ON MEROMORPHIC STARLIKE FUNCTIONS

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1. Introduction. Let $\mathfrak{S}(\alpha)$ $(0 \leqq \alpha < 1)$ denote the class of functions of the form

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$$

that are analytic in $1 < |z| < \infty$ and satisfy

$$Re\, z \frac{f'(z)}{f(z)} \geqq \alpha .$$

The class $\mathfrak{S}(0)$ is formed by the meromorphic starlike functions $f(z) = z + \cdots$, that is by the functions that map $\{|z| > 1\}$ onto a region whose compact complement is starlike with respect to the origin.

Apart from the case $\alpha = 0$ the most interesting case is $\alpha = 1/2$. We shall see (Corollary 1) that every $f \in \mathfrak{S}(\frac{1}{2})$ can be approximated by the roots of polynomials with zeros in the unit disk, that is by functions of the form

$$f(z) = \left(\prod_{\nu=1}^{n} (z - z_\nu)\right)^{1/n} \qquad (|z_\nu| \leqq 1) . \tag{1.1}$$

These functions belong to $\mathfrak{S}(\frac{1}{2})$.

The main results will be: *As* $r \rightarrow 1$,

$$\max_{f \in \mathfrak{S}(\alpha)} \max_{|z| \leqq r} |f'(z)| \sim \frac{2^{2(1-\alpha)} e^{-1}}{(1 - r^{-1}) \log 1/(1 - r^{-1})} . \tag{1.2}$$

If $f \in \mathfrak{S}(\alpha)$ *then*

$$\begin{aligned} |f'(z)| &\geqq \alpha |z^{-1} f(z)| + (1 - \alpha) |z^{-1} f(z)|^{-\alpha/(1-\alpha)} (1 - |z|^{-2}) \\ &\geqq (1 - |z|^{-2})^{1-\alpha} . \end{aligned} \tag{1.3}$$

For each fixed $f \in \mathfrak{S}(\alpha)$,

$$|f'(z)| \geqq \kappa (1 - |z|^{-1})^{\max(1-2\alpha, \varepsilon)}$$

for every $\varepsilon > 0$ *and some constant* $\kappa = \kappa(f, \varepsilon) > 0$.

The coefficients of $f(z)$ *satisfy*

$$|a_n| \leqq \frac{2(1 - \alpha)}{n + 1} , \tag{1.4}$$

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$$\inf_{n\to\infty}\sup n\,|a_n| < 2(1-\alpha)\,. \tag{1.5}$$

All these inequalities are best possible.

Inequality (1.3) is well-known for the case $\alpha = 0$ whereas (1.2) seems to be new even for $\alpha = 0$. Inequality (1.4) has been proved by Clunie [1] for $\alpha = 0$. Some of the problems can also be studied by the variation method developed by Royster [9].

We shall also prove inequalities analogous to (1.4) and (1.5) for functions $g(\zeta)$ that are analytic and bounded in $|\zeta| < 1$ and satisfy $Re\,\zeta g'(\zeta)/g(\zeta) \geqq \alpha$. (For $\alpha = 0$, see [2] and [7]).

2. A representation formula.

THEOREM 1. *Let* $f \in \mathfrak{S}(\alpha)$ $(0 \leqq \alpha < 1)$. *Then*

$$f(z) = z\exp\left(2(1-\alpha)\int_0^{2\pi}\log(1-e^{it}z^{-1})d\gamma(t)\right) \tag{2.1}$$

where $\gamma(t)$ *increases and* $\gamma(t+2\pi)-\gamma(t) = 1$. *Also* $\arg f(re^{it}) \to \pi t + 2\pi(1-\alpha)\gamma(t)$ *as* $r \to 1$, *and*

$$z\frac{f'(z)}{f(z)} = \alpha + (1-\alpha)\int_0^{2\pi}\frac{1+e^{it}z^{-1}}{1-e^{it}z^{-1}}d\gamma(t)\,. \tag{2.2}$$

Conversely, every function of the form (2.1) *belongs to* $\mathfrak{S}(\alpha)$.

Proof. We shall reduce (2.1) in the general case to the case $\alpha = 0$ where (2.1) is a known formula. If $f \in \mathfrak{S}(\alpha)$ let

$$g(z) = z^{-\alpha/(1-\alpha)}f(z)^{1/(1-\alpha)} = z + \cdots\,.$$

Then $Re\,zg'(z)/g(z) = -\alpha/(1-\alpha) + 1/(1-\alpha)\cdot Re\,zf'(z)/f(z) \geqq 0$. Hence $g(z)$ is starlike and (compare [7, Lemma 1])

$$g(z) = z\exp\left(\frac{1}{\pi}\int_0^{2\pi}\log(1-e^{it}z^{-1})dV(t)\right),$$

$$zg'(z)/g(z) = \frac{1}{2\pi}\int_0^{2\pi}(1+e^{it}z^{-1})/(1-e^{it}z^{-1})dV(t)$$

where $V(t) = \lim_{r\to 1}\arg g(re^{it})$ is monotone increasing and $V(t+2\pi) - V(t) = 2\pi$. Putting $\gamma(t) = V(t)/(2\pi)$ we see that $f(z) = z^\alpha g(z)^{1-\alpha}$ satisfies (2.1) and (2.2). Direct computation shows that every function of the form (2.1) belongs to $\mathfrak{S}(\alpha)$.

COROLLARY 1. *For given* $f \in \mathfrak{S}(\alpha)$, *there is a sequence of functions* $f_n \in \mathfrak{S}(\alpha)$ *of the form*

$$f_n(z) = z\prod_1^n(1-z_\nu z^{-1})^{2(1-\alpha)/n} \qquad (|z_\nu| \leqq 1)$$

that converges to $f(z)$ *locally uniformly in* $|z| > 1$. *All functions of this form belong to* $\mathfrak{S}(\alpha)$.

This corollary follows immediately from (2.1) by approximating $\gamma(t)$ by step-functions with n steps of height $1/n$. The last assertion of the corollary is established by computation. For $\alpha = \frac{1}{2}$, the corollary shows that f can be approximated by functions of the form (1.1).

3. Estimates of the function-values.

THEOREM 2. *Let* $f(z) = z + a_0 + a_1 z^{-1} + \cdots \in \mathfrak{S}(\alpha)$ *and* $|z| = r > 1$. *Then*

$$(3.1) \qquad \left|\frac{f(z)}{z}\right| \leqq \left(1 + \frac{|a_0|}{1-\alpha} r^{-1} + r^{-2}\right)^{1-\alpha} \leqq (1 + r^{-1})^{2(1-\alpha)} ,$$

$$(3.2) \qquad \left|\frac{f(z)}{z}\right| \geqq (1 - r^{-1})^{1-\alpha+|a_0|/2}(1 + r^{-1})^{1-\alpha-|a_0|/2} \geqq (1 - r^{-1})^{2(1-\alpha)} .$$

Equality can be attained in all inequalities.

For $\alpha = 0$ we find $(1 + r^{-1})^2 \geqq |z^{-1}f(z)| \geqq (1 - r^{-1})^2$, and these are well-known inequalities which hold for all functions $f(z) = z + \cdots$ univalent and $\neq 0$ in $|z| > 1$.

Proof of (3.1). Using (2.1) and the fact that the geometric mean is not greater than the arithmetic mean we see that

$$(3.3) \qquad \begin{aligned} |z^{-1}f(z)|^{1/(1-\alpha)} &= \exp\left(\int_0^{2\pi} \log|1 - e^{it}z^{-1}|^2 d\gamma(t)\right) \\ &\leqq \int_0^{2\pi} |1 - e^{it}z^{-1}|^2 d\gamma(t) \\ &= \int_0^{2\pi} (1 + r^{-2}) d\gamma(t) - 2Re\left[z^{-1}\int_0^{2\pi} e^{-it} d\gamma(t)\right] . \end{aligned}$$

Also by (2.1)

$$a_0 = -2(1 - \alpha)\int_0^{2\pi} e^{it} d\gamma(t) .$$

Hence $|a_0| \leqq 2(1 - \alpha)$, and by (3.3)

$$|z^{-1}f(z)|^{1/(1-\alpha)} \leqq 1 + r^{-2} + |a_0| r^{-1}/(1 - \alpha) \leqq (1 + r^{-1})^2$$

from which inequalities (3.1) follow. We have equality for the function

$$f(z) = z\left(1 + \frac{a_0}{1-\alpha} z^{-1} + z^{-2}\right)^{1-\alpha} = z + a_0 + \cdots \in \mathfrak{S}(\alpha)$$

where $0 \leqq a_0 \leqq 2(1 - \alpha)$, and $z = r > 1$.

Proof of (3.2). Since $zf'(z)/f(z) = 1 - a_0 z^{-1} + \cdots$ and $Re\, zf'(z)/f(z) \geqq \alpha$ the function

$$(3.4) \qquad \phi(z) = z \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) + (1 - 2\alpha)} = -\frac{a_0}{2(1-\alpha)} + \cdots$$

is analytic in $|z| > 1$ and satisfies $|\phi(z)| < 1$. Then

$$(3.5) \quad \frac{\partial}{\partial r} \log |z^{-1} f(z)| = -\frac{1}{r} + Re\left[\frac{z}{r}\frac{f'(z)}{f(z)}\right] \leqq 2(1-\alpha)\left|\frac{z^{-1}\phi(z)}{r(1 - z^{-1}\phi(z))}\right| .$$

If $b = |\phi(\infty)| = |a_0|/2(1-\alpha)$ then [3, p. 287]

$$|\phi(z)| \leqq (br + 1)/(b + r) ,$$

hence by (3.5)

$$\frac{\partial}{\partial r} \log |z^{-1} f(z)| \leqq 2(1-\alpha)\frac{br + 1}{r(r^2 - 1)} .$$

Integration over $[r, +\infty]$ gives

$$\log |z^{-1} f(z)| \geqq (1-\alpha)(1+b)\log(1 - r^{-1}) + (1-\alpha)(1-b)\log(1 + r^{-1}) ,$$

and inequalities (3.2) follow because $b = |a_0|/2(1-\alpha) = |\phi(\infty)| \leqq 1$.

Equality is attained for $z = r > 1$ by the function

$$f(z) = z(1 - z^{-1})^{1-\alpha+a_0/2}(1 + z^{-1})^{1-\alpha-a_0/2} = z - a_0 + \cdots$$

with $0 \leqq a_0 \leqq 2(1-\alpha)$.

We could have proved (3.1) by the same method as (3.2). But the proof directly from the representation formula seemed to be more interesting.

4. Upper estimate of the derivative. The best possible upper estimate of the derivative for the functions $f(z) = z + \cdots$ that are meromorphic and univalent in $|z| > 1$ is $|f'(re^{i\theta})| \leqq 1/(1 - r^{-2})$ (see for instance [3, p. 120]). We shall show that this inequality can be improved for starlike functions and small $r > 1$.

THEOREM 3. *Let* $0 \leqq \alpha < 1$. *Then as* $r \to 1$

$$\max_{f \in \mathfrak{S}(\alpha)} \max_{|z| \leqq r} |f'(z)| \sim \frac{2^{2(1-\alpha)} e^{-1}}{(1 - r^{-1}) \log 1/(1 - r^{-1})} .$$

The proof will show that the function $f(z)$ for which $|f'(z_0)|$ becomes maximal for a given z_0 has the form

$$(4.1) \qquad f(z) = z(1 - z_1 z^{-1})^{2(1-\alpha)\gamma_1}(1 - z_2 z^{-1})^{2(1-\alpha)\gamma_2}$$

with $0 \leqq \gamma_1 \leqq 1$, $\gamma_1 + \gamma_2 = 1$ and $|z_1| = |z_2| = 1$.

Proof. 1. Let $n \geqq 2$ and different z_ν $(\nu = 1, \cdots, n)$ with $|z_\nu| = 1$ be given. We consider the functions

$$f(z) = z \prod_{\nu=1}^{n} (1 - z_\nu z^{-1})^{2(1-\alpha)\gamma_\nu} \tag{4.2}$$

with $0 \leqq \gamma_\nu \leqq 1$, $\gamma_1 + \cdots + \gamma_n = 1$. For fixed $z_0(|z_0| > 1)$ let $f(z)$ be the function of this form for which $|f'(z_0)|$ becomes maximal. We shall show that only two of the γ_ν can be $\neq 0$ for this maximal function. We shall use an elementary variation method (compare [5]).

2. Suppose this were false, and $\gamma_\nu > 0$ $(\nu = 1, 2, 3)$. If $\beta_1 + \beta_2 + \beta_3 = 0$, $\beta_4 = \cdots = \beta_n = 0$ then $\gamma_\nu^* = \gamma_\nu \pm \delta\beta_\nu \geqq 0$ for sufficiently small $\delta > 0$, and $\Sigma\gamma_\nu^* = 1$. Let $f_*(z)$ be the function of the form (4.2) where γ_ν has been replaced by γ_ν^*. Let $\zeta = z_0^{-1}$. Then

$$\begin{aligned} |f_*'(z_0)|^2 &= |z_0 f_*'(z_0)/f_*(z_0)|^2 \cdot |z_0^{-1} f_*(z_0)|^2 \\ &= \left| \sum_{\nu=1}^{n} (\gamma_\nu \pm \delta\beta_\nu) \frac{1 + (1 - 2\alpha) z_\nu \zeta}{1 - z_\nu \zeta} \right|^2 \prod_{\nu=1}^{n} |1 - z_\nu \zeta|^{4(1-\alpha)(\gamma_\nu \pm \delta\beta_\nu)} . \end{aligned}$$

For abbreviation let

$$c = \sum_{\nu=1}^{n} \gamma_\nu \frac{1 + (1 - 2\alpha) z_\nu \zeta}{1 - z_\nu \zeta} , \qquad b = \sum_{\nu=1}^{3} \beta_\nu \frac{1 + (1 - 2\alpha) z_\nu \zeta}{1 - z_\nu \zeta}$$

and

$$l = \sum_{\nu=1}^{3} \beta_\nu \log |1 - z_\nu \zeta| .$$

Since $\beta_4 = \cdots = \beta_n = 0$ it follows that

$$\begin{aligned} |f_*'(z_0)|^2 &= (|c|^2 \pm 2\delta Re[b\bar{c}] + \delta^2 |b|^2) \cdot |z_0^{-1} f(z_0)|^2 \\ &\quad \cdot (1 \pm 4\delta(1 - \alpha) l + 8\delta^2 (1 - \alpha)^2 l^2 + 0(\delta^3)) \\ &= |z_0^{-1} f(z_0)|^2 [|c|^2 \pm 2\delta(Re[b\bar{c}] + 2(1 - \alpha)|c|^2 l) \\ &\quad + \delta^2(|b|^2 + 8(1 - \alpha) l Re[b\bar{c}] + 8(1 - \alpha)^2 |c|^2 l^2) + 0(\delta^3)] . \end{aligned}$$

Since $|f_*'(z_0)|^2 \leqq |f'(z_0)|^2 = |z_0^{-1} f(z_0)|^2 |c|^2$ by the maximal choice of f, it follows that

$$Re[b\bar{c}] + 2(1 - \alpha)|c|^2 l = 0$$

and

$$|b|^2 + 8(1 - \alpha) l Re[b\bar{c}] + 8(1 - \alpha)^2 |c|^2 l^2 \leqq 0 .$$

Eliminating $Re[b\bar{c}]$ we obtain

$$|b|^2 \leqq 8(1-\alpha)|c|^2 l^2 .$$

In particular, $l = 0$ implies $b = 0$.

The two homogeneous linear equations

$$l = \sum_1^3 \beta_\nu \log|1 - z_\nu \zeta| = 0$$

$$\beta_1 + \beta_2 + \beta_3 = 0$$

have a nontrivial solution β_1, β_2, β_3. With these values it follows, as we have proved, that

$$b = \sum_1^3 \beta_\nu \frac{1 + (1-2\alpha)z_\nu\zeta}{1 - z_\nu\zeta} = 0 .$$

Hence the three different points $w_\nu = (1 + (1-2\alpha)z_\nu\zeta)/(1 - z_\nu\zeta)$ ($\nu = 1, 2, 3$) lie on a straight line, which is impossible because $|z_\nu| = 1$ and $|\zeta| < 1$. We have therefore proved that in the maximal function all exponents γ_ν are 0 except possibly two.

3. We can write this maximal function in the form (4.1). Thus, by (2.1) the function $f \in \mathfrak{S}(\alpha)$ for which $|f'(z_0)|$ is maximal also has the form (4.1).

We may therefore assume that $f(z)$ has the form (4.1) and also that $\gamma_1 \leqq \gamma_2$, hence $\gamma_1 \leqq \frac{1}{2} \leqq \gamma_2$. Let $\rho = |z|^{-1}$. Then

$$|f'(z)| \leqq 2^{2(1-\alpha)} + 2(1-\alpha)\gamma_1(1-\rho)^{2(1-\alpha)\gamma_1 - 1}2^{2(1-\alpha)\gamma_2} + 2(1-\alpha)\gamma_2 \cdot 2^{2(1-\alpha)\gamma_1} \max[2^{2(1-\alpha)\gamma_2 - 1}, (1-\rho)^{2(1-\alpha)\gamma_2 - 1}] .$$

Since

$$\max_{\gamma \geqq 0} \gamma(1-\rho)^{2(1-\alpha)\gamma} = 1/[2(1-\alpha)e \log 1/(1-\rho)] \tag{4.3}$$

it follows that

$$|f'(z)| \leqq 4 + 2^{2(1-\alpha)}e^{-1}[(1-\rho)\log 1/(1-\rho)]^{-1} + 16(1-\rho)^{-\alpha} . \tag{4.4}$$

4. We finally consider the special function

$$f(z) = z(1 - z^{-1})^{2(1-\alpha)\gamma}(1 + z^{-1})^{2(1-\alpha)(1-\gamma)}$$

with $\gamma = 1/[2(1-\alpha)\log 1/(1-\rho)]$ for $\rho < 1$ near to 1. Then $f \in \mathfrak{S}(\alpha)$, and by (4.3)

$$f'(\rho^{-1}) \sim 2^{2(1-\alpha)}e^{-1}[(1-\rho)\log 1/(1-\rho)]^{-1} \qquad (\rho \to 1) .$$

Together with (4.4) this proves Theorem 3.

5. Lower estimates of the derivative. The best possible lower estimate of the derivative for functions $f(z) = z + \cdots$ meromorphic and univalent in $|z| > 1$ is $|f'(z)| \geqq 1 - |z|^{-2}$, with equality for $f(z) = z + z^{-1}$. This function belongs to $\mathfrak{S}(0)$. For $0 < \alpha < 1$ we can prove more.

THEOREM 4. *If* $f \in \mathfrak{S}(\alpha)$ $(0 \leqq \alpha < 1)$ *then*

$$(5.1) \quad |f'(z)| \geqq \alpha\left|\frac{f(z)}{z}\right| + (1-\alpha)\left|\frac{z}{f(z)}\right|^{\alpha/(1-\alpha)}\left(1 - \frac{1}{|z|^2}\right) \geqq \left(1 - \frac{1}{|z|^2}\right)^{1-\alpha}.$$

Equality can be attained in all inequalities.

Proof. By (2.1) we have

$$\left|\frac{z}{f(z)}\right|^{1/(1-\alpha)} = \exp\left(\int_0^{2\pi} \log \frac{1}{|1 - e^{it}z^{-1}|^2} d\gamma(t)\right)$$

with $d\gamma(t) \geqq 0$ and $\int d\gamma(t) = 1$. Since the geometric mean is not greater than the arithmetic mean it follows that

$$\left|\frac{z}{f(z)}\right|^{1/(1-\alpha)} \leqq \int_0^{2\pi} \frac{1}{|1 - e^{it}z^{-1}|^2} d\gamma(t) .$$

Hence by (2.2)

$$\left|\frac{z}{f(z)}\right|^{1/(1-\alpha)}(1 - |z|^{-2}) \leqq \int_0^{2\pi} \frac{1 - |z|^{-2}}{|1 - e^{it}z^{-1}|^2} d\gamma(t)$$

$$= \frac{1}{1-\alpha}\left(Re\, z\frac{f'(z)}{f(z)} - \alpha\right)$$

and therefore

$$\left|z\frac{f'(z)}{f(z)}\right| \geqq Re\, z\frac{f'(z)}{f(z)} \geqq \alpha + (1-\alpha)\left|\frac{z}{f(z)}\right|^{1/(1-\alpha)}(1 - |z|^{-2})$$

from which (5.1) follows. Equality is attained by

$$f(z) = z(1 - 2r^{-1}z^{-1} + z^{-2})^{1-\alpha} \in \mathfrak{S}(\alpha)$$

for $z = r > 1$.

Though (5.1) is best possible for the whole class $\mathfrak{S}(\alpha)$ more is true for each fixed function in $\mathfrak{S}(\alpha)$ if $\alpha > 0$.

THEOREM 5. *Let* $f \in \mathfrak{S}(\alpha)$. (i) *If* $0 \leqq \alpha < \frac{1}{2}$ *then there is a* $\kappa = \kappa(f) > 0$ *such that*

$$|f'(re^{i\theta})| \geqq \kappa(1 - r^{-1})^{1-2\alpha}$$

The exponent cannot be decreased. (ii) *If* $\frac{1}{2} \leqq \alpha < 1$ *then*

$$|f'(re^{i\theta})| \geqq \kappa(\varepsilon)(1 - r^{-1})^{\varepsilon}$$

for every $\varepsilon > 0$ *where* $\kappa(\varepsilon) = \kappa(f, \varepsilon) > 0$.

Proof. We may take $\alpha > 0$. Suppose Theorem 5 were false. Then there would be a sequence $z_k = r_k e^{i\theta_k}$ such that

$$f'(z_k) = o((1 - r_k^{-1})^{1-2\alpha}) \tag{5.2}$$

and also

$$f'(z_k) = o((1 - r_k^{-1})^{\varepsilon_0}) \tag{5.3}$$

for some $\varepsilon_0 > 0$. We may assume that $\theta_k \to 0$. We have to distinguish two cases:

Case 1. The point $t = 0$ is a discontinuity of the function $\gamma(t)$ of Theorem 1. Then let λ_0 be the height of the jump of $\gamma(t)$ at 0, and let λ_j $(j = 1, \cdots, m)$ be the jumps of height $\geqq \lambda_0$, occuring at $t_j \neq 0$. Let λ^* be the highest jump $< \lambda_0$, and let $\delta = \lambda_0 - \lambda^*$. Then $\delta > 0$. If $\sigma(t)$ is the function that is constant except for jumps of height 1 at $0, \pm 2\pi, \cdots$ let

$$\gamma^*(t) = \gamma(t) - \lambda_0\sigma(t) - \sum_{j=1}^{m} \lambda_j \sigma(t - t_j) . \tag{5.4}$$

This function increases and has highest jump λ^*. We see from Theorem 1 that

$$z_k \frac{f'(z_k)}{f(z_k)} = \alpha + (1 - \alpha)\lambda_0 \frac{1 + z_k^{-1}}{1 - z_k^{-1}} + (1 - \alpha) \sum_{\mu=1}^{m} \lambda_k \frac{1 + e^{it_\mu} z_k^{-1}}{1 - e^{it_\mu} z_k^{-1}} + F^*(z_k) \tag{5.5}$$

where

$$F^*(z) = (1 - \alpha) \int_0^{2\pi} \frac{1 + e^{it} z^{-1}}{1 - e^{it} z^{-1}} d\gamma^*(t)$$

is a function of positive real part. Hence [4, Theorem 2]

$$|F^*(z_k)| \leqq (1 - \alpha)(\lambda^* + \tfrac{1}{3}\delta)/(1 - r_k^{-1}) \tag{5.6}$$

for large k. The third term on the right side of (5.5) is bounded as $k \to \infty$ because $\theta_k \to 0 \neq t_j$ $(j = 1, \cdots, m)$. Hence by (5.5) and (5.6)

$$\begin{aligned} |z_k f'(z_k)/f(z_k)| &\geqq (1 - \alpha)(\lambda_0 - \tfrac{1}{3}\delta - \lambda^* - \tfrac{1}{3}\delta)/(1 - r_k^{-1}) \\ &= \tfrac{1}{3}\delta/(1 - r_k^{-1}) \end{aligned}$$

for large k. On the other hand, Theorem 2 implies $|z_k^{-1}f(z_k)| \geqq (1 - r_k^{-1})^{2-2\alpha}$. Therefore $|f'(z_k)| \geqq \frac{1}{3}\delta(1 - r_k^{-1})^{1-2\alpha}$, in contradiction to (5.2).

Case 2. The function $\gamma(t)$ is continuous at $t = 0$. Let now λ_j ($j = 1, \cdots, m$) be the jumps of height $\geqq \varepsilon_0/4(1-\alpha)$ occuring at $t_j \neq 0$. Let $\gamma^*(t)$ be defined as in (5.4) (with $\lambda_0 = 0$). Then we see from Theorem 1 that

$$f(z) = \prod_{j=1}^{m} (1 - e^{it_j}z^{-1})^{2(1-\alpha)\lambda_j} f^*(z) \tag{5.7}$$

where

$$f^*(z) = z \exp\left(2(1 - \alpha)\int_0^{2\pi} \log (1 - e^{it}z^{-1}) d\gamma^*(t)\right)$$

is a starlike function. Hence $1/f^*(\zeta^{-1})$ is starlike in $|\zeta| < 1$. Since all jumps of $\gamma^*(t)$ are $<\varepsilon_0/4(1 - \alpha)$ it follows [7, Theorem 1] that

$$1/f^*(\zeta^{-1}) = o((1 - |\zeta|)^{-\varepsilon_0}) .$$

Since $\theta_k \to 0 \neq t_j$ as $k \to \infty$ we therefore obtain from (5.7) that

$$|z_k^{-1}f(z_k)| \geqq (1 - r_k^{-1})^{\varepsilon_0}$$

for large k. Because $Re\, zf'(z)/f(z) \geqq \alpha > 0$ it follows that

$$|f'(z_k)| \geqq \alpha |z_k^{-1}f(z_k)| \geqq \alpha(1 - r_k^{-1})^{\varepsilon_0}$$

in contradiction to (5.3).

Finally, that function $f(z) = z(1 - z^{-1})^{2-2\alpha} \in \mathfrak{S}(\alpha)$ shows that $1 - 2\alpha$ cannot be made smaller.

6. Estimates of the coefficients If $f(z) = z + a_0 + a_1z^{-1} + \cdots$ is a starlike function then, as Clunie [1] has proved, $|a_n| \leqq 2/(n + 1)$. To generalise this inequality to the class $\mathfrak{S}(\alpha)$ we first prove the following lemma, using Clunie's method.

LEMMA 1. *If* $f \in \mathfrak{S}(\alpha)$ $(0 \leqq \alpha < 1)$ *and* $n = 0, 1, \cdots$ *then*

$$(n + 1)^2 |a_n|^2 \leqq 4(1 - \alpha)^2 - 4(1 - \alpha) \sum_{\nu=0}^{n-1} (\nu + \alpha) |a_\nu|^2 .$$

Proof. As in (3.4) let again

$$\phi(z) = z\frac{zf'(z) - f(z)}{zf'(z) + (1 - 2\alpha)f(z)} .$$

Then it follows that

$$-\sum_{\nu=0}^{\infty}(\nu+1)a_\nu z^{-\nu}=\phi(z)\Big[2(1-\alpha)-\sum_{\nu=0}^{\infty}(\nu-1+2\alpha)a_\nu z^{-\nu-1}\Big],$$

hence for $n=0,1,\cdots$

$$-\sum_{\nu=0}^{n}(\nu+1)a_\nu z^{-\nu}-\sum_{\nu=n+1}^{\infty}(\nu+1)a_\nu z^{-\nu}+\phi(z)\sum_{\nu=n}^{\infty}(\nu-1+2\alpha)a_\nu z^{-\nu-1}$$
$$=\phi(z)\Big[2(1-\alpha)-\sum_{\nu=0}^{n-1}(\nu-1+2\alpha)a_\nu z^{-\nu-1}\Big].$$

Since the second and third term on the left side involve only powers $z^{-\mu}$ with $\mu\geqq n+1$ and since $|\phi(z)|<1$ Parceval's formula gives

$$\sum_{\nu=0}^{n}(\nu+1)^2|a_\nu|^2\leqq 4(1-\alpha)^2+\sum_{\nu=0}^{n-1}(\nu-1+2\alpha)^2|a_\nu|^2,$$

and Lemma 1 follows at once.

COROLLARY 2. *Let $f\in\mathfrak{S}(\alpha)$, $0<\alpha<1$, and let A be the area of the compact complement of the image region $\{f(z):|z|>1\}$. Then*

$$\pi\geqq A>\pi\alpha,$$

and these inequalities are best possible.

Proof. The inequality $\pi\geqq A$ is of course classical. Lemma 1 implies

$$\sum_{\nu=0}^{\infty}(\nu+\alpha)|a_\nu|^2\leqq 1-\alpha,$$

hence

$$A=\pi\Big(1-\sum_{\nu=1}^{\infty}\nu|a_\nu|^2\Big)\geqq\pi\Big(\alpha+\alpha\sum_{\nu=0}^{\infty}|a_\nu|^2\Big)\geqq\pi\alpha.$$

Equality could only hold if $a_\nu=0$ for $\nu=0,1,\cdots$. But then $A=\pi$.

To show that $A>\pi\alpha$ is best possible we consider

$$f(z)=z(1+z^{-n-1})^{2(1-\alpha)/(n+1)}=z+\frac{2(1-\alpha)}{n+1}z^{-n}+\cdots\in\mathfrak{S}(\alpha). \tag{6.1}$$

The function $w=f(z)$ maps $|z|=1$ onto a set contained in

$$\{|w|\leqq 2^{2(1-\alpha)/(n+1)},\ |\arg w-2\pi k/(n+1)|$$
$$\leqq\pi\alpha/(n+1),k=0,\cdots,n\}.$$

Therefore A is smaller than the area $\pi\alpha\cdot 2^{4(1-\alpha)/(n+1)}$ of this last set which tends $\to\pi\alpha$ as $n\to\infty$.

THEOREM 6. *Let $f(z)=z+\sum_{n=0}^{\infty}a_n z^{-n}$ be in $\mathfrak{S}(\alpha)$ $(0\leqq\alpha<1)$.*

Then

$$|a_n| \leqq \frac{2(1-\alpha)}{n+1} \tag{6.2}$$

for $n = 0, 1, \cdots$, *with equality for the functions* (6.1).

Inequality (6.2) follows immediately from Lemma 1. The next theorem will show that (6.2) cannot be improved much even for a fixed function in $\mathfrak{S}(\alpha)$. In particular, there is a starlike function with $a_n \neq o(n^{-1})$.

THEOREM 7. *If* $f \in \mathfrak{S}(\alpha)$ *then*

$$\limsup_{n\to\infty} n\,|a_n| < 2(1-\alpha)\ . \tag{6.3}$$

For every $\varepsilon > 0$ *there is a function* $f \in \mathfrak{S}(\alpha)$ *such that*

$$\limsup_{n\to\infty} n\,|a_n| > 2(1-\alpha) - \varepsilon$$

Proof. It follows from Lemma 1 that

$$\limsup_{n\to\infty}\,(n+1)^2\,|a_n|^2 \leqq 4(1-\alpha)^2 - 4(1-\alpha)\sum_{\nu=0}^{\infty}(\nu+\alpha)\,|a_\nu|^2 < 4(1-\alpha)^2$$

except when $a_\nu = 0$ for $\nu = 0, 1, \cdots$. But then (6.3) is trivial. The last assertion will be proved at the end of this paper.

We shall now consider bounded starlike functions that are analytic in $|\zeta| < 1$.

THEOREM 8. *Let* $g(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^n$ *be analytic in* $|\zeta| < 1$ *and satisfy*

$$Re\,\zeta g'(\zeta)/g(\zeta) \geqq \alpha\ . \tag{6.4}$$

and $|g(\zeta)| < 1$. *Then*

$$\sum_{n=1}^{\infty}(n-\alpha)\,|b_n|^2 \leqq 1-\alpha\ . \tag{6.5}$$

For $n = 2, 3, \cdots$

$$|b_n| < \frac{2(1-\alpha)}{n+1-2\alpha}\ . \tag{6.6}$$

The factor $2(1-\alpha)$ *cannot be replaced by a smaller factor independent of* n. *For every* $\varepsilon > 0$ *there is a function* $g(\zeta)$ *such that*

$$\limsup_{n\to\infty} n\,|b_n| > 2(1-\alpha) - \varepsilon\ . \tag{6.7}$$

In the case $\alpha = 0$ inequality (6.6) has been proved (in a slightly weaker form) by Clunie and Keogh [2], and in [7, Theorem 3]. Clunie and Keogh also gave an example of a starlike bounded function with $b_n \neq o(n^{-1})$.

Proof. For $0 < \rho < 1$

$$\sum_{n=1}^{\infty} (n-\alpha)\,|\,b_n\,|^2 \rho^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |\,g(\rho e^{i\theta})\,|^2 (Re[\rho e^{i\theta} g'/g] - \alpha) d\theta \; . \tag{6.8}$$

Since $|\,g(\rho e^{i\theta})\,| < 1$ it follows by (6.4) that

$$\sum_{n=1}^{\infty} (n-\alpha)\,|\,b_n\,|^2 \rho^{2n} \leqq \frac{1}{2\pi}\int_0^{2\pi} (Re[\rho e^{i\theta} g'/g] - \alpha) d\theta = 1 - \alpha$$

which implies (6.5)

As in the proof of Lemma 1 we obtain

$$(n + 1 - 2\alpha)^2\,|\,b_n\,|^2 \leqq 4(1-\alpha) \sum_{\nu=1}^{n} (\nu - \alpha)\,|\,b_\nu\,|^2 \; .$$

By (6.5) this expression is $\leqq 4(1-\alpha)^2$. If we had equality it would follow from (6.5) that $g(\zeta) = b_1\zeta + \cdots$ $(b_1 \neq 0)$ is a polynomial of degree $n > 1$. Then $\zeta g'(\zeta)/g(\zeta)$ is continuous and $|\,g(\zeta)\,| < 1$ almost everywhere on $|\,\zeta\,| = 1$, and (6.8) shows that (6.5) holds with strict inequality.

Let $g_0(\zeta) = n^{1/(n-1)}(\zeta + 2n^{-1}\zeta^n + \cdots)$ be the starlike function of Example 1 in [7]. It satisfies $|\,g_0(\zeta)\,| < 1$. Then

$$g(\zeta) = \zeta^\alpha g_0(\zeta)^{1-\alpha} = n^{-(1-\alpha)/(n-1)}(\zeta + 2(1-\alpha)n^{-1}\zeta^n + \cdots)$$

satisfies (6.4) and $|\,g(\zeta)\,| < 1$. Hence the factor $2(1-\alpha)$ in (6.6) cannot be made smaller. The existence of a function with (6.7) will be proved in the following.

7. Construction of examples. We shall now complete the proofs of Theorems 7 and 8 by showing: For every $\varepsilon > 0$ there exist functions

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n} \qquad (|\,z\,| > 1)$$

in $\mathfrak{S}(\alpha)$ and

$$g(\zeta) = \sum_{1}^{\infty} b_n \zeta^n \qquad (|\,\zeta\,| < 1)$$

with $Re\,\zeta g'(\zeta)/g(\zeta) \geqq \alpha$ and $|\,g(\zeta)\,| < 1$ such that

$$\limsup_{n\to\infty} n\,|\,a_n\,| > 2(1-\alpha) - \varepsilon\;, \qquad \limsup_{n\to\infty} n\,|\,b_n\,| > 2(1-\alpha) - \varepsilon\;.$$

We shall need the following lemma.

LEMMA 2. *Given $\delta > 0$ and $\eta > 0$ there exists a function*

$$h(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$$

with $c_n \geqq 0$ that is analytic and has positive real part in $|\zeta| < 1$ such that

$$\limsup_{n\to\infty} c_n > 2 - \delta \,, \tag{7.1}$$

$$\sum_1^\infty \frac{c_n}{n} < \eta \,. \tag{7.2}$$

This lemma was first proved by F. Riesz [8] but only with $\limsup c_n \geqq 1$. It seems not to be known in the general form given here. It will be seen that the weaker form is not sufficient to prove the existence of an $f \in \mathfrak{S}(0)$ with $a_n \neq o(n^{-1})$. But the weaker form is sufficient to prove the existence of a bounded starlike function $g(\zeta)$ with $b_n \neq o(n^{-1})$ [2].

Proof. Let $1 > \lambda > 1 - \frac{1}{2}\delta$ and let p be so large that

$$\begin{aligned} P(\theta) &= 1 + 2\lambda \cos \theta + \cdots + 2\lambda^p \cos p\theta \\ &= Re[1 + 2\lambda e^{i\theta} + \cdots + 2(\lambda e^{i\theta})^p] > 0 \,. \end{aligned}$$

Let q be such that

$$(2p + 1)2^{-q} < \min(\tfrac{1}{8}\eta, \tfrac{1}{2}) \,. \tag{7.3}$$

For $m = 1, 2, \cdots$ let

$$Q_m(\theta) = \prod_{k=1}^{m} P(2^{qk}\theta) \,. \tag{7.4}$$

Then $Q_m(\theta) > 0$. Because of (7.3) induction shows that

$$Q_m(\theta) = 1 + \sum_{k=1}^{2p\cdot 2^{qm}} c_n \cos n\theta$$

where c_n is independent of m and $0 \leqq c_n \leqq 2$, $c_{2^{qm}} = 2\lambda$, and $c_n > 0$ only if

$$n = \Sigma \mu_k 2^{qk} \qquad (-p \leqq \mu_k \leqq p) \,. \tag{7.5}$$

It follows that

$$\sum_1^\infty \frac{c_n}{n} \leqq \Sigma' \frac{2}{n}$$

where the last sum is taken over all n of the form (7.5). For $j = 1, 2, \cdots$, we group together all those n for which $\mu_j \neq 0$ but $\mu_k = 0$ for $k > j$.

Then $n \geqq 2^{qj} - p2^{q(j-1)} - \cdots - p2^q > 2^{qj}(1 - 2p \cdot 2^{-q}) > 2^{qj-1}$ by (7.3). The number of these n is $<(2p+1)^j$. Hence again by (7.3)

$$\sum_1^\infty \frac{c_n}{n} \leqq 2 \sum_{j=1}^\infty (2p+1)^j 2^{-qj+1} < 8(2p+1)2^{-q} < \eta .$$

The function

$$h(\zeta) = 1 + \sum_{n=1}^\infty c_n \zeta^n$$

is analytic in $|\zeta| < 1$, and $Re\, h(\zeta) \geqq 0$ by (7.4). Also $c_{2^{qm}} = 2\lambda > 2 - \delta$, and Lemma 2 is proved.

We shall now construct the starlike functions. Let $0 < \varepsilon < 1$ and $\delta = \varepsilon/6$, $e^{(1-\alpha)\eta} = 1 + \varepsilon/6$. If $h(\zeta)$ is the function of Lemma 2 let

$$H(\zeta) = \int_0^\zeta s^{-1}(h(s) - 1)ds = \sum_{n=1}^\infty \frac{c_n}{n} \zeta^n .$$

Let $\beta = 1 - \alpha$ and

$$f(z) = z \exp[-\beta H(z^{-1})] , \qquad g(\zeta) = \zeta \exp[\beta(H(\zeta) - \eta)] . \tag{7.6}$$

Then $zf'(z)/f(z) = 1 + \beta z^{-1} H'(z^{-1}) = \alpha + (1-\alpha)h(z^{-1})$, hence $f(z) = z + \cdots \in \mathfrak{S}(\alpha)$. Also, $g(\zeta)$ satisfies (6.4), and by (7.2)

$$|g(\zeta)| \leqq \exp\left[\beta\left(\sum_1^\infty n^{-1}c_n - \eta\right)\right] < 1 .$$

By (7.6),

$$\zeta^{-1} g(\zeta) e^{\beta\eta} - \zeta f(\zeta^{-1}) = 2\left(\beta H(\zeta) + \frac{1}{3!}\beta^3 H(\zeta)^3 + \cdots\right) . \tag{7.7}$$

Since the coefficients $n^{-1}c_n$ of $H(\zeta)$ are nonnegative the function on the right side of (7.7) has coefficients $\geqq 2\beta n^{-1} c_n$. Hence

$$-a_{n-1} + b_{n+1} e^{\beta\eta} \geqq 2\beta n^{-1} c_n$$

and therefore

$$n\,|a_{n-1}| \geqq 2\beta c_n - n\,|b_{n+1}|\, e^{\beta\eta} , \tag{7.8}$$

$$n\,|b_{n+1}| \geqq (2\beta c_n - n\,|a_{n-1}|) e^{-\beta\eta} . \tag{7.9}$$

By Theorems 6 and 8 we have $n\,|a_{n-1}| \leqq 2\beta$ and $n\,|b_{n+1}| \leqq 2\beta$. Therefore (7.8) and (7.9) together with (7.1) give

$$\limsup_{n\to\infty} n\,|a_n| \geqq 2\beta(2 - \delta - e^{\beta\eta}) = 2\beta(1 - \varepsilon/3) > 2\beta - \varepsilon ,$$

$$\limsup_{n\to\infty} n\,|b_n| \geqq 2\beta(1-\delta)e^{-\beta\eta} \geqq 2\beta(1 - \varepsilon/6)/(1 + \varepsilon/6) > 2\beta - \varepsilon .$$

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