A CONE OF SUPER-(L) FUNCTIONS

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1. Introduction. Bonsall [2] introduced the following generalization of the concept of a real-valued concave function of one real variable on a closed interval $[a, b]$, where $a < b$:

DEFINITION 1.1. Let y_1 and y_2 be arbitrary real numbers, and let x_1 and x_2 be real numbers such that $a \leqq x_1 < x_2 \leqq b$. Let $L(y) = d^2y/dx^2 + p(x)dy/dx + q(x)y = 0$ be such that there exists a unique solution F on $[a, b]$ (where the appropriate one-sided derivatives are used at the end-points) for which $F(x_i) = y_i$, $i = 1, 2$. Then a real-valued function f is *super*-(L) on $[a, b]$ if $f(x) \geqq F(f, x_1, x_2; x)$ for all x, x_1, and x_2 such that $a \leqq x_1 < x_2 \leqq b$ and $x_1 \leqq x \leqq x_2$, where $F(f, x_1, x_2; x)$ is the solution of $L(y) = 0$ such that $F(f, x_1, x_2; x_i) = f(x_i)$, $i = 1, 2$.

This definition is a special case of the generalized concave function introduced by Beckenbach [1].

In this paper, it will be shown that the set of non-negative continuous super-(L) functions on $[a, b]$ is a convex cone, and the extremal structure of the cone will be characterized. A result due to Choquet [4] will then be used to prove the existence of a type of integral representation for the elements of the cone in terms of the extremal elements of the cone. It will be assumed throughout this paper that the functions considered are continuous on $[a, b]$.

DEFINITION 1.2. Let A be a set in a real linear space. Then A is a *convex cone* if

(1) for every f and g in A and every nonnegative real number k, $f + g$ and kf belong to A, and

(2) f in A and $-f$ in A imply $f = 0$, the origin of the real linear space.

It is easy to check that if f and g are super-(L) functions on $[a, b]$ and k is a nonnegative real number, then kf and $f + g$ are super-(L) functions on $[a, b]$, and hence it follows that the set C of nonnegative super-(L) functions on $[a, b]$ forms a convex cone.

2. Extremal structure of C. McLachlan [5] has characterized the extremal structure of the convex cone of nonnegative concave functions on $[a, b]$. It will be shown in this section that the extremal structure of C is analogous to that obtained by McLachlan.

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DEFINITION 2.1. A real-valued function f on $[a, b]$ is *(L)-linear on* $[x_1, x_2]$ if $f(x) = F(f, x_1, x_2; x)$ for all x in $[x_1, x_2]$, where $a \leqq x_1 < x_2 \leqq b$.

LEMMA 2.1. *If f, f_1, and f_2 are super-(L) functions on $[a, b]$ such that $f(x) = f_1(x) + f_2(x)$ for all x in $[x_1, x_2]$, where $a \leqq x_1 < x_2 \leqq b$, and f is (L)-linear on $[x_1, x_2]$, then f_1 and f_2 are (L)-linear on $[x_1, x_2]$.*

The proof is straightforward and will be omitted.

DEFINITION 2.2. A real-valued function f on $[a, b]$ is an *(L)-conical* function with its vertex over w in $[a, b]$ if

(1) $f(w) > 0$,

(2) $f(a) = f(b) = 0$ if $w \neq a, b$; $f(a) = 0$ if $w = b$; or $f(b) = 0$ if $w = a$; and

(3) f is (L)-linear on $[a, w]$ and on $[w, b]$.

LEMMA 2.2 [2, p. 101]. *If f is super-(L) on $[a, b]$, then $f(x) \leqq F(f, x_1, x_2; x)$ for all x in $[a, x_1]$ and $[x_2, b]$.*

LEMMA 2.3. *If f in C is such that $f(x_0) = 0$ for some x_0 in (a, b), then $f = 0$.*

Proof. Suppose there exists an x' in $[a, b]$ such that $f(x') > 0$. There is no loss in generality in assuming that $x' < x_0$. Since f is super-(L), $F(f, a, b: x_0) \leqq f(x_0) = 0$. If $F(f, a, b; x_0) < 0$, then $F(f, a, b; b) \geqq 0$ and $F(f, a, b; a) \geqq 0$ imply $F(f, a, b; x)$ has zero function value at two distinct points, and hence is zero on $[a, b]$, a contradiction. Thus $F(f, a, b; x_0) = 0 = f(x_0)$, and it follows that $F(f, a, x_0; x) = F(f, a, b; x) = F(f, x_0, b; x)$ on $[a, b]$. Since f is super-(L), $f(x) \geqq F(f, a, x_0; x)$ on $[a, x_0]$ and $f(x) \geqq F(f, x_0, b; x)$ on $[x_0, b]$. By Lemma 2.2, $f(x) \leqq F(f, a, x_0; x)$ on $[x_0, b]$ and $f(x) \leqq F(f, x_0, b; x)$ on $[a, x_0]$. Thus $f(x) = F(f, a, b; x)$ on $[a, b]$. Let x_1 and x_2 be such that $a \leqq x_1 < x_0 < x_2 \leqq b$, and let y_1 and y_2 be positive real numbers. Since $F(f, a, b; x') > 0$, it follows that $F(f, a, b; x) > 0$ for all $x \neq x_0$ in $[a, b]$. Then there exist real numbers r_1 and r_2 such that $y_i = r_i F(f, a, b; x_i)$, $i = 1, 2$. Let G be the solution of $L(y) = 0$ such that $G(x_i) = y_i$, $i = 1, 2$. Assume $r_1 \geqq r_2$, since the proof for the other case is similar. Then $r_1 F(f, a, b; x_2) \geqq r_2 F(f, a, b; x_2) = y_2 = G(x_2)$. If $G(x_0) < 0$, G is zero at two distinct points and hence is identically zero on $[a, b]$, a contradiction. Then $G(x_0) \geqq F(f, a, b; x_0)$ and $G(x_2) \leqq r_1 F(f, a, b; x_2)$ imply the existence of an x_3 in $[x_0, x_2]$ for which $G(x_3) = r_1 F(f, a, b; x_3)$. Hence $G(x) = r_1 F(f, a, b; x)$ on $[a, b]$ since $G(x_1) = r_1 F(f, a, b, x_1)$. This contradicts the existence of solutions of $L(y) = 0$ taking arbitrary positive function values at x_0 and $x_1 \neq x_0$.

DEFINITION 2.3. Let A be a convex cone. An element f of A is an *extremal element* of A if for every pair of elements f_1 and f_2 of A such that $f = f_1 + f_2$ there exists a real number k such that $f_1 = kf$.

THEOREM 2.1. *A function $f\,(\neq 0)$ is an extremal element of C if and only if f is an (L)-conical function.*

Proof. It is easy to check that an (L)-conical function is super-(L). The result follows in a straightforward fashion upon applying Lemma 2.1.

If f in C is not (L)-linear on $[a, b]$ and is such that either $f(a) > 0$ or $f(b) > 0$, then f is not an extremal element of C since a nonproportional decomposition for f is $F(f, a, b; x)$ and $f(x) - F(f, a, b; x)$.

If f in C is (L)-linear on $[a, b]$ and $f(a) > 0$ and $f(b) > 0$, then f is not an extremal element of C since a nonproportional decomposition for f is the (L)-linear function f_1 such that $f_1(a) = f(a)$ and $f_1(b) = 0$ and the (L)-linear function f_2 such that $f_2(a) = 0$ and $f_2(b) = f(b)$.

To complete the proof of the theorem, let $f \neq 0$ be an element of C which is not (L)-conical and is such that $f(a) = f(b) = 0$. Let x_0 be such that $a < x_0 < b$. Assume f is not (L)-linear on $[x_0, b]$, since the proof for the other case is similar. Let $g(x) = f(x) - F(f, x_0, b; x)$ on $[a, b]$. For each positive real number y, let F_y be the (L)-linear function determined by $(a, 0)$ and (b, y). Let $u = \inf\{y\colon F_y(x) > g(x)$ for all x in $[a, b]\}$. Clearly u exists and is positive. The assumption that $F_u(x) > g(x)$ for all x in $[a, b]$ leads to a contradiction, so $\{x\colon F_u(x) = g(x)\}$ is not empty. Let $\bar{x} = \sup\{x\colon F_u(x) = g(x)\}$. Since $F_u(b) > 0$, there exists an x' in $(\bar{x}, b)$ such that $F_u(x) = F(f, a, x'; x)$ on $[a, b]$. Let $f_1(x) = F(f, a, x'; x)$ on $[a, \bar{x}]$ and $f_1(x) = f(x) - F(f, x_0, b; x)$ on $[\bar{x}, b]$. Let $f_2 = f - f_1$. It will be shown that f_1 and f_2 form a nonproportional decomposition of f. Clearly f_1 and f_2 are nonnegative. Let x_1 and x_2 be in $[a, b]$. Since f_1 is super-(L) on $[a, \bar{x}]$ and on $[\bar{x}, b]$, it will be assumed that $a \leqq x_1 < \bar{x}$ and $\bar{x} < x_2 \leqq b$. If $f_1(x_3) < F(f_1, x_1, x_2; x_3)$ for some x_3 in (x_1, x_2), then $F(f_1, x_1, x_2; x)$ must intersect $F(f, a, x'; x)$ for some x in $[\bar{x}, x_2)$, a contradiction. Thus f_1 is super-(L) on $[a, b]$. By observing that $f_2(x) \lesseqq F(f, x_0, b; x)$ on $[a, \bar{x}]$, a similar argument may be used to prove f_2 is super-(L) on $[a, b]$. Suppose there exists a real number k such that $f_1 = kf$. Then f is (L)-linear on $[a, \bar{x}]$ since f_1 is and $k \neq 0$. Since $f = f_1 + f_2$ and $f_2 \neq 0$, it follows that $k \neq 1$, and so $f_2 = (1 - k)f$ implies f is (L)-linear on $[\bar{x}, b]$. This contradicts the assumption that f is not (L)-conical.

3. Integral representation.

The existence of an integral representation (Radon measure) for the elements of the cone C in terms of its extremal elements will be based on the following theorem due to Choquet:

THEOREM 3.1 [4, p. 237]. *If the linear space L is a locally convex Hausdorff space, and if A is a convex compact subset of L, then for each x in A there exists a nonnegative Radon measure on the closure of the set of extreme points of A whose center of gravity is x.*

The theorem will be applied in the following way: First, it is known that $C - C$ is a real linear space such that the vertex of C is the origin of $C - C$ [3]. It is also known that when $C - C$ is topologized with the topology of simple convergence (the induced product topology of $R^{[a,b]}$), it is a locally convex Hausdorff topological linear space [4]. It will be shown that $B = \{f : f \text{ is in } C, f(x_0) = 1\}$, where x_0 is a fixed real number in (a, b), is a convex compact subset of $C - C$ which meets each ray of C once and only once and does not contain 0, the origin of $C - C$, and that the set of extreme points of B is closed in $C - C$. Then by the theorem there will exist an integral representation for each element of B in terms of extreme points of B. It will then follow that there exists an integral representation for each element of C in terms of extremal elements of C since B meets each ray of C once and only once and does not contain 0.

LEMMA 3.1. *Let F_1 be the (L)-linear function determined by the points $(x_0, 1)$ and $(b, 0)$, and let F_2 be the (L)-linear function determined by the points $(a, 0)$ and $(x_0, 1)$. Then $\{f(x) : f \text{ is in } B\} = [F_2(x), F_1(x)]$ for each x in $[a, x_0]$ and $\{f(x) : f \text{ is in } B\} = [F_1(x), F_2(x)]$ for each x in $[x_0, b]$.*

Proof. Clearly F_1 and F_2 belong to B, $F_1(x) > F_2(x)$ on $[a, x_0)$, and $F_1(x) < F_2(x)$ on $(x_0, b]$. Let f belong to B. The assumption that there exists an x_1 in $[a, x_0)$ such that $f(x_1) > F_1(x_1)$ leads to a contradiction through an application of Lemma 2.3. Similarly, $f(x) \leqq F_2(x)$ on $[x_0, b]$. The assumption that $f(x_2) < F_2(x_2)$ for some x_2 in $[a, x_0)$ or $f(x_3) < F_1(x_3)$ for some x_3 in $(x_0, b]$ either contradicts f being super-(L) or f being nonnegative. Therefore $\{f(x) : f \text{ is in } B\} \subset [F_2(x), F_1(x)]$ for each x in $[a, x_0]$ and $\{f(x) : f \text{ is in } B\} \subset [F_1(x), F_2(x)]$ for each x in $[x_0, b]$. Given any x in $[a, b]$ and y between $F_1(x)$ and $F_2(x)$, there exists an (L)-linear function in B passing through that point. Hence $\{f(x) : f \text{ is in } B\} = [F_2(x), F_1(x)]$ for each x in $[a, x_0]$ and $\{f(x) : f \text{ is in } B\} = [F_1(x), F_2(x)]$ for each x in $[x_0, b]$.

The Tychonoff product theorem may now be applied to show that B is a subset of a compact set in $R^{[a,b]}$, so that to prove B is compact it is only necessary to prove it is closed in $R^{[a,b]}$.

LEMMA 3.2. *The convex cone C is closed in $R^{[a,b]}$ for the topology of simple convergence.*

Proof. Let f belong to the complement of C. If there exists an x_1 in $[a, b]$ such that $f(x_1) < 0$, let $\varepsilon = -f(x_1)/2$. Let g belong to $U(f; x_1; \varepsilon)$, a neighborhood of f in the topology of simple convergence. Then $|g(x_1) - f(x_1)| < \varepsilon$ implies $g(x_1) - f(x_1) < -f(x_1)/2$, so that $g(x_1) < f(x_1)/2 < 0$. Thus g is not in C, and hence $U(f; x_1; \varepsilon)$ is in the complement of C.

If f is nonnegative, then f is not super-(L). Hence there exist x_1, x_2, and x_3 such that $x_1 < x_3 < x_2$ and $f(x_3) < F(f, x_1, x_2; x_3)$. Let $k_i = f(x_i)[F(f, x_1, x_2; x_3) - f(x_3)]/[f(x_i) + F(f, x_1, x_2; x_3)]$, $i = 1, 2$. Take $\varepsilon = \min\{k_1, k_2\}/2$ if $k_1 > 0$ and $k_2 > 0$, $\varepsilon = k_1/2$ if $k_2 = 0$, or $\varepsilon = k_2/2$ if $k_1 = 0$. Then $U(f; x_1, x_2, x_3; \varepsilon)$ is in the complement of C.

THEOREM 3.2. *The set B is a convex compact subset of $C - C$ which meets each ray of C once and only once and which does not contain* 0.

Proof. Let f_1 and f_2 belong to B, and let k be any real number such that $0<k<1$. Then $kf_1+(1-k)f_2$ belongs to C and $(kf_1+(1-k)f_2)(x_0)=1$, so that $kf_1 + (1 - k)f_2$ belongs to B. Thus B is convex.

Let f be in the complement of B relative to C. Since f is in C, $f(x_0) \neq 1$. Let $\varepsilon = |f(x_0) - 1|$. Then $U(f; x_0; \varepsilon) \cap C$ is in the complement of B relative to C, and hence B is closed in $R^{[a,b]}$ by Lemma 3.2. It now follows that B is a compact subset of $C - C$.

Let H be any ray in C. Let f in C be such that $H = \{kf : k$ is a nonnegative real number$\}$. By Lemma 2.3, $f(x_0) \neq 0$ since x_0 is in (a, b), so that $k_1 = 1/f(x_0)$ is such that $k_1 f$ belongs to B. Thus the intersection of B with H exists and is unique. Obviously 0 is not in B.

THEOREM 3.3. *The set $e(B)$ of extreme points of B is closed in $C - C$ for the topology of simple convergence.*

Proof. By Theorem 3.2, B is closed relative to $C - C$, so that it is only necessary to prove $e(B)$ is closed relative to B. Let f be in the complement of $e(B)$ relative to B. Then clearly there exists an x_1 in (a, b) such that $f(x_1) \neq F_1(x_1)$ and $f(x_1) \neq F_2(x_1)$, where F_1 and F_2 are the functions defined in Lemma 3.1. It will be assumed that x_1 is in (a, x_0) since the proof for the other case is similar. Let G_1 be the (L)-conical function in $e(B)$ determined by the points $(a, 0)$ and $(x_1, f(x_1))$. Let $\bar{x}$ be the x-coordinate of the vertex of G_1, and observe that $\bar{x} < x_0$. Suppose $f(x) = G_1(x)$ on $[a, \bar{x}]$. Then f super-(L) and $F(f, \bar{x}, x_0; x) = F_1(x)$ on $[a, b]$ imply $f(x) \geqq F_1(x)$ on $[\bar{x}, x_0]$ and $f(x) \leqq F_1(x)$ on $[x_0, b]$ by Lemma 2.2. By Lemma 3.1, $f(x) \leqq F_1(x)$ on $[\bar{x}, x_0]$ and $f(x) \geqq F_1(x)$ on $[x_0, b]$. Thus $f = G_1$, which contradicts f being in the complement of $e(B)$. Therefore there exists an x_2 in $[a, \bar{x}]$ such that $f(x_2) \neq G_1(x_2)$. Let G_2 be the (L)-conical function in $e(B)$ determined by the points $(a, 0)$ and $(x_2, [f(x_2) + G_1(x_2)]/2)$. Since $G_2(a) = G_1(a)$ and $G_2(x_2) \neq G_1(x_2)$, it follows that $G_2(x_1) \neq G_1(x_1) = f(x_1)$. Let $\varepsilon = (1/2)\min \{|f(x_1) - G_2(x_1)|, |f(x_2) - G_1(x_2)|\}$. Then $U(f: x_1, x_2; \varepsilon) \cap B$ is in the complement of $e(B)$ relative to B, and hence $e(B)$ is closed relative to B.

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