ANALYTIC METHODS IN THE STUDY OF ZEROS OF POLYNOMIALS

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Several analytic methods are used to obtain estimates for part or all zeros of a polynomial with complex coefficients and for linear combinations of polynomials. Some results of Biernacki, Montel and Specht are strengthened or generalized. Some results about the location of zeros of linear combinations of polynomials are also obtained.

1. Cauchy type estimates.

THEOREM 1. *Let* $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ *be a polynomial with complex coefficients. Let* $\beta'_2 \geqq \beta'_3 \geqq \cdots \geqq \beta'_n$ *be the ordered positive numbers* $|b_i| = |a_i \gamma^{-i}|, \gamma > 0, i = 2, \cdots, n,$ *then all the zeros of the polynomial* $P(z)$ *are in the union of the two circles*:

$$|z| < \gamma(1 + \sigma_1) \text{ and } |z + a_1| \leqq \gamma$$

where

$$\sigma_1 = \beta'_2 - \frac{\delta'_2}{1 + \beta'_2} - \frac{\delta'_3}{(1 + \beta'_2)^2} - \cdots - \frac{\delta'_n}{(1 + \beta'_2)^{n-1}}$$

with

$$\delta'_i = \beta'_i - \beta'_{i+1}, \beta'_{n+1} = 0 .$$

Proof. It is well known (See e.g. [4]), that all zeros of the polynomial $P(z)$ are in the union of the two circles $|z + a_1| \leqq \gamma$ and $|z| \leqq \gamma(1 + \beta'_2)$. Let ζ be a zero of the polynomial $P(z)$. We may assume that $|\zeta| = \gamma r$, where $1 < r \leqq 1 + \beta'_2$. The inequality

$$|\zeta^n + a_1\zeta^{n-1}| \leqq |a_2||\zeta|^{n-2} + \cdots + |a_n|$$

yields

$$(1) \quad r^{n-1}|\zeta + a_1| \leqq \gamma(|b_2|r^{n-2} + \cdots + |b_n|) \leqq \gamma(\beta'_2 r^{n-2} + \cdots + \beta'_n) ,$$

since β'_i, $i = 2, \cdots, n,$ are decreasing and $r > 1$. Multiplying both sides of the inequality (1) by $(r - 1)r^{-(n-1)}$ we get

$$(r - 1)|\zeta + a_1| \leqq \gamma\left(\beta'_2 - \frac{\delta'_2}{r} - \cdots - \frac{\delta'_n}{r^{n-1}}\right) .$$

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Since $r \leqq 1 + \beta_2'$ and $\delta_i' \geqq 0$,

$$(r-1)\,|\,\zeta + a_1\,| \leqq \gamma\Big(\beta_2' - \frac{\delta_2'}{1+\beta_2'} - \cdots - \frac{\delta_n'}{(1+\beta_2')^{n-1}}\Big)\,.$$

Suppose $|\,\zeta + a_1\,| > \gamma$, then

$$r < 1 + \beta_2' - \frac{\delta_2'}{1+\beta_2'} - \cdots - \frac{\delta_n'}{(1+\beta_2')^{n-1}}$$

and

$$|\,\zeta\,| = r\gamma < \gamma + \gamma\Big(\beta_2' - \frac{\delta_2'}{1+\beta_2'} - \cdots - \frac{\delta_n'}{(1+\beta_2')^{n-1}}\Big)\,. \tag{2}$$

Consequently all the zeros of the polynomial $P(z)$ which are outside the circle $|\,\zeta + a_1\,| \leqq \gamma$ lie inside the circle (2).

One notes that this result can be repeatedly improved replacing β_2' by $\sigma_1 \leqq \beta_2'$ in the proof of Theorem 1. The last result improves the known estimate used. As an immediate consequence from Theorem 1 we obtain that all the zeros of the polynomial $P(z)$ are in the region

$$[|\,z\,| < \gamma(1+\sigma)] \cap ([|\,z + a_1\,| \leqq \gamma] \cup [|\,z\,| < \gamma(1+\sigma_1)])\,.$$

(See remark following the proof of theorem 2).

THEOREM 2. *Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, be a polynomial with complex coefficients. Let $\gamma > 0$, $b_i = a_i\gamma^{-i}$, $i = 1, \cdots, n$. Assume furthermore that β_i, $i = 1, \cdots, n$, $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n$, are the ordered numbers $|\,b_i\,|$.*

Define

$$0 \leqq \alpha = \max_{2 \leqq i \leqq n} \frac{\beta_i}{\beta_{i-1}} \leqq 1\,,$$

where the maximum is taken over all i such that $\beta_{i-1} \neq 0$. Denote

$$\sigma_2 = \beta_1 - \frac{\delta_1''}{1+\beta_1} - \frac{\delta_2''}{(1+\beta_1)^2} - \cdots - \frac{\delta_n''}{(1+\beta_1)^n} \tag{3}$$

where

$$\delta_i'' = \alpha\beta_i - \beta_{i+1} \geqq 0,\ i = 1, \cdots, n,\ \beta_{n+1} = 0\,;$$

then all the zeros of the polynomial $P(z)$ are in the circle

$$|\,z\,| \leqq \max\,(\gamma(\alpha + \sigma_2), \gamma)\,.$$

Proof. Let ζ be a zero of $P(z)$. We may assume that

$$\gamma(\alpha + \sigma_2) \leqq |\zeta| = \gamma r < \gamma(1 + \beta_1)\,, \qquad r \geqq 1\,. \tag{4}$$

The equality $\zeta^n = -a_1\zeta^{n-1} - \cdots - a_n$ implies

$$\gamma^n r^n \leqq |a_1|\gamma^{n-1}r^{n-1} + \cdots + |a_n| = \gamma^n(|b_1|r^{n-1} + \cdots + |b_n|)\,.$$

Hence,

$$r^n \leqq \beta_1 r^{n-1} + \beta_2 r^{n-2} + \cdots + \beta_n\,.$$

Also since $r \geqq \alpha$, we have

$$\begin{aligned} r^n(r - \alpha) &\leqq \beta_1 r^n - r^{n-1}(\alpha\beta_1 - \beta_2) - \cdots - r^2(\alpha\beta_{n-2} - \beta_{n-1}) \\ &- r(\alpha\beta_{n-1} - \beta_n) - \alpha\beta_n = \beta_1 r^n - \delta_1'' r^{n-1} - \cdots - \delta_{n-1}'' - \delta_n''\,. \end{aligned} \tag{5}$$

Since $\alpha + \sigma_2 \leqq r$, (5) implies

$$\frac{\delta_1''}{r} + \cdots + \frac{\delta_n''}{r^n} \leqq \beta_1 - r + \alpha \leqq \beta_1 - \sigma_2\,. \tag{6}$$

Taking into account (3) and (6) we get

$$\frac{\delta_1''}{r} + \cdots + \frac{\delta_n''}{r^n} \leqq \frac{\delta_1''}{1 + \beta_1} + \cdots + \frac{\delta_n''}{(1 + \beta_1)^n}\,.$$

Since $\delta_i'' \geqq 0$ it follows now that $r \geqq 1 + \beta_1$ which contradicts the assumption made in (4). Hence $|\zeta| \leqq \gamma(1 + \sigma_2)$. If however, $r < 1$ then $|\zeta| < \gamma$. Theorem 2 strengthens a result due to Specht [6], $|z| < \gamma(1 + \sigma)$, where

$$\sigma = \beta_1\left(\frac{\beta_1}{1 + \beta_1} + \frac{\beta_2}{(1 + \beta_1)^2} + \cdots + \frac{\beta_n}{(1 + \beta_1)^n}\right)\,.$$

One verifies easily that $\alpha + \sigma_2 \leqq 1 + \sigma$.

2. Estimates of at least p zeros ($1 \leqq p \leqq n$) for a polynomial of degree n. It is known, [2] p. 110, that is the coefficients $a_0, a_1, \cdots, a_{p-1}, a_{p+h}$ are fixed, then p zeros of the polynomial

$$Q(z) = a_n z^n + \cdots + a_0 \tag{7}$$

are bounded. Various bounds for at least p zeros of $Q(z)$, as functions of these coefficients, were obtained by different authors ([2] Chap. VIII).

LEMMA 1. (Montel) [2] p. 111–112. *Let the polynomial $Q(z)$, defined in (7), have zeros $z_1, \cdots, z_n$, such that $|z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|$. Let*

$$Q_p(z) = \frac{Q(z)}{(z_1 - z) \cdots (z_{n-p} - z)} = \sum_{j=0}^{p} a_j^{(p)} z^j .$$

Define $r_p = |z_{n-p+1}|$. Then

(8) $$|a_k^{(p)}| \leqq r_p^{-(n-p)} \sum_{j=0}^{k} C(n-p+j-1, j) |a_{k-j}| r_p^{-j} , \quad k = 0, 1, \cdots, p .$$

THEOREM 3. *Let $Q(z)$ be a complex polynomial of degree n defined in (7), $a_n \neq 0, 1 \leqq p \leqq n$. Let $r(p)$ be the greatest positive root of the equation*

(9) $$(z - \rho)(z - 1)^{n-p} - S(p, q', \rho) = 0$$

where $0 < \rho \leqq 1$ and

$$S(p, q', \rho) = \max_{0 \leqq k \leqq p-1} \left(\sum_{j=0}^{k} \left(\left| \frac{a_j}{a_n} \right|^{q'} \right)^{1/q'} \rho^{-(p-k-1)} \right), \; q' \geqq 1 .$$

Then at least p zeros of the polynomial $Q(z)$ are in the circle

(10) $$|z| \leqq r(p) .$$

Proof. We apply the Holder inequality to (8) and after some simple transformations we obtain

(11) $$|a_k^{(p)}| \leqq r_p^{-(n-p)} \left[\sum_{j=0}^{k} (C(n-p+j-1, j) r_p^{-j})^{p'} \right]^{1/p'} \left[\sum_{j=0}^{k} |a_{k-j}|^{q'} \right]^{1/q'}$$
$$\leqq r_p^{-(n-p)} \sum_{j=0}^{k} C(n-p+j-1, j) r_p^{-j} \left[\sum_{j=0}^{k} |a_{k-j}|^{q'} \right]^{1/q'},$$
$$1/q' + 1/q' = 1 .$$

Since $r(p) \geqq 1$, without loss of generality we may assume that $r_p > 1$. We replace the first sum in the right hand side of (11) by

(12) $$\sum_{j=0}^{\infty} C(n-p+j-1, j) r_p^{-j} = \left(1 - \frac{1}{r_p} \right)^{-(n-p)} .$$

By (11) and (12) we get

(13) $$|a_k^{(p)}| \leqq (r_p - 1)^{-(n-p)} \left[\sum_{j=0}^{k} |a_j|^{q'} \right]^{1/q'} .$$

It is known (See e.g. [4]) that all the zeros of $Q_p(z)$, and in particular z_{n-p+1}, satisfy the inequality

(14) $$r_p = |z_{n-p+1}| \leqq \max_{0 \leqq k \leqq p-1} \left(\left| \frac{a_k^{(p)}}{a_p^{(p)}} \right| \frac{x_1}{x_{p-k}} + \frac{x_{p-k+1}}{x_{p-k}} \right)$$

for any $x_i > 0, i = 1, \cdots, p, x_{p+1} = 0$. Taking into account that $a_p^{(p)}$

$= (-1)^{n-p}a_n$ and using (13) and (14), with $x_i = \rho^i, i = 1, \cdots, p, 0 < \rho \leqq 1$, we obtain

$$r_p \leqq \max_{0 \leqq k \leqq p-1}\left[(r_p - 1)^{-(n-p)}\left(\sum_{j=0}^{k}\left|\frac{a_j}{a_n}\right|^{q'}\right)^{1/q'}\rho^{-(p-k-1)} + \rho\right]. \tag{15}$$

Using the definition of $S(p, q', \rho)$, the equivalent to (15) is

$$(r_p - \rho)(r_p - 1)^{n-p} \leqq S(p, q', \rho) .$$

It follows now easily that $r_p \leqq r(p)$.

COROLLARY 1. *Theorem 3 includes, as particular cases, a result due to Marden* ([2] p. 113) *for* $\rho = 1$, *and a result due to Montel* ([2] Th. 32,1) *for* $q' \rightarrow \infty, \rho = 1$.

Proceeding as in the proof of Theorem 3, using this time the estimate

$$\left|\frac{a_k^{(p)}}{a_p^{(p)}}\right| < \frac{N_p r_p^{n-p}}{(1 + N_p)(r_p - 1)^{n-p} - N_p r_p^{n-p}}$$

for all k, $r_p > 1$ and

$$N_p = \max_{0 \leqq j \leqq p-1}\left|\frac{a_j}{a_p}\right| ,$$

due to Montel (See [2] p. 115), one obtains:

THEOREM 4. *Let $Q(z)$ be the polynomial defined in* (7). *At least p zeros of $Q(z)$ are in the circle $|z| \leqq r_1(p)$, where $r_1(p)$ is the positive root of the equation*

$$\left(1 + \frac{1}{N_p}\right)\frac{z - \rho}{z + \rho^{1-p} - \rho}\left(1 - \frac{1}{z}\right)^{n-p} - 1 = 0 .$$

For $\rho = 1$, Theorem 4 gives an estimate due to Montel (See [2] p. 115). *We remark that by a minimum argument it follows, that for $(p - 1)N_p \leqq 1$ and $\rho = [(p - 1)N_p]^{1/p}$, Theorem 4 yields results better than those obtained by the classical formula.*

Using estimates which involve a number of arbitrary parameters we obtain bounds for at least p zeros for lacunary polynomials. We quote first a lemma.

LEMMA 2. *At least p zeros, $1 \leqq p \leqq n$, of the polynomial $Q(z) = a_n z^n + \cdots + a_0, a_n \neq 0$, lie in or on the circle $|z| = \rho'$ where ρ' is the positive root of either of the two equations:*

$$(16)\qquad |a_n|z^n - \sum_{k=0}^{p-1} C(n-k-1, p-k-1)|a_k|z^k = 0$$

and

$$(17)\qquad |a_p|z^p - \sum_{k=0}^{p-1} C(n-k, p-k)|a_k z^k = 0 .$$

(16) *is due to Montel,* (17) *is due to Van Vleck. A simultaneous proof of* (16) *and* (17) *was given by Markovitch* [3]. *We prove now the following*:

THEOREM 5. *Let* $Q(z) = a_n z^n + \cdots + a_0$, $a_p a_n \neq 0, 1 \leqq p \leqq n$. *Let* r *and* s *be two numbers having the properties*: $r \geqq n - p + 1$ *is the smallest number such that* $a_{n-r} \neq 0$; s, $1 \leqq s \leqq p$, *is the smallest number such that* $a_{p-s} \neq 0$. *Then at least* p *zeros of the polynomial* $Q(z)$ *lie in, either of the two circles*:

$$(18)\qquad |z| \leqq \max\left[\rho^r, \left(\sum_{j=r}^{n} C(j-1, p-n+j-1)\rho^{-(j-r)}\left|\frac{a_{n-j}}{a_n}\right|\right)^{1/r}\right]$$

and

$$(19)\qquad |z| \leqq \max\left[\rho^s, \left(\sum_{j=s}^{p} C(n-p+j, j)\rho^{-(j-s)}\left|\frac{a_{p-j}}{a_p}\right|\right)^{1/s}\right]$$

for any $\rho > 0$.

Proof. Denote $c_k = -C(k-1, p-n+k-1)|a_{n-k}/a_n|$. The left hand side of (16) can be written as

$$(20)\qquad z^n + c_q z^{n-q} + \cdots + c_n, q = n - p + 1 .$$

By our assumption, (20) is equivalent to

$$(21)\qquad z^n + c_r z^{n-r} + \cdots + c_n, c_r \neq 0 .$$

It follows from the result proved in [5], with $x_i = \rho^i$, $i = 1, \cdots, n$, $\rho > 0$, that all the zeros of the polynomial (21) are in the union of the circle and the lemniscate defined by the inequalities

$$|z| \leqq \rho^r$$

and

$$|z^r + c_r| \leqq \sum_{j=r+1}^{n} C(j-1, p-n+j-1)\left|\frac{a_{n-j}}{a_n}\right|\rho^{-(j-r)} .$$

The inequality (18) follows applying (16) and the last result. To prove (19), we define $c_{p-k} = -C(n-k, p-k)|a_k/a_p|$ and proceed as before, using this time equality (17).

COROLLARY 2. *Let* $Q(z) = a_n z^n + \cdots + a_p z^p + a_0$, $a_0 a_p a_n \neq 0$, $1 \leqq p \leqq n$. *At least* p *zeros of* $Q(z)$ *are in or on, either of the two circles:*

$$|z| = \left(C(n-1, p-1)\left|\frac{a_0}{a_n}\right|\right)^{1/n}$$

and

$$|z| = \left(C(n, p)\left|\frac{a_0}{a_p}\right|\right)^{1/p}.$$

These results are obtained from Theorem 5, *with* $r = n$ *and* $r = p$ *respectively. Both results are due to Van Vleck. (See* [2]).

3. Linear combinations of polynomials.

LEMMA 3. *Let the polynomials* $R(z) = z^n + \cdots + a_n$ *and* $S(z) = z^k + \cdots + b_k$, $a_n b_k \neq 0$, *have zeros* z_i, $i = 1, \cdots, n$, *and* ζ_j, $j = 1, \cdots, k$ *respectively.*

Let $F(z; \lambda) = R(z) + \lambda S(z)$ have zeros $\eta_1(\lambda), \cdots, \eta_l(\lambda)$, $|\eta_1| \leqq |\eta_2| \leqq \cdots \leqq |\eta_l|$. If the circle $|z| \leqq \tau$ contains all the zeros of the polynomials $R(z)$ and $S(z)$ and m zeros, $0 \leqq m < l$, of $F(z; \lambda)$, then

$$\prod_{i=m+1}^{l} |\eta_i(\lambda)| \leqq \frac{1}{C(\lambda)} \tau^{[(1+(|\lambda|/\mu))n-(|\lambda|/\mu)k-m]}, \tag{22}$$

where $C(\lambda)$ equals $|1+\lambda|, |\lambda|$, or 1 according to whether $k = n$, k $> n$, or $k < n$ respectively, provided

$$\mu = \min_{0 \leqq \theta \leqq 2\pi} \left|\frac{R(\tau e^{i\theta})}{S(\tau e^{i\theta})}\right| - 1 > 0.$$

Proof. Applying Jensen's formula to the polynomials $R(z)$, $S(z)$ and $F(z; \lambda)$ we obtain (Omitting the parameter λ in the notation for F):

$$\begin{aligned} n \log \tau &= \frac{1}{2\pi}\int_0^{2\pi} \log |R(\tau e^{i\theta})|\, d\theta \\ k \log \tau &= \frac{1}{2\pi}\int_0^{2\pi} \log |S(\tau e^{i\theta})|\, d\theta \end{aligned} \tag{23}$$

$$\log\left(|C(\lambda)| \prod_{i=m+1}^{l} |\eta_i(\lambda)|\right) + m \log \tau = \frac{1}{2\pi}\int_0^{2\pi} \log |F(\tau e^{i\theta})|\, d\theta.$$

Using the formulas (23) and the inequality $\log(\alpha x_1 + \beta x_2) \geqq \alpha \log x_1 + \beta \log x_2$ with $x_1 = |R| + |\lambda S|$, $x_2 = |S|$ and

$$\alpha = \frac{|R| - |S|}{|R| - |S| + |\lambda S|}, \quad \alpha + \beta = 1 ,$$

after a few transformations we deduce the inequalities

$$\int_0^{2\pi} \log|F(z)|\,d\theta \leqq \int_0^{2\pi} \log(|R(z)| + |\lambda S(z)|)d\theta$$
$$\leqq \int_0^{2\pi} \frac{\log|R(\tau e^{i\theta})| - \log|S(\tau e^{i\theta})|}{\alpha(\theta, \lambda)} d\theta + \int_0^{2\pi} \log|S(\tau e^{i\theta})|\,d\theta .$$

On the other hand, by the assumption of the theorem it follows that

$$\frac{1}{\alpha(\theta, \lambda)} \leqq 1 + \frac{|\lambda|}{\mu} .$$

Hence

$$(24) \quad \int_0^{2\pi} \log|F(z)|\,d\theta \leqq \left(1 + \frac{|\lambda|}{\mu}\right)\int_0^{2\pi} \log|R(\tau e^{i\theta})|\,d\theta - \frac{|\lambda|}{\mu}\int_0^{2\pi} \log|S(\tau e^{io})|\,d\theta .$$

Dividing (24) by 2π and substituting the values from (23), we deduce the inequality

$$\log\left(|C(\lambda)| \prod_{i=m+1}^{l} |\eta_i(\lambda)|\right) + m \log\tau \leqq \left(1 + \frac{|\lambda|}{\mu}\right) n \log\tau - \frac{|\lambda|}{\mu} k \log\tau .$$

The desired follows now after simple transformations. We remark that in case $\mu < 0$, Lemma 3 is true interchanging R and S, λ and $1/\lambda$. As a consequence of the lemma we have:

THEOREM 6. *Under the assumptions of Lemma* 3 *all the zeros of the polynomial* $F(z; \lambda)$ *are in the disc*

$$|z| \leqq \frac{1}{|C(\lambda)|} \tau^{[(1+|\lambda|/\mu)n - (|\lambda|/\mu)k - l + 1]} .$$

Proof.

$$\max_{m+1 \leqq i \leqq l} |\eta_i(\lambda)| = \prod_{i=m+1}^{l} |\eta_i(\lambda)| \Big(\prod_{i \neq i_{\max}} |\eta_i(\lambda)|\Big)^{-1}$$
$$\leqq \frac{1}{|C(\lambda)|} \tau^{[(1+|\lambda|/\mu)n - (|\lambda|/\mu)k - l + 1]} ,$$

since $|\eta_i(\lambda)| \geqq \tau$ for $m + 1 \leqq i \leqq l$. Some estimates for the zeros of linear combinations of polynomials can be derived by a continuity argument. We denote by $D(k, n-k)$ the open domain in the extended complex λ-plane, for which the polynomial $F(z; \lambda) = R(z) + \lambda S(z)$ has exactly k zeros with negative real part ($n-k$ zeros with positive real part), where $R(z)$ and $S(z)$ are fixed and $\deg(R + \lambda S) = n$, $k = 0, 1, \cdots, n$. Some of the domains $D(k, n-k)$ may be empty. We quote two results to be used later.

LEMMA 4. (Marden [2] p. 54). *The zeros of the linear combination*

$$f(z) = \lambda_1 f_1(z) + \cdots + \lambda_p f_p(z)$$

where $\lambda_j \neq 0$, $j = 1, 2, \cdots, p$, *lie in the locus* Γ *of the roots of the equation*

$$\lambda_1(z-\alpha_1)^{n_1} + \cdots + \lambda_p(z-\alpha_p)^{n_p} = 0$$

when the $\alpha_1, \cdots, \alpha_p$ vary independently over the circular regions $C_1, \cdots, C_p$ and where C_j contains all the zeros of the polynomial $f_j(z)$. The following result is due to Walsh (see [2] p. 55).

LEMMA 5. *If the points* $\alpha_1, \cdots, \alpha_p$ *vary independently over the closed interiors of the circles* $C_1, \cdots, C_p$ *respectively, then the locus of the point.*

$$\alpha = \sum_{j=1}^{p} m_j \alpha_j$$

where the m_j *are arbitrary complex numbers, will be the closed interior of a circle* C *of center* c *and radius* r, *where*

$$c = \sum_{j=1}^{p} m_j c_j, \quad r = \sum_{j=1}^{p} |m_j| r_j$$

and c_j *and* r_j *denote respectively the center and radius of the circle* C_j. *We prove a preliminary result.*

LEMMA 6. *Let*

$$(25) \qquad f(z, \lambda) = (z+a)^n + \lambda z^p, \ |Re\, a| \geqq 1, \ \arg a = \alpha, \ p \leqq n .$$

Then for all λ, $|\lambda| \leqq |\cos^p \alpha|$, *the polynomial* $f(z, \lambda)$ *has all its zeros in the same right, or left, half plane as the polynomial* $(z+a)^n$.

Proof. By the remarks made following the proof of Theorem 6 it is sufficient to prove that the domain $D(k, n-k)$, $k = 0$ or n, which contains the origin, contains also the circle about the origin with radius $|\cos^p \alpha|$. Substituting $z = iy$ in (25) and solving for λ

in the equation $f(iy, \lambda) = 0$ we get

$$|\lambda| = |iy + a|^{n-p}\left|\frac{iy + a}{y}\right|^{p} \geqq |Re\, a|^{n-p}\left|i + \frac{a}{y}\right|^{p} .$$

Since by elementary geometric considerations $|i + a/y| \geqq |Re\, a|/|a|$, we get $|\lambda| \geqq |Re\, a|^{n}/|a|^{p}$, hence the region D which contains the origin contains also the circle $|\lambda| < \cos^{p}\alpha\, ||Re\, a|^{n-p}$. Taking into account the conditions (25) the desired result follows.

THEOREM 7. *Let $R(z) = z^n + \cdots + a_n$, $S(z) = z^k + \cdots + b_k$, $k \leqq n$. Let the zeros of $R(z)$ and $S(z)$ lie in the discs C_i: $|z - c_i| \leqq r_i$, $i = 1, 2$ respectively, such that one of the following conditions holds*:

$$\text{(a)} \quad Re\,[(c_2 - c_1)] - (r_1 + r_2) \geqq 1$$
$$\text{(b)} \quad Re\,[(c_2 - c_1)] + r_1 + r_2 \leqq -1 .$$

Denote the circle $|z - (c_2 - c_1)| \leqq r_1 + r_2$ by C_3 and let $\min_{z\in C_3}|\cos \arg z| = A$. Then in case (a) *all the zeros of the polynomial $R(z) + \lambda S(z)$ for $|\lambda| \leqq A^k$, are in the region $Re\, z \leqq r_2 + Re\, c_2$, and in case* (b)*—in the region $Re\, z \geqq Re\, c_2 - r_2$.*

Proof. By Lemma 4 all the zeros of the polynomial $R(z) + \lambda S(z)$ are in the locus of the zeros of the polynomial $g(z) = (z - \alpha_1)^n + \lambda(z - \alpha_2)^k$, where α_i, $i = 1, 2$ vary independently in C_i. By Lemma 5, $\alpha_2 - \alpha_1 \in C_3$ for any $\alpha_i \in C_i$. We may apply therefore Lemma 6 to the polynomial $g(\zeta + \alpha_2) = [\zeta + (\alpha_2 - \alpha_1)]^n + \lambda\zeta^k$. It results that in case (a) all the zeros of the polynomial $g(\zeta + \alpha_2)$ are in the region $Re\, \zeta \leqq 0$ for $|\lambda| \leqq A^k$, and $Re\, z = Re\, \zeta + Re\, \alpha_2 \leqq Re\, \alpha_2 \leqq r_2 + Re\, c_2$.

Similarly, in case (b), the zeros of $g(\zeta + \alpha_2)$ are in the region $Re\, \zeta \geqq 0$. It is clear that if $k > n$, similar results can be obtained replacing α_2 by α_1, λ by $1/\lambda$ and R by S.

Combining the last result with a similar result for the imaginary part of the zeros of $R(z) + \lambda S(z)$ we obtain:

COROLLARY 3. *With the notations and assumptions of Theorem 7 suppose that one of the following holds*

$$\text{(a}'\text{)} \quad Re\,[(c_2 - c_1)] - (r_1 + r_2) \geqq 1$$
$$\text{(b}'\text{)} \quad Im\,[(c_2 - c_1)] - (r_1 + r_2) \geqq 1 .$$

Denote

$$A_1 = \min\,(\min_{z\in C_3}|\cos \arg z|, \min_{z\in C_3}|\sin \arg z|) ,$$

Then for $|\lambda| \leqq A_1^k$, *in case* (a′) *all the zeros of the polynomial* $R(z) + \lambda S(z)$ *are in the quadrant* $Re\, z \leqq r_2 + Re\, c_2$, $Im\, z \leqq r_2 + Im\, c_2$, *and in case* (b′) *all the zeros of the above polynomial are in the quadrant* $Re\, z \geqq Re\, c_2 - r_2$, $Im\, z \geqq Im\, c_2 - r_2$.

The estimates based on the continuity argument can be further developed and it is possible to obtain bounded regions for the zeros under suitable restrictions upon the parameter λ.

We prove finally a result concerning the location of at least p zeros of linear combinations of polynomials.

THEOREM 8. *Let* $R(z)$ *and* $S(z)$ *be two polynomials of degree* n *and* k *with zeros* z_i *and* ζ_j *respectively,* $i = 1, \cdots, n, j = 1 \cdots, k$. *Let the numbers* $a, r, r_1, r_2, r_3, n_1, n_2, k_1, k_2$; $n_1 + n_2 = n$, $k_1 + k_2 = k$ *satisfy the following conditions*:

(a) *The polynomial* $R(z)$ *has* n_1 *zeros in the disc* C_1: $|z - a| \leqq r_1 < r$ *and* n_2 *zeros outside the disc* C_1 *which are in the disc* $|z| \leqq r$.

(b) *The polynomial* $S(z)$ *has* k_1 *zeros in the disc* C_2; $|z - a| \leqq r_2 < r$ *which are also in the disc* $|z| \leqq r$ *and* k_2 *zeros outside the disc* C_2 *which are also outside the disc* $|z - a| \geqq r_3 > r$.

Suppose furthermore that one of the following conditions is satisfied:

$$(c_1)\ r < a_0 \text{ and } n_1\frac{r + a_0}{r - r_1} + n_2\frac{r - a_0}{2r + a_0} \leqq k_1\frac{r - a_0}{r} + k_2\frac{r + a_0}{r - r_3}$$

$$(c_2)\ r > a_0 \text{ and } n_1\frac{r + a_0}{r - r_1} + n_2\frac{r - a_0}{2r} \leqq k_1\frac{r - a_0}{r + r_2} + k_2\frac{r + a_0}{r - r_3}\ ,$$

then the polynomial $F(z; \lambda) = R(z) + \lambda S(z)$ *has at least* n_1 *zeros in the disc* $|z - a| \leqq r$, *for any complex number* λ.

Proof. A straightforward calculation yields the following results:
(1) Let $z = a + re^{i\theta}$, then

$$M = \max_{0 \leqq \theta \leqq 2\pi} Re\left(\frac{z}{z - u}\right) = \frac{r^2 - |a^2| + Re\,(a\bar{u}) \pm r\,|u|}{r^2 - |u - a|^2}$$

$$m = \min_{0 \leqq \theta \leqq 2\pi} Re\left(\frac{z}{z - u}\right) = \frac{r^2 - |a|^2 + Re\,(a\bar{u}) \mp r\,|u|}{r^2 - |u - a|^2}$$

according to $|u - a| \lessgtr r$ respectively.
(2) Let $a = a_0e^{i\varphi}$, $z - a = re^{i\theta}$, $u - a = \rho e^{i\varphi}$, then

$$M_1 = \max_{0\leqq\theta,\varphi\leqq 2\pi} Re\left(\frac{z}{z-u}\right) = \begin{cases} \dfrac{r+a_0}{r-\rho} & \text{for } \rho < r \\ \dfrac{r+a_0}{r-\rho} & \text{for } \rho > r,\ |u| < r \\ \dfrac{r^2 - a_0\rho - r|a_0-\rho|}{r^2-\rho^2} & \text{for } \rho > r,\ |u| > r \end{cases}$$

(3)

$$m_1 = \min_{0\leqq\theta,\varphi\leqq 2\pi} Re\left(\frac{z}{z-u}\right) = \begin{cases} \dfrac{r-a_0}{r+\rho} & \text{for } r > \rho,\ |u| < r \\ \dfrac{r^2 - a_0\rho - r|a_0-\rho|}{r^2-\rho^2} & \text{for } r > \rho,\ |u| > r \\ \dfrac{r+a_0}{r-\rho} & \text{for } r < \rho\ . \end{cases}$$

(4) Let the polynomial $R(z)$ satisfy condition (a) of Theorem 8, then

$$M_2 = \max_{|z-a|=r} \frac{d}{d\theta} \arg R(z) \leqq n_1 \frac{r+a_0}{r-r_1} + n_2 \frac{r-a_0}{2r+a_0} \quad \text{for } r < a_0$$

$$M_2 \leqq n_1 \frac{r+a_0}{r-r_1} + n_2 \frac{r-a_0}{2r} \quad \text{for } r > a_0\ .$$

(5) Let the polynomial $S(z)$ satisfy condition (b) of Theorem 8, then

$$m_2 = \min_{|z-a|=r} \frac{d}{d\theta} \arg S(z) \geqq k_1 \frac{r-a_0}{r} + k_2 \frac{r+a_0}{r-r_3} \quad \text{for } r < a_0$$

$$m_2 \geqq k_1 \frac{r-a_0}{r+r_2} + k_2 \frac{r+a_0}{r-r_2} \quad \text{for } r > a_0\ .$$

By the results (4) and (5), the condition (c_1) or (c_2) implies that

$$\max_{|z-a|=r} \frac{d}{d\theta} \arg \frac{R(z)}{S(z)} \leqq \max_{|z-a|=r} \frac{d}{d\theta} \arg R(z) - \min_{|z-a|=r} \frac{d}{d\theta} \arg S(z) \leqq 0\ .$$

Hence the $\arg R(z)/S(z)$, as $|z-a| = r$ and z makes one turn in the positive direction, decreases monotonically.

$$\mathop{\Delta}_{|z-a|=r} \arg \frac{R(z)}{S(z)} = 2\pi(n_1 - k_1)\ .$$

It follows now that

$$\mathop{\Delta}_{|z-a|=r} \left(\arg \frac{R(z)}{S(z)} + \lambda\right) \geqq 2(n_1 - k_1)$$

for any complex number λ. Hence

$$\mathop{\Delta}_{|z-a|=r} \arg\left(R(z) + \lambda S(z)\right) \geqq 2\pi(n_1 - k_1) + 2\pi k_1 = 2\pi n_1\ .$$

This completes the proof.

In the particular case $a = 0$, $n_2 = k_2 = 0$, at least n zeros of the polynomial $F(z; \lambda)$ are in the disc $|z| \leqq \max((nr_2 + kr_1)/(k-n), r_2)$ for $k > n$. The last result is due to Biernacki (See. [1]). If $a = 0$ and the zeros of $R(z)$ and $S(z)$ are in the discs $|z - c_i| \leqq d_i$, it results from Theorem 8, with $r_i = |c_i| + d_i$, $i = 1, 2$, that in the case $k > n$, at least n zeros of the polynomial $F(z; \lambda)$ are in the disc

$$|z| \leqq \max\left(\frac{nd_2 + kd_1}{k-n} + \frac{n|c_2| + k|c_1|}{k-n}, d_2 + |c_2|\right).$$

This result is due to Jankowski [1].

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