$\left. - \int_{\pi/n}^{\delta}\frac{\bar{\Phi}_1(t)}{t^2}\,dV\left(\frac{1}{t} - 1\right)\right\}.$$

The integrated part is $o(1)$, by (1.7) and the fact that

$$W_n \equiv \sum_{k=0}^{n}\frac{k\,|p_k - p_{k-1}|}{(\log(k+1))^r}\,; \qquad W_0 \equiv 0\,.$$

Then, by (1.8) we have

$$V_n = \sum_{k=0}^{n}|p_k - p_{k-1}|$$
$$= \sum_{k=0}^{n}\frac{\{\log(k+1)\}^r}{k}(W_k - W_{k-1})$$
$$= \sum_{k=0}^{n-1}W_k\left\{\Delta\left[\frac{\{\log(k+1\}^r}{k}\right]\right\} + \frac{W_n\{\log(n+1)\}}{n}$$
$$= o\{R(n)\}\,.$$

Now the second term is

$$o\left[\frac{1}{P(n)}\int_{\pi/n}^{\delta}\left\{V(n)-V\left(\frac{1}{t}-1\right)\right\}\frac{dt}{t^2(\log 1/t)^r}\right]$$
$$=o\left[\frac{1}{P(n)}\int_{1/\delta}^{n/\pi}\frac{ds}{(\log s)^r}[V(n)-V(s-1)]\right]$$
$$=o\left\{\frac{1}{P(n)}\int_0^n\frac{s}{\{\log(s+1)\}^r}dV(s)\right\}$$
$$+o\left[\frac{1}{R(n)}\{V(n)-V(s-1)\}\frac{s}{\{\log(s+1)\}^r}\right]_0^n$$
$$=o\left\{\frac{1}{P(n)}\sum_{k=0}^{n}\frac{k\,|\,p_k-p_{k-1}\,|}{\{\log(k+1)\}^r}\right\}+o\left\{\frac{1}{P(n)}\frac{n\,|\,p_n-p_{n-1}\,|}{\{\log(n+1)\}^r}\right\}$$

which is $o(1)$, by virtue of (1.7), (1.8) and the fact that $V_n=o\{P(n)\}$.

The third term is

$$o\left\{\frac{1}{P(n)}\int_{\pi/n}^{\delta}\frac{1}{t(\log 1/t)^r}\left|dV\left(\frac{1}{t}-1\right)\right|\right\},\quad \text{by (1.9)}$$
$$=o\left\{\frac{1}{P(n)}\right\}\int_0^n\frac{s\,|\,dV(s-1)\,|}{\{\log(s+1)\}^r}$$

which is $o(1)$, as in the case of second term.

Thus we have

(5.20) $$Q_3=o(1)\quad\text{as}\quad n\to\infty\,.$$

From (5.17), (5.18), (5.19) and (5.20), we have

(5.21) $$I_{2,1,2}=o(1)\quad\text{as}\quad n\to\infty\,.$$

From (3.4) (3.5), (3.16) and (5.21), we see that

(5.22) $$I_{2,1}=o(1)\quad\text{as}\quad n\to\infty\,.$$

Similarly, we can show that

(5.23) $$I_{2,2}=o(1)\quad\text{as}\quad n\to\infty\,.$$

From (3.3), (5.22) and (5.23) we get

(5.24) $$I_2=o(1)\quad\text{as}\quad n\to\infty\,.$$

Lastly by Riemann Lebesgue Theorem and the regularity of the method of summation, we have, as $n\to\infty$

(5.25) $$I_3=o(1)\,.$$

Collection of (3.2), (5.24) and (5.25) as $n\to\infty$, completes the proof of the theorem.

I am much indebted to Professor M. L. Misra for his valuable guidance during the preparation of this paper.

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