ON HARMONIC FUNCTIONS OF FOUR VARIABLES WITH RATIONAL p_4-ASSOCIATES*

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1. Introduction. In this note we shall investigate the solutions of the four dimensional Laplace equation,

$$\Box H \equiv \sum_{\nu=1}^{4} H_{x_\nu x_\nu} = 0 , \tag{1}$$

by means of the integral operator approach as developed by S. Bergman and some others ([1] [2] [3] [4] [6] [7] [8] [9] [10] [11]). In particular, we shall use the operator which transforms analytic functions of three complex variables into solutions of (1) [7] [10].

$$H(X) = p_4(f;\mathscr{D};X^\circ) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} f(\tau, \eta, \xi)\frac{d\xi}{\xi}\frac{d\eta}{\eta} , \tag{2}$$

where

$$\tau = x_1\left(1 + \frac{1}{\eta\xi}\right) + ix_2\left(1 - \frac{1}{\eta\xi}\right) + x_3\left(\frac{1}{\xi} - \frac{1}{\eta}\right) + ix_4\left(\frac{1}{\xi} + \frac{1}{\eta}\right) ,$$

$$\| X - X^\circ \| < \varepsilon,\ X \equiv (x_1, x_2, x_3, x_4),\ X^\circ \equiv (x_1^0, x_2^0, x_3^0, x_4^0)$$

where $\mathscr{D} \equiv C \times \Gamma$ is the product of a contour C in the ξ-plane and a contour Γ in the η-plane, and $\varepsilon > 0$ is taken to be sufficiently small. The domain $\mathscr{D}$ is further restricted for a particular choice of $f(\tau, \eta, \xi)$ so that the integrand is *absolutely integrable* [3] [13]; in this case the double integral may be regarded as an iterated integral, and the orders of integration may be interchanged.[1] The function $f(\tau, \eta, \xi)$ is called the p_4-associate of $H(X)$.

The operator $p_4(f)$ was first introduced by *R.* Gilbert [7]; however, certain improvements in the notation, which are employed here are due to E. Kreyszig [10]. Kreyszig has also obtained an inverse operator for $p_4(f)$, and investigated in detail the representation of harmonic polynomials generated by this operator.

In order to understand how the operator p_4 transforms analytic functions into harmonic functions it is useful to consider the powers of τ, which act as generating functions for the homogenous, harmonic

Received March 3, 1962. This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AFOSR 62-454.

[1] It is also possible to give a meaning to p_4 in the case where the integrand is not absolutely integrable, but one of the iterated integrals exists.

polynomial [5] [7] [10][2]

$$\tau^n = \left[x_1\left(1+\frac{1}{\eta\xi}\right)+ix_2\left(1-\frac{1}{\eta\xi}\right)+x_3\left(\frac{1}{\xi}-\frac{1}{\eta}\right)+ix_4\left(\frac{1}{\xi}+\frac{1}{\eta}\right)\right]^n$$
$$=\left[Y+Z\frac{1}{\xi}+Z^*\frac{1}{\eta}+Y^*\frac{1}{\eta\xi}\right]^n=\sum_{k_1l=0}^{n}H_n^{k,l}(X)\xi^{-k}\eta^{-l}\ ,$$

where

(3) $$H_n^{k\ l}(X)\equiv H_n^{k,l}(x_1,x_2,x_3,x_4)=H_{nkl}(Y,Y^*,Z,Z^*)\ ,$$

and

$$Y=x_1+ix_2,\ Z=x_3+ix_4$$
$$Y^*=x_1-ix_2,\ Z^*=-(x_3-ix_4)\ .$$

The $H_n^{k\ l}(X)$ are linearly independent polynomials, which form a complete system. From (3) it is clear then that there are just $(n+1)^2$ independent, homogeneous, harmonic polynomials of degree n. These polynomials have an integral representation (in view of (3))

(4) $$H_n^{k,l}(X)=-\frac{1}{4\pi^2}\int_{|\xi|=1}\int_{|\eta|=1}\tau^n\eta^{k-1}\xi^{l-1}d\eta d\xi\ ,$$

where k,l are integers from 0 to n. Because of this representation it is clear that we must consider a special class of analytic functions $\{f(\tau,\eta,\xi)\}$, which are transformed into harmonic functions $H(X)$. For instance, as Kreyszig points out both the functions

(5) $$\tilde{f}(\tau,\eta,\xi)=\sum_{n,m,p=0}^{\infty}a_{nmp}\tau^n\eta^m\xi^p\ ,$$

and

(6) $$f(\tau,\eta,\xi)=\sum_{n=0}^{\infty}\sum_{m,p=0}^{n}a_{nmp}\tau^n\eta^m\xi^p\ ,$$

are transformed into the harmonic function

(7) $$H(X)=\sum_{n=0}^{\infty}\sum_{m,p=0}^{n}a_{nmp}H_n^{mp}(X)\ .$$

Following the notation of Bergman [4], Kreyszig [10] refers to (6) as the *normalized associated function of $H(Z)$ with respect to p_4*. Kreyszig [10] also give an inverse operator for p_4 (which is similar to Bergman's [4] inverse of p_3), that maps $H(Z)$ back onto its normalized associate $f(\tau,\eta,\xi)$.

[2] The introduction of the variables Y, Y^*, Z, Z^* is due to Kreyszig. In this form Laplace's equation may be written as $H_{yy^*}=H_{zz^*}$.

(8) $$f(\tau, \eta, \xi) = p_4^{-1}(H) \equiv \int_0^1\int_0^1 (\tau(\tau\tilde{H}]_\tau)_\tau d\alpha\, d\beta \,,$$

where the subscripts τ denote partial derivatives,

$$\tilde{H} = H^0(\tau(1-\alpha)(1-\beta), \tau\xi\beta(1-\alpha), \tau\eta\alpha(1-\beta)) \,,$$

and $H^0(Y, Z, Z^*)$ is $H(Y, Y^*, Z, Z^*)$ restricted to the set

$$E\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \,.$$

2. A class of harmonic vectors in four variables. It is possible to introduce an integral operator which generates a class of harmonic vectors $\vec{u} \equiv (u_1, u_2, u_3, u_4)$ (where $\Box u_k = 0$), from analytic functions of three complex variables. Let $\{f(\tau, \eta, \xi)\}$ be the class of analytic function described earlier, and let us define the components of $\vec{u}$ as follows,

(9)
$$\begin{aligned}
u_1 &= -\frac{1}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1} f(\tau, \eta, \xi)\Big(1 + \frac{1}{\eta\xi}\Big)\frac{d\eta}{\eta}\frac{d\xi}{\xi} \,,\\
u_2 &= -\frac{i}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1} f(\tau, \eta, \xi)\Big(1 - \frac{1}{\eta\xi}\Big)\frac{d\eta}{\eta}\frac{d\xi}{\xi} \,,\\
u_3 &= -\frac{1}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1} f(\tau, \eta, \xi)\Big(\frac{1}{\xi} - \frac{1}{\eta}\Big)\frac{d\eta}{\eta}\frac{d\xi}{\xi} \,,\\
u_4 &= -\frac{i}{4\pi^2}\int_{|\eta|=1}\int_{|\xi|=1} f(\tau, \eta, \xi)\Big(\frac{1}{\xi} + \frac{1}{\eta}\Big)\frac{d\eta}{\eta}\frac{d\xi}{\xi} \,,
\end{aligned}$$

then it may be shown that $\vec{u}$ satisfies the four dimensional analogue of the vanishing of the curl and divergence. (This property is simiar to that given by Prof. Bergman [3] in the case of three dimensional harmonic vectors). By direct computation it follows that $\sum_{\nu=1}^4 \partial u_\nu/\partial x_\nu = 0$. As a generalization of the curl of $\vec{u}$, we introduce the skew-symmetric tensor

(10) $$P_{mn} = \varepsilon_{mnrs}\frac{\partial u_r}{\partial x_s} \,,$$

where ε_{mnrs} is a permutation symbol, and we are using the summation convention for repeated indices. The components of the four-vector $\vec{u}$ may be expressed as

(11) $$u_r(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} f(\tau, \eta, \xi) N_r(\eta, \xi)\frac{d\eta}{n}\frac{d\xi}{\xi} \,,$$

where N_r is the rth component of

(12) $$\vec{N} \equiv \Big(1 + \frac{1}{\eta\xi}, i\Big(1 - \frac{1}{n\xi}\Big), \frac{1}{\xi} - \frac{1}{n}, i\Big(\frac{1}{\xi} + \frac{1}{\eta}\Big)\Big) \,;$$

consequently, $\partial u_r/\partial x_s$ has the representation

$$\frac{\partial u_r}{\partial x_s} = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} f_{x_s}(\tau, \eta, \xi) N_r \frac{dn}{n}\frac{d\xi}{\xi} \tag{13}$$
$$= -\frac{1}{4\pi^2}\iint_{\mathscr{D}} f_\tau(\tau, \eta, \xi) N_r N_s \frac{d\eta}{\eta}\frac{d\xi}{\xi},$$

since τ may be written as the scalar product of $\vec{X}$ and $\vec{N}$. It follows from (13) that

$$P_{mn} \equiv 0 . \tag{14}$$

The class of harmonic vectors, whose components are defined by (11) play an interesting role in the development of a residue calculus for harmonic functions of four variables. This aspect will be presented in a later paper.

3. Integral representations for harmonic functions with rational associates. The introduction of the operator p_4 allows a simple method for constructing harmonic functions with standard singularities. For example, let us suppose that the p_4-associate $f(\tau, \eta, \xi)$ of $H(X)$ is a rational function, that is $f(\tau, \eta, \xi) = p(\tau, \eta, \xi)/q(\tau, \eta, \xi)$ where p and q are polynomals. It is convenient for some of our formulae and no real loss in generality to assume further, that $q(\tau, \eta, \xi) = \tau - \phi(\eta, \xi)\eta^{-1}\xi^{-1}$. In order to investigate the harmonic, function-element

$$H(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{p(\tau, \eta, \xi)}{\tau\eta\xi - \phi(\eta, \xi)} d\eta d\xi , \tag{15}$$

and the connection between the branches of the whole harmonic function it is useful to consider the singularity manifold of the integrand,

$$Z^4 \equiv E\{\tau\xi\eta - \phi(\eta, \xi) = 0\} \tag{16}$$
$$\equiv E\left\{Y\eta\xi + Z\eta + Z^*\xi + Y^* - \phi(\eta, \xi) \equiv \sum_{\nu=0}^{n} \phi_\nu(X;\eta)\xi^\nu = 0\right\},$$

where n is the degree of $\phi(\eta, \xi)$ in ξ. Alternately, one might represent the singularity manifold as

$$Z^4 \equiv E\{\xi = A_\nu(X;\eta);\ \nu = 1, 2, \cdots, \eta\} ,$$

where the $A_\nu(X; \eta)$ are algebraic functions of X and η. We choose an initial point X^0 (about which we define our harmonic function element) and a domain of integration $\mathscr{D}$, such that

$$E\{\xi = A_\nu(X^0; \eta)\} \cap \mathscr{D} = 0, \text{ for } \nu = 1, 2, \cdots, n .$$

Suppose $\mathscr{D} = C \times \Gamma$ (where C, Γ lie in the ξ, η planes respectively), and let η^0 be a fixed value of $\eta \in \Gamma$. Furthermore, let us suppose that the denominator $\eta^0 \xi q(\tau, \eta^0, \xi) \equiv Q(X; \eta^0, \xi)$ vanishes for μ roots, $\xi_1, \xi_2, \cdots, \xi_\mu$ inside of C, that as η varies about Γ the μ roots do not cross over or meet C, and as η returns to η^0 after a circuit over Γ the new roots $\xi_{k_1}, \xi_{k_2}, \cdots, \xi_{k_\mu}$ are simply a permutation of the $\xi_1, \xi_2, \cdots, \xi_\mu$. In this case the integral may be evaluated as follows [12]

$$(17) \qquad -\frac{1}{4\pi}\int_c \frac{p(\tau^0, \eta^0, \xi)}{q(\tau^0, \eta^0, \xi)} \frac{d\xi}{\eta^0 \xi} = -\frac{1}{4\pi^2}\int_c \frac{P(X^0; \eta^0, \xi)}{Q(X^0; \eta^0, \xi)} d\xi$$
$$= \frac{1}{2\pi i}\left(\frac{P(X^0; \eta^0, \xi)}{Q_\xi(X^0; \eta^0, \xi)} + \cdots + \frac{P(X^0; \eta^0, \xi_\mu)}{Q_\xi(X^0; \eta^0, \xi_\mu)}\right),$$

and hence one may write

$$(18) \quad H(X^0) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{P(X^0; \eta, \xi_1(\eta))}{Q(X^0. \eta, \xi_1(\eta))} + \cdots + \frac{P(X^0; \eta, \xi_\mu(\eta))}{Q(X^0; \eta, \xi_\mu(\eta))}\right) d\eta$$

$$(19) \quad = \frac{1}{2\pi i}\int_{\gamma \sim \mu\Gamma} \frac{P(X^0; \eta, \xi(n))}{Q(X^0; \eta, \xi(\eta))} d\eta = \frac{1}{2\pi i}\int_\gamma \frac{P(X^0; \eta, \xi(\eta))\, d\eta}{Y^0\eta + Z^{0*} - \phi_\xi(\eta, \xi(\eta))},$$

where the individual terms in (18) correspond to different circuits about Γ in (19). This expression is thus seen to be equivalent to a period of an Abelian integral [14].

We next consider the domain of definition of the function element defined by $p_4((p/q), \mathscr{D}, X^0)$. Certainly

$$(20) \quad H^0(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{P(X; \eta, \xi)}{Q(X; \eta, \xi)} d\xi\, d\eta = \frac{1}{2\pi i}\int_\gamma \frac{P(X; \eta, \xi) d\eta}{Y\eta + Z^* - \phi_\xi(\eta, \xi)},$$

will be valid for all points X, which may be reached by continuation along a contour $\mathscr{L}(X)$ originating at X^0 provided that at no time a point of $\mathscr{L}(X)$ corresponds to a singularity of $P(X; \eta, \xi)/Q(X; \eta, \xi)$ on $\mathscr{D}$. The domain of definition $\mathscr{R}_0^4$ of the function element is then seen to contain at least the points $\tilde{X}$ whic may be reached from X^0 along $\mathscr{L}(\tilde{X})$ such that

$$(21) \qquad E\{X \in \mathscr{L}(\tilde{X})\} \cap E\{Q(X; \eta, \xi) = 0; (\eta, \xi) \in \mathscr{D}\} = 0 .$$

Certainly $\mathscr{R}_0^4$ may be enlarged to include a point X^1 on the set $\Delta^4 \equiv E\{Q(X; \eta, \xi) = 0; (\eta, \xi) \in \mathscr{D}\}$ if it is possible to deform $\mathscr{D}$ continuously so as to not pass over a point of

$$(22) \qquad E\{X \in \mathscr{L}(X^1) \subset \mathscr{R}_0^4\} \cap E\{Q(X; \eta, \xi) = 0\} .$$

However, there are instances where the integral representation (20) for $H^0(X)$ is defined for $X^1 \in \Delta^4$, and in these cases it may be possible to continue the function element $H^0(X)$ along a contour $\mathscr{L}(X)$ which passes through X^1. This is the case, when the intersection $E\{Q(X^1; \eta, \xi) = 0\} \cap \mathscr{D}$ consists only of isolated points and these points are inessential singularities of the first kind for $P(X^1; \eta, \xi)/Q(X^1; \eta, \xi)$ whose pole-like behavior is of order one.[3] [13] We are now in a position to consider the representation

$$H^0(X) = -\frac{1}{4\pi}\iint_{\mathscr{D}} \frac{P(X; \eta, \xi)}{Q(X; \eta_1 \xi)} d\xi d\eta, \quad \mathscr{D} = C \times \Gamma_1 ,$$

under general circumstances. As before we assume that the singularity manifold $Z^4 \equiv E\{\xi = A_\nu(X; \eta); \nu = 1, 2, \cdots, \eta\}$ has n branches; then for a particular $\eta^0 \in \Gamma$ there will be μ roots $\xi_1, \xi_2, \cdots,$ inside C and $n - \mu$ roots $\xi_{u+1}, \xi_{u+2}, \cdots, \xi_n$ outside C. If the discriminant,

$$\prod_{0 \leqq \mu < \nu \leqq n} [A_\nu(X; \eta^0) - A_\mu(X; \eta^0)] \neq 0 \tag{23}$$

the branches of the singularity manifold are unique. Now if X is some point in a neighborhood of X^0, $N(X^0)$, such that

$$X \notin B^3 \equiv E\{\prod_{0 \leqq \mu < \nu \leqq n} [A_\nu(X; \eta) - A_\mu(X; \eta)] = 0; \eta \in \Gamma\} , \tag{24}$$

then as η transcribes Γ the branches $\xi_\mu = A_\mu(X; \eta)$ move in the ξ-plane and may cross C, but no point $\xi_u \in C$ can be more than a first order pole of the integrand. Furthermore, if $\mathscr{D} \equiv C \times \Gamma$ has been chosen such that $C \cap E\{\xi_\mu = A_\mu(X; \eta); \eta \in \Gamma\}$ consists only of isolated points the integral (20) is defined, and we may write

$$H^0(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{P(X; \eta, \xi)}{Q(X; \eta, \xi)} d\xi d\eta = \frac{1}{2\pi i}\sum_{\mu=1}^{n}\int_{\Gamma_\mu} \frac{P(X; \eta, \xi_\mu(n))}{Q_\xi(X; \eta, \xi_\mu(n)}d\eta \tag{25}$$

where Γ_μ is that subset of Γ for which $\xi_\mu = A_\mu(X; \eta)$ lies inside of C.

4. Illustrations of integral representations.

EXAMPLE 1. Let us consider the double integral

$$\begin{aligned} H(X) &= -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{d\xi d\eta}{Y\eta\xi + Z\eta + Z^*\xi + Y^* - \alpha} \\ &\equiv -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{d\xi d\eta}{\tau\xi\eta - \alpha} , \end{aligned} \tag{26}$$

where α is a complex constant, $\mathscr{D} = C \times \Gamma$ and C, Γ are unit circles in the ξ, η planes respectively. $H(X)$ may be rewritten as the

[3] If these points are inessential singularities of the second kind they do not have a pole-like behavior.

iterated integral,

$$H(X) = -\frac{1}{4\pi^2}\int_{\Gamma}\frac{d\eta}{Y\eta + Z^*}\int_{C}\frac{d\xi}{\left(\xi + \dfrac{Z\eta + Y^* - \alpha}{Y\eta + Z^*}\right)}, \tag{27}$$

which is absolutely integrable providing, that the linear transformation

$$\xi(\eta) \equiv -\frac{Z\eta + (Y^* - \alpha)}{Y\eta + Z^*}, \quad \text{(for } \Delta \equiv ZZ^* - YY^* + Y\alpha \neq 0) \tag{28}$$

does not map the unit circle $|\eta| = 1$ into itself. This may be seen to happen only when $Y = \frac{1}{2}\bar{\alpha}$, if x_1, x_2, x_3, x_4 are taken to be real. One may readily evaluate integral (27) by realizing how the transformation (28) maps the unit circle $|\eta| = 1$. We distinguish three case:

$$\begin{aligned} &\text{(i)} && |\xi(\eta)| \leqq 1 \quad \text{for all } \eta \in \Gamma \equiv \{|\eta| = 1\} \\ &\text{(ii)} && |\xi(\eta)| \geqq 1 \quad \text{for all } \quad '' \quad '' \\ &\text{(iii)} && |\xi(\eta)| < 1 \quad \text{for some subset } \tilde{\Gamma} \subset \Gamma, \text{ and } |\xi(\eta)| \geqq 1 \end{aligned} \tag{29}$$

for η a point of the complement of $\tilde{\Gamma}$ with respect to Γ. Since, (28) is a linear transformation it maps circles into circles, and if the image circle of $|\eta| = 1$ touches $|\xi| = 1$ at all it must do so in two points or be tangent to $|\xi| = 1$. If it touches in two points we shall call these points $\xi(\eta_1)$, $\xi(\eta_2)$, and they will be the images of η, η_2 going around $|\eta| = 1$ in a positives sense. In case (i) the integral may be evaluated as follows,

$$H(X) = \frac{1}{2\pi i}\int_{\Gamma}\frac{d\eta}{Y\eta + Z^*} = \begin{cases} \dfrac{1}{Y}, \text{ if } \left|\dfrac{Z}{Y}\right| < 1, \\ 0, \text{ if } \left|\dfrac{Z}{Y}\right| > 1, \\ \text{undefined, if } \left|\dfrac{Z}{Y}\right| = 1. \end{cases} \tag{30}$$

In case (ii) $H(X) \equiv 0$, and in case (iii)

$$\begin{aligned} H(X) &= \frac{1}{2\pi i}\int_{\tilde{\Gamma}}\frac{d\eta}{Y\eta + Z^*} = \frac{1}{2\pi i Y}\int_{\eta_1 |\eta|=1}^{\eta_2}\frac{d\eta}{\eta + Z^*/Y} \\ &= \frac{1}{2\pi i}\log\left\{\frac{Y\eta_2 + Z^*}{Y\eta_1 + Z^*}\right\}, \end{aligned} \tag{31}$$

providing that $\bar{Z}/Y \notin \tilde{\Gamma}$.

EXAMPLE 2. Picard and Simart [14] give some interesting cases of double integrals with higher than first order inessential singularities, and evaluate the residue by using infinitesimal domains of integration. We shall apply some of these methods below to evaluate integral representations for harmonic functions. Let us consider the harmonic function element

$$\tag{32} \begin{aligned} H(X) &= -\frac{1}{4\pi^2}\iint_{\mathscr{D}}\frac{d\xi d\eta}{[\tau\xi\eta + f(\xi,\eta)][\tau\xi\eta + g(\xi,\eta)]} \\ &= \frac{-1}{4\pi^2}\iint_{\mathscr{D}}\Big[\frac{1}{[Y\eta\xi + Z\eta + Z^*\xi + Y^* + f(\xi,\eta)]}\cdot \\ &\quad \cdot\frac{d\xi d\eta}{[Y\eta\xi + Z\eta + Z^*\xi + Y^* + g(\xi,\eta)]}\Big], \end{aligned}$$

where $\mathscr{D}$ shall be specified below.

For a fixed $X \equiv (Y, Y^*, Z, Z^*)$, let us assume that

$$\tag{33} \begin{aligned} &\text{(i)} \quad Y\eta\xi + Z\eta + Z^*\xi + Y^* + f(\xi,\eta) = 0\,, \\ &\text{(ii)} \quad Y\eta\xi + Z\eta + Z^*\xi + Y^* + g(\xi,\eta) = 0\,, \end{aligned}$$

are two curves which intersect in a simple manner at the point $\xi = \alpha, \eta = \beta$. We now choose a suitably small contour Γ about β, such that for $\eta \in \Gamma$ there correspond points $\xi_1(\eta), \xi_2(\eta)$ near α, which satisfy equations 33(i), (ii) respectively. We choose for the contour C a suitably small circle about $\xi_1(\eta)$ such that as η traverses $\Gamma, \xi_1(\eta)$ remains inside C and $\xi_2(\eta)$ remains outside. One then obtains from the ξ-integration,

$$\begin{aligned} &-\frac{1}{4\pi^2}\int_C\frac{d\xi}{(\gamma\xi\eta + f)(\tau\xi\eta + g)} \\ &= \frac{1}{2\pi i}\frac{1}{[Y\eta + Z^* + f_\xi(\xi,\eta)][Y\eta\xi_1 + Z\eta + Z^*\xi_1 + Y^* + g(\xi_1,\eta)]}\,. \end{aligned}$$

With the η-integration we have

$$\begin{aligned} H(X) = &\frac{1}{Y\eta + Z^* + f_\xi(\xi_1,\eta)}\cdot \\ &\frac{1}{\left[Y\xi_1 + Z + g_\eta(\xi_1,\eta) + \dfrac{d\xi_1}{d\eta}(Y\eta + Z^* + g_\xi(\xi_1,\eta))\right]} \end{aligned}$$

where

$$\frac{d\xi_1}{d\eta} = -\frac{Y\xi_1 + Z + f_\eta(\xi_1,\eta)}{Y\eta + Z^* + f_\xi(\xi_1,\eta)}\,,$$

from which it follows that,

$$H(X) = \frac{1}{(Y\eta + Z^* + f_\xi)(Y\xi_1 + Z + g_\eta) - (Y\xi_1 + Z + f_\eta)(Y\eta + Z^* + g_\xi)}\Bigg|_{\substack{\xi=\alpha \\ \eta=\beta}}$$

$$(34) \qquad H(X) = \Bigg[\frac{1}{(Y\beta + Z^* + f_\alpha(\alpha, \beta))(Y\alpha + Z + g_\beta(\alpha, \beta))} \cdot \\ \cdot \frac{1}{-(Y\alpha + Z + f_\beta(\alpha, \beta))(Y\beta + Z^* + g_\alpha(\alpha, \beta))}\Bigg] .$$

EXAMPLE 3. As another illustration we consider the integral

$$(35) \qquad H(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{d\xi d\eta}{\tau\eta\xi + f(\eta, \xi)} \\ \equiv -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{d\xi d\eta}{Y\eta\xi + Z\eta + Z^*\xi + Y^* + f(\eta, \xi)}$$

where $Y\eta\xi + Z\eta + Z^*\xi + Y^* + f(\eta, \xi) = 0$ has a double point at $\xi = \alpha, \eta = \beta$. As before, we choose a suitably small contour Γ such that for $\eta \in \Gamma$ there correspond two roots $\xi_1(\eta), \xi_2(\eta)$, with ξ_1 inside and ξ_2 outside of C; one has then

$$(36) \quad H(X) = \frac{1}{2\pi i}\int_\Gamma \frac{d\eta}{Y\eta + Z^* + f_\xi(\xi_1[\eta], \eta)} \\ = \left[\frac{1}{\dfrac{d\xi_1}{d\eta} f_{\xi\xi}(\xi_1[\eta], \eta) + Y + f_{\xi\eta}(\xi_1[\eta], \eta)}\right]_{\substack{\xi=\alpha \\ \eta=\beta}} \\ = \frac{1}{\sqrt{[Y - f_{\xi\eta}(\xi_1[\eta], \eta)]^2 - f_{\xi\xi}(\xi_1[\eta], \eta) f_{\eta\eta}(\xi_1[\eta], \eta)}}\Bigg|_{\substack{\xi=\alpha \\ \eta=\beta}}$$

since $d\xi_1[\eta]/d\eta$ satisfies the equation

$$(37) \qquad \left(\frac{d\xi_1}{d\eta}\right)^2 f_{\xi\xi} + 2(Y + f_{\xi\eta})\frac{d\xi_1}{d\eta} + f_{\eta\eta} = 0 .$$

5. Singularities of harmonic functions with Rational p_4 associates. In an earlier paper [7] this author proved the following theorem.

THEOREM 0. *Let $Z^4 \equiv E\{S(X; \eta, \xi) = 0\}$ be the singularity manifold of $(1/n\xi)f(\tau, \eta, \xi)$,* then

$$H(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} f(\tau, \eta, \xi)\frac{d\eta}{\eta}\frac{d\xi}{\xi}$$

is regular at $X \in C^4$ (where C^4 is complex, four-dimenional space) providing

$$X \notin E[S(X; \eta, \xi) = 0\} \cap E\left\{\frac{\partial}{\partial\xi}S(X; \eta, \xi) + \pi'(\eta)\frac{\partial}{\partial\xi}S(X; \eta, \xi) = 0\right\} ,$$

where $\xi = \pi(\eta)$ *is an arbitrary analytic function of* η.

The set of points, however, which are contained in the above intersection may consist of a four dimensional region. Consequently, this is hardly a restriction on the possible singularities of $H(X)$. In order to locate the singularities more accurately we consider the case where $f(\tau, \eta, \xi)$ is a rational function, that is

$$\frac{1}{\eta\xi} f(\tau, \eta, \xi) = \frac{p(\tau, \eta, \xi)}{q(\tau, \eta, \xi)} = \frac{P(X; \eta, \xi)}{Q(X; \eta, \xi)} ,$$

where P and Q are polynomials. Now, a singularity of the interand occurs for a value of $X \in E\{Q(X; \eta, \xi) = 0\}$. If $\mathscr{D} = C \times \Gamma$, then for a fixed $\eta^0 \in \Gamma$, $Q(X; \eta^0, \xi)$ has the decomposition

$$(38) \quad Q(X; \eta^0, \xi) = [\xi - A_1(X; \eta^0)]^{m_1}[\xi - A_2(X; \eta^0)]^{m_2} \cdots [\xi - A_r(X; \eta^0)]^{m_r} ,$$

where the m_k are integers $\geqq 0$, and $m_1 + m_2 + \cdots + m_r =$ degree of ξ in Q. The criteria for an $m_k > 1$ is that $\partial Q/\partial\xi = 0$, for some $\xi = A_k(X; \eta^0)$; this is equivalent to a multiple pole singularity of the integrand.

Since, $\xi = \pi(\eta)$ is an analytic function of η, and furthermore, since only poles of order equal to or greater than two are non-integrable on $\mathscr{D}$, X is a regular point providing

$$(39) \quad X \notin E\{Q(X; \eta, \xi) = 0\} \cap E\left\{\frac{\partial Q}{\partial\eta} + \pi'(\eta)\frac{\partial Q}{\partial\xi} = 0\right\} \cap E\left\{\frac{\partial Q}{\partial\xi}\right\}$$
$$\equiv E\{Q(X; \eta, \xi) = 0\} \cap E\left\{\frac{\partial Q}{\partial\eta} = 0\right\} \cap E\left\{\frac{\partial Q}{\partial\xi} = 0\right\} .$$

By interchanging the roles played by η and ξ in the above theorem (either may be considered independent), and by considering the decomposition

$$Q(X; \eta, \xi^0) = [\eta - B_1(X; \xi^0)]^{k_1} \cdots [\eta - B_s(X; \xi^0)]^{k_s}$$

(where the k_j are integers $\geqq 0$, and $k_1 + k_2 + \cdots + k_s =$ degree η, $\xi^0 \in C$ is fixed) one sees that $\eta = B_l(X; \xi^0)$ is a multiple pole of the integrand if and only if $\{(\partial/\partial\eta)Q(X; \eta, \xi^0)\} \cap \{\eta = B_l(X; \xi^0)\} = 0$. In this case we realize that X is a regular point providing it does not lie on

$$E\{Q = 0\} \cap E\left\{\lambda'(\xi)\frac{\partial Q}{\partial\eta} + \frac{\partial Q}{\partial\xi} = 0\right\} \cap E\left\{\frac{\partial Q}{\partial\eta} = 0\right\}$$
$$\equiv E\{Q = 0\} \cap E\left\{\frac{\partial Q}{\partial\xi} = 0\right\} \cap E\left\{\frac{\partial Q}{\partial\eta} = 0\right\} ,$$

since $\eta = \lambda(\xi)$ is analytic. From this we have the following result.

THEOREM 1. *Let*

$$\frac{1}{\eta\xi}f(\tau, \eta, \xi) \equiv \frac{P(X; \eta, \xi)}{Q(X, \eta, \xi)}$$

be a rational function of τ, η, ξ, then

$$H(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{P(X; \eta, \xi)}{Q(X, \eta, \xi)} d\eta d\xi$$

is regular at $X \in C^4$, *providing*

$$X \notin E\{Q(X; \eta, \xi) = 0\} \cap E\left\{\frac{\partial Q}{\partial \eta} = 0\right\} \cap E\left\{\frac{\partial Q}{\partial \xi} = 0\right\}.$$

To illustrate this result let us consider the representation

$$H(X) = -\iint_{\mathscr{D}} \frac{d\eta d\xi}{(\tau\eta\xi - \alpha)^2}.$$

In this case we may take the singularity manifold to be

$$Z^4 \equiv E\{\tau\eta\xi - \alpha \equiv Y\eta\xi + Z\eta + Z^*\xi + Y^* - \alpha = 0\}.$$

We may eliminate by computing the intersection

$$E\{Y\eta\xi + Z\eta + Z^*\xi + Y^* - \alpha = 0\} \cap E\{Y\xi + Z = 0\} \cap E\{Y\eta + Z^* = 0\}$$
$$\equiv E\{YY^* - ZZ^* = \alpha Y\}.$$

If x_1, x_2, x_3, x_4 are real this is seen to become

$$E\{(x_1 - a)^2 + (x_2 - b)^2 + x_3^2 + x_4^2 = a^2 + b^2\} \cap E\{bx_1 + ax_2 = 0\},$$
$$\frac{\alpha}{2} = a + ib.$$

The proceding theorem also gives us an insight into the singularities of the harmonic function

$$H(X) = -\frac{1}{4\pi^2}\iint_{\mathscr{D}} \frac{d\eta d\xi}{\tau\eta\xi - \alpha} = -\frac{1}{4\pi^2}\int_{\Gamma} \frac{d\eta}{Y\eta + Z^*}\int_{C} \frac{d\xi}{\xi + \dfrac{Z\eta + Y^* - \alpha}{Y\eta + Z^*}},$$

which we discussed in the last section. The linear transformation

$$\xi(\eta) = \frac{-Z\eta + \alpha - Y^*}{Y\eta + Z^*}$$

is a one-to-one mapping of the η-plane onto the ξ-plane if and only if $\Delta \equiv ZZ^* - YY^* + Y\alpha \neq 0$. If $\Delta = 0$, then this transformation reduces to a constant, that is the entire η-plane is mapped into

$\xi = (\alpha - Y^*)/Z^*$. The singularities of the integrand are at most first-order, pole-like, inessential singularities. It is clear then, that if $X^0 \notin E\{\Delta = 0\}$ there exists a domain $\mathscr{D} = C \times \Gamma$, for which the integral representation of $H(X^0)$ is defined, that is it is possible to choose $\mathscr{D}$ in such a manner that no line segment of $\mathscr{D}$ is contained in the set $E\{\tau\eta\xi - \alpha = 0; X = X^0\}$ $H(X)$ may be defined at points X in the neighborhood of X^0, $N(X^0)$ providing that $N(X^0) \subset E\{\Delta \neq 0\}$, and that no line segment of $\mathscr{D}$ is contanined in $S_{X \in N(X^0)} E\{\pi\eta\xi - \alpha = 0\}$. ($S$ denotes the topological sum.) Furthermore, $H(X)$ then may be continued to any point X, which can be reached by a contour $\mathscr{L}(X)$ originating at X^0, and which lies entirely within $E\{\Delta \neq 0\}$, providing no set of singularities for the integrand corresponds to a line segment of $\mathscr{D}$ for any value of $X \in \mathscr{L}(X)$. It is possible, however, to extend this region of definition for $H(X)$ to include other points X' if $\mathscr{D}$ may be continuously deformed so as to not have a line segment lie on $E\{\tau\eta\xi - \alpha = 0; X = X'\}$ for any stage of the deformation. Such a deformation is always possible if $X' \notin E\{\Delta = 0\}$. When $X' \in E\{\Delta = 0\}$, however, the entire η-plane is mapped onto $\xi = (\alpha - Y^*)/Z^*$ by $\xi(\eta)$, hence for all values of $\eta \in \Gamma$, $\xi = (\alpha - Y^*)/Z^*$ will be a singularity of the integrand in the ξ-plane. In this case it is impossible to deform $\mathscr{D}$ in a continuous manner so as to pass over the singularities of the integrand without it at some time having a line segment corresponding to $E\{\tau\eta\xi - \alpha = 0; X = X'\}$. With this we have obtained

Result 1. *The harmonic function*

$$H(X) = -\frac{1}{4\pi^2} \iint_{\mathscr{D}} \frac{d\eta d\xi}{\tau\eta\xi - \alpha}$$

is regular at X, *providing* $X \notin E\{ZZ^* - YY^* + Y\alpha = 0\}$[4].

6. An inverse for the operator. As mentioned earlier, Kreyszig [10] gave an inverse operator for p_4. (see expression 8). In an earlier paper the present author introduced an operator which was not an inverse for p_4, but did generate a function of three complex variables closely related to the normalized associate. This was done by using the orthonormal property of the spherical harmonics on the unit hypersphere. Since, there are distinct advantages to inverse operator of both types we shall develop an inverse for p_4 which

[4] It is interesting to consider the three categories, $|Z/Y| < 1, > 1, = 1$, of expression (30) in view of this result. On $E\{|Z|^2 + |Y|^2 - \alpha Y = 0\}$ (x_1, x_2, x_3, x_4 real) these categories become respectively $|Y| > |\alpha|/2, < |\alpha|/2, = |\alpha|/2$. $Y = \bar{\alpha}/2$, was seen to coincide with the case where $\xi(\eta)$ maps the unit circle into itself and this corresponds to the third category, $|Y| = |\alpha|/2$.

depends on the orthonormal property.

The variable τ used in this paper is somewhat different than the original variable

$$(40)\quad t = iy_1\Big(1 - \frac{1}{\xi\eta}\Big) - iy_2\Big(\frac{1}{\xi} + \frac{1}{\eta}\Big) + y_3\Big(\frac{1}{\xi} - \frac{1}{\eta}\Big) + y_4\Big(1 + \frac{1}{\eta\xi}\Big),$$

and was a modification of Kreysig's[5], which lead to a more unified presentation of formulae. The original variable t is, however, useful because of its connection with the surface harmonics, $S_n^{k,l}(\eta_1, \theta_2, \varphi)$, [5]

$$(41)\quad t^n = \sum_{k_1 l=0}^{n} \binom{n}{k} \rho^n S_n^{k,l}(\theta_1, \theta_2, \varphi)\eta^{-k}\xi^{-l}$$

where $\rho, \theta_1, \theta_2, \varphi$ are the hyperspherical coordinates defined by

$$(42)\quad \begin{aligned} y_1 &= \rho \cos\theta_1 , \\ y_2 &= \rho \sin\theta_1 \cos\theta_2 , \\ y_3 &= \rho \sin\theta_1 \sin\theta_2 \cos\varphi , \\ y_4 &= \rho \sin\theta_1 \sin\theta_2 \sin\varphi , \end{aligned}$$

and

$$\rho \geqq 0,\ 0 \leqq \theta_j \leqq \pi\ (j = 1, 2),\ 0 \leqq \varphi \leqq 2\pi .$$

We construct the kernel

$$(43)\quad K(\sigma\rho, \eta, \xi; \theta_j, \varphi) \equiv \sum_{n=0}^{\infty} \sum_{k_1 l=0}^{n} (n+1) \frac{\binom{n}{k}}{\binom{n}{l}} (\sigma\rho)^n S_n^{k,l}(\theta_j; \varphi)\eta^k\xi^l ,$$

which because of the orthogonality relations over the hypersphere,

$$(44)\quad \frac{1}{2\pi^2}\iiint_\Omega S_n^{k,l} S_n^{k',l'} d\Omega = \delta_{kk'}\delta_{ll'} \frac{\binom{\eta}{l}}{\binom{n}{k}} ,$$

(where $d\Omega = \sin^2\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\varphi$) may be used to generate to normalized, p_4-associate of a harmonic function $H(X)$. For instance, suppose

$$\begin{aligned} H(X^*) \equiv V(\rho, \theta_1, \theta_2, \varphi) &= \sum_{n=0}^{\infty} \sum_{k_1 l=0}^{n} a_{nkl} \binom{n}{k} \rho^n S_n^{k,l}(\theta_j; \varphi) \\ &= \sum_{n=0}^{\infty} \sum_{k\ l=0}^{n} a_{nkl} H_n^{*k,l}(X^*), \end{aligned}$$

[5] Kreyszig's form and mine are related by the permutation and reflection of coordinates, $X^*T = X$:

$$y_1 \to x_2,\ y_2 \to -x_4,\ y_3 \to x_3,\ y_4 \to X_1 .$$

where

$$H_n^{*k,l}(X^* T^{-1}) = H_n^{k,l}(X) ,$$

then one may generate the p_4-associate,

$$f(\sigma, \eta, \xi) = \sum_{n=0}^{\infty} \sum_{k,l=0}^{n} a_{nkl} \sigma^n \eta^k \xi^l$$

by the integral operator

$$f(\sigma, \eta, \xi) = \frac{1}{2\pi^2} \iiint_{\Omega} \overline{V(\rho, \theta_j, \varphi)} K(\sigma\rho; \eta, \xi; \theta_j, \varphi) d\Omega . \tag{45}$$

One may sum the series formally for the kernel $K(\sigma\rho, \eta, \xi; \theta_j, \varphi)$ as follows. The term

$$\frac{1}{\binom{\eta}{l}} = \frac{(n-l)!l!}{n!} = (n+1)\frac{\Gamma(n-l+1)\Gamma(l+1)}{\Gamma(n+2)}$$

may be replaced by

$$(n+1)\int_0^1 \zeta^{n-l}(1-\zeta)^l d\zeta .$$

Hence, one has

$$\begin{aligned} K(\sigma\rho, \eta, \xi; \theta_j, \varphi) &= \sum_{n=0}^{\infty} \sum_{k,l=0}^{n} \int_0^1 \zeta^{n-l}(1-\zeta)^l d\zeta \binom{n}{k} (n+1)^2 (\sigma\rho)^n S_n^{k,l}(\theta_j; \varphi) \eta^k \xi^l \\ &= \sum_{n=0}^{\infty} (n+1)^2 \int_0^1 d\zeta \Big[\sum_{k,l=0}^{n} (\zeta\sigma\rho)^n \Big(\frac{\zeta}{\xi[1-\zeta]}\Big)^{-l} \binom{n}{k} S_n^{k,l}(\theta_j; \varphi)\eta^k \Big] , \\ &= \sum_{n=0}^{\infty} (n+1)^2 \int_0^1 [s(\zeta)]^n d\zeta , \end{aligned}$$

where

$$[s(\zeta)]^n = \sum_{k,l=0}^{n} \binom{n}{k} r^n S_n^{k,l}(\theta_j, \varphi) \eta^k \lambda^{-l} , \tag{46}$$

and

$$r = \zeta\sigma\rho, \lambda = \frac{\zeta}{\xi(1-\zeta)} .$$

If $|s(\zeta)| < 1$, then we may interchange the orders of summation and integration, and formally sum the series. One has in this case

$$K(\sigma\rho, \eta, \xi; \theta_j, \varphi) = \int_0^1 \sum_{n=0}^{\infty} (n+1)^2 s^n d\zeta \tag{47}$$

$$= \int_0^1 \frac{d}{ds}\left(\frac{sd}{ds}\frac{1}{1-s}\right)d\zeta = \int_0^1 \frac{1+s}{(1-s)^3}d\delta .$$

If the representation (40) is denoted by $t(X^*; \eta, \xi)$ then we may express

$$s(\zeta) = t\left(X^*; \eta^{-1}, \frac{\zeta\xi^{-1}}{1-\zeta}\right)\zeta\sigma = \sigma[A(X^*)\zeta + B(X^*)] ,$$

where

$$\begin{aligned} A(X^*) &= iy_1(1+\eta\xi) + iy_2(\xi - \eta) - y_3(\xi + \eta) + y_4(1 - \eta\xi) , \\ B(X^*) &= \xi(-iy_1\eta - iy_2 + y_3 + y_4\eta) . \end{aligned} \tag{48}$$

When $|\sigma| < 1/|A\zeta + B|$ (i.e. $|s| < 1$), the expression for the kernel is itegrable and one obtains

$$K(\sigma\rho, \eta, \xi; \theta_j, \varphi) = -\frac{1}{\sigma^2}\int_0^1 \frac{A\zeta + B + \sigma^{-1}}{(A\zeta + B + \sigma^{-1})^3}d\zeta$$
$$= \frac{B(2\sigma^{-1} - A - 2B)}{\sigma^2(\sigma^{-1} - B)^2(\sigma^{-1} - B - A)^2} .$$

Hence we have the result.

THEOREM 2. *Let $H(X^*)$ be a harmonic function regular at the origin, and let $V(\rho, \theta_j, \varphi)$ be the function obtained by replacing y_1, y_2, y_3, y_4, by the hyperspherical polar coordinates then*

$$f(\sigma, \eta, \xi) = \frac{1}{2\pi^2}\iiint_\Omega \frac{\overline{V(\rho, \theta_j, \varphi)}B(2\sigma^{-1} - A - 2B)}{\sigma^2(\sigma^{-1} - B)^2(\sigma^{-1} - B - A)^2}d\Omega , \tag{50}$$

where the integration is over the unit hypersphere.

There are two particular uses for this type of inverse operator, (i) obtaining integral solutions to boundary value problems, and (ii) formulating necessary and sufficient criteria for singularities of $H(X^*)$.

Occasionally it is useful to extend the arguments of $H(X^*)$ to complex values; if we introduce the complex, hyperspherical, polar coordinates

$$\begin{aligned} u_1 &= \frac{y_1}{\rho} , \\ u_2 &= \frac{y_2}{\sqrt{\rho^2 - y_1^2}} , \\ v &= +\sqrt{\frac{y_3 + iy_4}{y_3 - iy_4}} , \\ \rho &= +\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} , \end{aligned} \tag{51}$$

which reduces to $u_1 = \cos\theta_1$, $u_2 = \cos\theta_2$, $v = e^{i\varphi}$, for y_1, y_2, y_3, y_4 real, we may represent $p_4^{-1}(H)$ as a Cauchy integral. Indeed, since $(1+s)(1-s)^3$ is analytic in u_1, u_2, v, ζ for $|s| < 1$, $K(\sigma\rho, \eta, \zeta \mid u_1, u_2, v)$ is analytic in u_1, u_2, v, $(u_j \neq \pm 1, v \neq 0)$ for $|\rho|$ sufficiently small. The integral (50) may then by expressed as a triple-Cauchy integral

$$(52)\quad f(\sigma, \eta, \xi) = \frac{i}{2\pi^2} \iiint_{R=\gamma_1\times\gamma_2\times\delta} \frac{U(\rho, u_1, u_2, v)B(2\sigma^{-1} - A - 2B)}{\sigma^2(\sigma^{-1} - B)^2(\sigma^{-1} - B - A)^2} \sqrt{1-u_1^2}\, du_1 du_2 \frac{dv}{v},$$

where the domain of integration is a product of contours, $\gamma_1 \times \gamma_2 \times \delta$, in the u_1, u_2, v-planes respectively; γ_1, γ_2 are paths joining $+1$ to-1, and δ is a closed loop encircling the origin in the v-plane.

Since (25) is a Cauchy integral we may deform the product of contours continuously providing we do not pass over a third-order pole of u_j, or v. (We note that the integrand has branch-point like singularities only at $u_j = \pm 1$, which corresponds to points on the boundary of $\mathscr{R}$). Let us start with an initial point of definition for $f(\sigma, \eta, \xi)$ say the origin $\eta = \xi = \sigma = 0$, and let us choose $\mathscr{R}$ such that $p_4^{-1}(H)$ is absolutely integrable. $f(\sigma, \eta, \xi)$ will then be defined in some neighborhood of the origin N_0 providing that all points $(\sigma, \eta, \xi) \equiv \Sigma \in N_0$ lie on a curve $\mathscr{L}(\Sigma)$ originating at the origin and such that no point of $\mathscr{L}(\Sigma)$ corresponds to a third order pole of the integrand on the domain of integration etc. Having thus established a domain of definition for $f(\sigma, \eta, \xi)$ we may extend this region by continuously deforming $\mathscr{R}$ according to the usual precautions. We recognize, however, that the singularities of the integrand are of a more complicated type than occur for the operator $p_4(f)$. For instance, there are singularities of the kernel, which move as we continue along a curve $\mathscr{L}(\Sigma)$, and there are the fixed singularities of the harmonic function $H(X^*)$. The singularities of the kernel are those points Σ, which lie on

$$(53)\qquad E\{\sigma^{-1} - B = 0\} \cup E\{\sigma^{-1} - B - A = 0\}\,.$$

Both A and B are linear in y_1, y_2, y_3, y_4, hence the zeros of $(\sigma^{-1} - B)$ and $(\sigma^{-1} - B - A)$ are first order for u_1, u_2, or v, where ever the transformation of coordinates (51) is a one-to-one, that is for $u_j \neq \pm 1$, $v \neq 0$. The kernel is then seen not to have a pole-like singularity of third order unless $A(X^*) = 0$, and in this case one has

$$(54)\qquad K(\sigma\rho, \eta, \xi \mid u_1, u_2, v) = \frac{2B(X^*)}{\sigma^2(\sigma^{-1} - B(X^*))^3}\,.$$

Now we may continue $f(\sigma, \eta, \xi)$ along $\mathscr{L}(\Sigma)$ as long as it is possible to deform $\mathscr{R}$ so that it does not cross over a third-order, pole-like singularity of (54). This may always be accomplished except when the singularities of (54) coincide with the fixed (third-order, pole-like) singularities of $H(X^*)$ [8] [6] [7] [12]. Hence, we have proved the following result.

THEOREM 3. *Let $H(X^*)$ be a meromorphic harmonic function regular at the origin, and let $U(\rho, u_1, u_2, v)$ be the function obtained by replacing y_1, y_2, y_3, y_4 by the complex, hyperspherical coordinates, then*

$$f(\sigma, \eta, \xi) = \frac{i}{2\pi^2}\iiint_{\mathscr{R}} \frac{U(\rho, u_1, u_2, u_3)B(2\sigma^{-1} - A - 2B)}{\sigma^2(\sigma^{-1} - B)^2(\sigma^{-1} - B - A)^2} \sqrt{1 - u_1^2}\, du_1 du_2 \frac{du_3}{u_3}$$

is regular at $\Sigma \equiv (\sigma, \eta, \xi)$ providing this point does not lie on

$$E\{A(X^*) = 0\} \cap E\{\sigma^{-1} - B(X^*)\} \cap \left[\bigcup_{j=1}^{3} E\left\{\frac{\partial G}{\partial u_j} = 0\right\} \cap E\left\{\frac{\partial^2 G}{\partial u_j^2} = 0\right\}\right],$$

where

$$U(\rho, u_1, u_2, u_3) = \frac{F(\rho, u_1, u_2, u_3)}{G(\rho, u_1, u_2, u_3)}$$

is a decomposition of U into entire functions.

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