ON THE COMPACTNESS OF INTEGRAL CLASSES

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1. Introduction. In a previous paper, [8], integral currents were used to develop a concept for non-oriented domains of integration in Euclidean n-space. This concept has been designed to be useful in the calculus of variations and this, therefore, demands that the domains of integration satisfy certain "smoothness" and "compactness" conditions. It was shown in [8] that these non-oriented domains, which are called integral classes, do possess the desired smoothness property and it was also shown that the integral classes possess the following compactness property: every N-bounded sequence of k-dimensional integral classes has a subsequence which converges to some flat class. In the case that $k = 0, 1, n-1$, or n, it was shown that the limiting flat class is, in fact, a rectifiable class, and therefore, a desirable compactness property is obtained.

The main purpose of this paper is to extend this compactness property to integral classes of arbitrary dimension under the assumption that certain "irregular" sets have zero measure (3.1). This is accomplished with the help of a theorem concerned with the behavior of the density of a measure associated with a minimizing sequence (2.8), and by relying heavily on the tangential properties of rectifiable sets. In the case of the Plateau Problem, two theorems concerning densities are proved (2.3, 2.4) which are analogous to results obtained in [6] and [3; 9.13].

Most of this work depends upon the paper [8], and therefore, the terminology and notation of [8] is readopted here without change. It will be assumed throughout that $1 < k < n - 1$.

2. Densities. In this section, the Plateau Problem is formulated in terms of integral classes and two theorems are proved which are analogous to results obtained in [3; 9.13] and [6]. Theorem 2.8 asserts that a portion of the irregular set, A_3, which appears in (3) below, has zero measure. A similar result, which states that $D_k^*(\mu, R^n, x) < \infty$ μ-almost everywhere and therefore that $\mu(A_3) = 0$, is still lacking.

2.1. DEFINITION. If μ is a measure over R^n, $A \subset R^n$, $\alpha(k)$ the volume of the unit k-ball, and $x \in R^n$, then

$$D_k(\mu, A, x) = \lim_{r \to 0} \alpha(k)^{-1} r^{-k} \mu(A \cap \{y: |y - x| < r\})$$

is the k-dimensional μ density of A at x; the upper and lower densities

Received February 14, 1963.

$$D_k^*(\mu, A, x) \text{ and } D_{*k}(\mu, A, x)$$

are defined as the corresponding lim sup and lim inf.

2.2. REMARK. Recall that if $\{\tau_i\}$ is a sequence of integral classes with the property that sup $\{M(\tau_i): i = 1, 2, \cdots\} < \infty$, then the sequence of total variation measures, $\{\|\tau_i\|\}$, possess a subsequence that converges weakly to some non-negative Radon measure μ, [8; (3.2), (3.3)], [2; Chapter III].

2.3. THEOREM. *Suppose $\sigma \in I_{k-1}(R^n, 2)$ is a cycle and let*

$$\Omega(\sigma) = \inf\{M(\tau): \tau \in I_k(R^n, 2), \partial\tau = \sigma\} .$$

Let $\{\tau_i\}$ be a sequence of integral classes such that

$$\partial\tau_i = \sigma , \qquad \lim_{i\to\infty} M(\tau_i) = \Omega(\sigma) ,$$

and $\{\|\tau_i\|\}$ converges weakly to a non-negative Radon measure μ. Then, for all $x \notin spt\ \sigma$,

$$D_k^*(\mu, R^n, x) \leqq \Omega(\sigma)(\alpha(k)r_0^k)^{-1}$$

where $r_0 = distance\ (x, spt\ \sigma)$.

Proof. Let ε_i be a sequence of real numbers tending to zero where

$$\Omega(\sigma) \leqq M(\tau_i) < \Omega(\sigma) + \varepsilon_i .$$

Let B_0 be the set of all $0 < r < r_0$ with the property

$$\begin{gathered} \|\tau_i\|[S(x, r)] \to \mu[S(x, r)] , \\ spt\,[\partial(\tau_i \cap S(x, r)] \subseteqq \{y: \text{distance}\,(y, x) = r\} , \\ \tau_i \cap S(x, r) \text{ is integral for } i = 1, 2, 3, \cdots, \end{gathered}$$

and notice that $L_1[(0, r_0) - B_0] = 0$. For $r \in B_0$ and $i = 1, 2, 3, \cdots$,

$$\tau_i = \tau_i \cap S(x, r) + \tau_i \cap [R^n - S(x, r)] .$$

Letting

$$\zeta_i = x\partial[\tau_i \cap S(x, r)] + \tau_i \cap [R^n - S(x, r)] ,$$

[8; 3.14] implies

$$\begin{gathered} \partial\zeta_i = \partial\tau_i = \sigma , \\ M[x\partial(\tau_i \cap S(x, r))] \leqq r/kM[\partial(\tau_i \cap S(x, r))] . \end{gathered}$$

Therefore,

$$M(\tau_i) < M(\zeta_1) + \varepsilon_i ,$$

which implies

$$(1) \qquad M[\tau_i \cap S(x, r)] \leqq M[x\partial(\tau_i \cap S(x, r))] + \varepsilon_i$$
$$\leqq r/kM[\partial(\tau_i \cap S(x, r))] + \varepsilon_i ,$$

for $r \in B_0$ and $i = 1, 2, 3, \cdots$. For $0 < r < r_0$ and $i = 1, 2. 3, \cdots$. let

$$\varphi_i(r) = M[\tau_i \cap S(x, r)] = \| \tau_i \| [S(x, r)] ,$$
$$\psi_i(r) = M[\partial(\tau_i \cap S(x, r))] ,$$

and note that $F_i(r) = \int_0^r \psi_i(t)dt \leqq \varphi_i(r)$, by [8; 4.1]. Again by [8; 4.1], $F_i(r + h) - F_i(r) \leqq \varphi_i(r + h) - \varphi_i(r)$ and therefore $F_i'(r) \leqq \varphi_i'(r)$ for L_1-almost all $0 < r < r_0$. This implies

$$\psi_i(r) \leqq \varphi_i'(r)$$

which, along with (1), implies

$$\varphi_i(r) \leqq rk^{-1}\varphi_i'(r) + \varepsilon_i$$

for $i = 1, 2, 3, \cdots$, and for $r \in B_1 \subset B_0$ where $L_1(B_0 - B_1) = 0$.

After passing to a subsequence, we may assume by Helly's theorem that $\varphi(r) = \lim_{1\to\infty} \varphi_i(r)$ exists whenever $0 < r < r_0$, and therefore by [3; 9.7], we have

$$\liminf_{i\to\infty} \varphi_i'(r) \leqq \varphi'(r)$$

for $r \in B_2 \subset B_1$ where $L_1(B_1 - B_2) = 0$. Since

$$\varphi_i(r) \leqq rk^{-1}\varphi_i'(r) + \varepsilon_i \text{ for } r \in B_2 ,$$

it follows that

$$\varphi(r) = \lim_{i\to\infty} \varphi_i(r) \leqq \liminf_{i\to\infty} rk^{-1}\varphi_i'(r) \leqq rk^{-1}\varphi'(r)$$

for $r \in B_2$. Therefore, for L_1-almost all $0 < r < r_0$,

$$(2) \qquad \varphi'(r)/\varphi(r) \geqq k/r .$$

Letting $\theta(r) = \mu[S(x, r)]$, we have that $\theta(r) = \varphi(r)$ for L_1-almost all $0 < r < r_0$ and thus, from (2), it follows that

$$\theta'(r)/\theta(r) \leqq k/r \text{ for } L_1\text{-almost all } 0 < r < r_0 .$$

Since $\log \circ \theta$ is non-decreasing, one finds by integrating this inequality that $\theta(r)r^{-k}$ is non-decreasing on $\{r: 0 < r < r_0\}$ and therefore, establishes the theorem.

2.4. Theorem. *With the same hypotheses and notations as in*

2.3, *for* μ*-almost* $x \in R^n - spt\, \sigma$,

$$D_{*k}(\mu, R^n, x) \geqq (k \cdot \alpha(k)^{1/k} \cdot 2^{k-1} \cdot c_2)^{-k}$$

where c_2 *is as in* [8; 4.6] *with* k *replaced by* $k - 1$.

Proof. Choose $x \not\ni spt\ \sigma$ so that $\varphi(r) \neq 0$ provided $r \neq 0$. For each $r \in B_2$, and for $i = 1, 2, 3, \cdots$, from [8; 4.7] one obtains $\sigma_i \in I_k$ ($\{y$: distance $(x, y) \leqq r\}$, 2) such that

$$\partial \sigma_i = \partial[\tau_i \cap S(x, r)] ,$$

$$(1) \qquad [M(\sigma_i)]^{k-1/k} \leqq 2^{k-1} c_2 \psi_i(r) .$$

Hence, $\varphi_i(r) < M(\sigma_i) + \varepsilon_i$ which implies

$$(2) \qquad [\varphi_i(r)]^{k-1/k} < [M(\sigma_i) + \varepsilon_i]^{k-1/k} < [M(\sigma_i)]^{k-1/k} + \varepsilon_i^* ,$$

where $\varepsilon_i^* \to 0$ for appropriate subsequences. From the fact that $\psi_i(r) \leqq \varphi_i'(r)$ and from (1) and (2), we have

$$[\varphi_i(r)]^{k-1/k} < 2^{k-1} c_2 \varphi_i'(r) + \varepsilon_i^*$$

and therefore, from [3; (9.7)]

$$[\varphi(r)]^{k-1/k} = \lim_{i \to \infty} [\varphi_i(r)]^{k-1/k} \leqq 2^{k-1} c_2 \liminf_{i \to \infty} \varphi_i'(r) \leqq 2^{k-1} c_2 \varphi'(r) .$$

That is, for L_1-almost all $0 < r < r_0$,

$$[\varphi(r)]^{k-1/k} \leqq 2^{k-1} c_2 \varphi'(r) ,$$

$$[\varphi^{1/k}]'(r) \geqq (k 2^{k-1} c_2)^{-1} .$$

Now, integration of this inequality implies

$$\varphi(r)/r^k \geqq (k 2^{k-1} c_2)^{-k}$$

and therefore establishes the theorem since $\varphi(r) = \mu[S(x, r)] = \theta(r)$ for L_1 almost all $0 < r < r_0$, and since θ is left-continuous.

2.5. LEMMA. *If* μ *is a non-negative Radon measure over* R^n, *then for* μ*-almost all* $x \in R^n$,

$$\limsup_{r \to 0} \frac{\mu[S(x, r/2)]}{\mu[S(x, r)]} > 0 .$$

Proof. For μ-almost all $x \in R^n$, we have

$$\lim_{r \to 0} \mu[S(x, r)] \cdot r^{-m} = \infty$$

where $m > n$. For all such x the lemma must hold for, if not, there would exist an $r_0 > 0$ such that for $r \leqq r_0$,

$$\frac{\mu[S(x, r/2)]}{\mu[S(x, r)]} < 2^{-m} .$$

This would imply

$$\frac{\mu[S(x, r2^{-n})]}{(r2^{-n})^m} = \frac{\mu[S(x, r2^{-n})]}{r^m 2^{-mn}} \leqq \frac{2^{-mn}\mu[S(x, r)]}{2^{-mn} r^m} = \frac{\mu[S(x, r)]}{r^m} ;$$

hence, it would follow that

$$\liminf_{r \to 0} \mu[S(x, r)] \cdot r^{-m} < \infty ,$$

a contradiction.

2.6. A COVERING THEOREM. From [1] and [5], we have the following theorem:

If $E \subset R^n$, F is a family of closed spherical balls in R^n such that each point of E is the center of arbitrarily small members of F, and μ is a non-negative Radon measure over R^n, then F has a disjointed subfamily covering μ-almost all of E.

2.7. DEFINITION. If $\tau \in W_k(R^n, 2)$, then let

$$L(\tau) = \inf \left\{ \liminf_{i \to \infty} M(\tau_i) \colon \tau_i \in I_k(R^n, 2),\ \partial \tau_i = 0,\ W(\tau_i - \tau) \longrightarrow 0 \right\} .$$

2.8. THEOREM. *Suppose $\tau_i \in I_k(R^n, 2)$ are cycles for $i=1, 2, 3, \cdots$, $\{\|\tau_i\|\}$ converges weakly to a non-negative Radon measure μ over R^n, $\lim_{i \to \infty} W(\tau_i - \tau) = 0$ and $\lim_{i \to \infty} M(\tau_i) = L(\tau)$ where $\tau \in W_k(R^n)$. Then, for μ-almost all $x \in R^n$,*

$$D_k^*(\mu, R^n, x) > 0 .$$

Proof. The proof is by contradiction: by 2.5 we may assume the existence of a set E and a real number $a > 0$ such that for $x \in E$,

$$\mu(E) > 0 ,$$
$$D_k^*(\mu, R^n, x) = 0 ,$$
$$\limsup_{r \to 0} \frac{\mu[S(x, r/2)]}{\mu[S(x, r)]} > a .$$

Therefore, for a given $\varepsilon > 0$ and for $x \in E$, there exists a set $B_x \subset (0, 1)$ such that B_x contains at least a denumerable number of elements and such that, if $r \in B_x$,

$$\begin{aligned}(1)\qquad &\mu[S(x, r)] < \varepsilon\alpha(k)r^k ,\\ &\mu[S(x, r/2)] > a\mu[S(x, r)] ,\\ &\mu[\{y\colon \text{distance}\,(x, y) = r\}] = 0 ,\\ &\mu[\{y\colon \text{distance}\,(x, y) = r/2\}] = 0 ,\\ &\tau_i \cap S(x, r) \text{ is integral for } i = 1, 2, 3, \cdots,\\ &\inf\{r\colon r \in B_x\} = 0.\end{aligned}$$

Hence, by 2.6, there exist points $x_1, x_2, \cdots, x_m$ and numbers $r_1, r_2, \cdots, r_m$ such that $S(x_i, r_i) \cap S(x_j, r_j) = 0$ for $i \neq j$, $r_i \in B_{x_i}$ and

$$\mu\Big[E - \bigcup_{j=1}^{m} S(x_j, r_j)\Big] < \varepsilon .$$

From (1), we have the existence of an integer $i_0 > 0$ such that for $i > i_0$ and $1 \leqq j \leqq m$,

$$\begin{aligned}(2)\qquad &\|\tau_i\|[S(x_j, r_j/2)] > a\,\|\tau_i\|[S(x_j, r_j)] ,\\ &\|\tau_i\|[S(x_j, r_j)] < \varepsilon\alpha(k)r_j^k .\end{aligned}$$

For $i \geqq i_0$ and $1 \leqq j < m$, [8, 4.1] provides a ball $S_{i,j} \subset S(x_j, r_j)$ with radius between $r_j/2$ and r_j and center at x_j such that

$$\begin{aligned}(3)\qquad &\tau_i \cap S_{i,j} \text{ is integral} ,\\ &M[\partial(\tau_i \cap S_{i,j})] \leqq 2r_j^{-1}\|\tau_i\|[S(x_j, r_j)] .\end{aligned}$$

Now by use of [8; 4.6] one can find a constant c and integral classes $\rho_{i,j}$, $\sigma_{i,j}$ such that,

$$\begin{aligned}(4)\qquad &\tau_i \cap S_{i,j} - \rho_{i,j} = \partial\sigma_{i,j} ,\\ &M(\rho_{i,j}) \leqq cM[\partial(\tau_i \cap S_{i,j})]^{k/k-1}\\ &M(\sigma_{i,j}) \leqq c[M(\tau_i \cap S_{i,j}) + M(\rho_{i,j})]^{k+1/k} .\end{aligned}$$

For $i \geqq i_0$ and $1 \leqq j \leqq m$, (2) and (3) imply

$$\begin{aligned}M[\partial(\tau_i \cap S_{i,j})]^{k/k-1} &\leqq (2r_j^{-1})^{k/k-1}\cdot\|\tau_i\|[S(x_j, r_j)]^{k/k-1}\\ &\leqq (2r_j^{-1})^{k/k-1}\cdot\|\tau_i\|[S(x_i, r_j)]\cdot\|\tau_i\|[S(x_j, r_j)]^{1/k-1}\\ &\leqq \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot\|\tau_i\|[S(x_j, r_j)] ;\end{aligned}$$

hence, from (4),

$$(5)\qquad M(\rho_{i,j}) \leqq \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c\cdot\|\tau_i\|[S(x_j\cdot r_j)] .$$

Similarly,

$$(6)\quad M(\sigma_{i,j}) \leqq c[M(\tau_i \cap S_{i,j}) + M(\rho_{i,j})]^{k+1/k}$$
$$\leqq c\{\|\tau_i\|[S(x_j, r_j)] + \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c\cdot\|\tau_i\|[S(x_j, r_j)]\}^{k+1/k}$$
$$\leqq c\{(1 + \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c)\|\tau\|[S(x_j, r_j)]^{k/k+1}\cdot\|\tau_i\|[S(x_j, r_j)]^{1/k+1}\}^{k+1/k}$$
$$\leqq c\|\tau_i\|[S(x_j, r_j)]\cdot(1 + \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c)^{k+1/k}\|\tau_i\|[S(x_j, r_j)]^{1/k}$$
$$\leqq \varepsilon^{1/k}\cdot c(1 + \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c)^{k+1/k}\alpha(k)^{1/k}\|\tau_i\|[S(x_j, r_j)]\ .$$

Let $\alpha = 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c$, $\beta(\varepsilon) = c(1 + \varepsilon^{1/k-1}\cdot 2^{k/k-1}\cdot\alpha(k)^{1/k-1}\cdot c)^{k+1/k}\cdot\alpha(k)^{1/k}$. Notice that $\beta(\varepsilon) \to c^{k+1/k+1}\cdot\alpha(k)^{1/k}$ as $\varepsilon \to 0$. If we let

$$\zeta_i = \tau_i + \sum_{j=1}^{m}(\rho_{i,j} - \tau_i \cap S_{i,j})\ ,$$

(4) and (6) imply that, for $i \geqq i_0$,

$$(7)\qquad W(\zeta_i - \tau_i) \leqq \varepsilon^{1/k}\beta(\varepsilon)M(\tau_i)\ .$$

Since

$$\zeta_i = \tau_i - \sum_{j=1}^{m}\tau_i \cap S_{i,j} + \sum_{j=1}^{m}\rho_{i,j}\ ,$$

it follows from (5) that, for $i \geqq i_0$,

$$(8)\qquad M(\zeta_1) \leqq M\Big(\tau_i - \sum_{j=1}^{m}\tau_i \cap S_{i,j}\Big) + \varepsilon^{1/k-1}\alpha M(\tau_i)\ .$$

Now, with $U = \bigcup_{j=1}^{m} S(x_j, r_j)$, we have from (1)

$$(9)\qquad M\Big(\tau_i - \sum_{j=0}^{m}\tau_i \cap S_{i,j}\Big) = \|\tau_i\|(R^n) - \sum_{j=1}^{m}\|\tau_i\|(S_{i,j})$$
$$\leqq \|\tau_i\|(R^n) - \sum_{j=1}^{m}\|\tau_i\|[S(x_j, r_{j/2})]$$
$$\leqq \|\tau_i\|(R^n) - a\sum_{j=1}^{m}\|\tau_i\|[S(x_j, r_j)]$$
$$\leqq \|\tau_i\|(R^n) - a\|\tau_i\|(U)\ \text{for}\ i \geqq i_0\ .$$

There exists an integer $i_1 \geqq i_0$ such that for $i \geqq i_1$,

$$|\,\|\tau_i\|(U) - \mu(U)\,| < \varepsilon\ ;$$

therefore, from (8) and (9),

$$M(\zeta_i) + a\|\tau_1\|(U) \leqq M(\tau_i) + \varepsilon^{1/k-1}\cdot\alpha\cdot M(\tau_i)\ ,$$
$$M(\zeta_i) + a\mu(U) \leqq M(\tau_i) + \varepsilon^{1/k-1}\cdot\alpha\cdot M(\tau_i) + a\varepsilon\ .$$

But $\mu(U) > \mu(E) - \varepsilon$, and therefore we finally obtain, for $i \geqq i_1$,

$$M(\zeta_i) + a\mu(E) \leqq M(\tau_i) + \varepsilon^{1/k-1}\cdot\alpha\cdot M(\tau_i) + 2a\varepsilon\ .$$

Hence, from this inequality and (7), it is now clear that we can find a sequence of integral cycles ψ_i such that

$$\lim_{i\to\infty} W(\psi_i - \tau) = 0 ,$$
$$\limsup_{i\to\infty} M(\psi_i) + a\mu(E) \leqq \lim_{i\to\infty} M(\tau_i) = L(\tau) ,$$

which is a contradiction since $\mu(E) > 0$ and $a > 0$.

3. The main theorem. In this section, the main theorem, which is concerned with the compactness of integral classes, is established. In the proof, an essential role is played by a decomposition theorem due to Federer [4], which will now be discussed.

Recall from [8; 5.13], that if A is a compact subset of R^n, $\tau_i \in I_k(A, 2)$ for $i = 1, 2, \cdots$, and $\sup\{M(\tau_i) + M(\partial\tau_i)\colon i = 1, 2, \cdots\} < \infty$, then there exists a subsequence, $\{\tau_{i_j}\}$, and a $\tau \in W_k(R^n, 2)$ such that $W(\tau_{i_j} - \tau) \to 0$. Of course, it would be desirable to show that τ is, in fact, a rectifiable class. To this end, assume without loss of generality that $\partial\tau = 0$ and let $\{\tau_i\}$ be a sequence of integral cycles for which

$$\lim_{i\to\infty} W(\tau_i - \tau) = 0 \text{ and } \lim_{i\to\infty} M(\tau_i) = L(\tau) .$$

Hence, by passing to a suitable subsequence if necessary, we have the existence of a non-negative Radon measure μ such that $\{\|\tau_i\|\}$ converges weakly to μ. Then, from [4; § 9] we know that R^n can be decomposed into four μ-measurable sets A_1, A_2, A_3, A_4 such that:

(1) $R^n = A_1 \cup A_2 \cup A_3 \cup A_4$,

(2) $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = 0$,

(3) A_1 is a countably k-rectifiable set and at each point $x \in A_1$, there exists a μ-approximate tangent k-plane to A_1 at x; for this, (2.8) is needed,

(4) Either $\mu(A_2) = 0$ or A_2 contains no k-rectifiable set B for which $\mu(B) > 0$,

(5) $L_k[p(A_2)] = 0$ for almost all orthogonal projections of R^n onto R^k,

(6) $A_3 = \{x\colon D_k(\mu, R^n, x) = 0 \text{ or } D_k^*(\mu, R^n, x) = \infty\}$. Observe that from 2.8, $\mu[\{x\colon D_k(\mu, R^n, x) = 0\}] = 0$,

(7) $\mu(A_4) = 0$.

Now let $A = A_1 \cap \{x\colon D_k(H^k, A_1, x) = 1\} \cap B$, where B is the set described in the proof of [8; 5.14]. We now are in a position to state the main theorem.

3.1. THEOREM. *Suppose that* $H^k(A_1) < \infty$ *and that* $\mu(A_2) = \mu(A_3) = 0$. *Then, there exists a* μ*-measurable set* $E \subset R^n$ *such that* $\mu(A - E) =$

0 *and*

$$\lim_{i\to\infty} W(\tau_i - A \cap E) = 0 .$$

Since $A \cap E$ is a Hausdorff k-rectifiable set, by [8; § 3], we can identify $A \cap E$ with a rectifiable class. Hence, the theorem asserts the existence of a rectifiable class to which $\{\tau_i\}$ converges.

For the proof of the theorem, we will need the following lemma.

3.2. LEMMA. *Suppose that A is a countably k-rectifiable set with $H^k(A) < \infty$. Then, for any real number $0 < a < 1$, and for any real number b such that $b > 1$ and $b^k < a^{-1}$,*

$$\lim_{r\to 0} \frac{H^k[S(x, r/b) \cap A]}{H^k[S(x, r) \cap A]} > a ,$$

for H^k almost all $x \in A$.

Proof. Since $D_k(H^k, A, x) = 1$ for H^k almost all $x \in A$, we have the following at all such x:

$$\lim_{r\to 0} b^{-k} \frac{H^k[S(x, r/b) \cap A]}{(r/b)^k} \cdot \frac{r^k}{H^k[S(x, r) \cap A]} = b^{-k} > a .$$

Proof of the theorem. It is sufficient to show that the conclusion holds for a subsequence of the given sequence. Passage to subsequences, which often occurs in what follows, will be indicated by words but not notationally. The proof will be divided into four main parts.

Choose $0 < \delta < 1$ and let $\gamma(B) = H^k(A \cap B)$ where B is any H^k measurable set. In view of the assumptions and with the aid of [4; (3.8)] we know that μ is absolutely continuous with respect to γ. Let $P(x)$ be the μ approximate tangent k-plane to A at x and let $K(x, r)$ be the open n-cube with center at x, side length $2r$ and one of its k-faces parallel to $P(x)$. In this proof, densities will be computed by using these cubes and 2.6 will be used with cubes instead of spheres; this does not change anything. Using the methods of [8; 5.14], for $\varepsilon > 0$ we have the existence of a positive number $r_1(x, \varepsilon)$ such that for $r \leqq r_1(x, \varepsilon)$,

$$(1) \qquad W[P(x) \cap K(x, r)) - A \cap K(x, r)] < \varepsilon^2/8 \cdot \beta(k) r^k$$

where $\beta(k)$ is volume of a k-cube with side length 2. Also, if $D(x) = D_k^*(\mu, R^n, x)$, then for each $x \in A$ there exists a number $r_0(x, \varepsilon) \leqq r_1(x, \varepsilon)$ such that for $r \leqq r_0(x, \varepsilon)$,

$$(2)\qquad \begin{aligned} &\gamma[K(x, r)] \geqq (1-\varepsilon)\beta(k)r^k\,, \\ &\mu[K(x, r)] < c(x)\beta(k)(r/b)^k\,, \\ &\mu[K(x, r) - S(P(x), \varepsilon/2\cdot r] < \varepsilon^2 2^{-k}\beta(k)r^k\,, \\ &\gamma[K(x, r/b)] > a\gamma[K(x, r)]\,, \end{aligned}$$

where $c(x) = b^k D(x)$, $a > 1-\varepsilon$, and where b is the number provided by 3.2. We will consider only those r for which $u[K^*(x, r)] = \mu[K^*(x, r/b)] = 0$, where $K^*(x, r)$ denotes the boundary of the cube. Since this omits at most a denumerable number of cubes, we have a Vitali covering of A and therefore, by 2.6, there exists a finite number of disjoint cubes $K(x_1, r_1)$, $K(x_2, r_2)$, $\cdots$, $K(x_m, r_m)$ such that $\gamma[A - \bigcup_{i=1}^m K(x_i, r_i)] < \varepsilon$ and $\mu[A - \bigcup_{i=1}^m K(x_i, r_i)] < \varepsilon$. Let $c = \max[c(x_1), \cdots, c(x_m)]$, $d = c/(b-1)$, and assume ε to be chosen so as to satisfy the following inequalities, where c_1 and c_2 are the constants in [8; 4.6]:

$$\varepsilon < [c_1 2^{2-2k}\beta(k)]^{-1}\,, \qquad [\varepsilon c_2 4^{k-1}2^{1-k}\beta(k)]^{k/k-1} + \varepsilon^2 < \delta^2/8\,,$$

(3) $\varepsilon c + \varepsilon L d + 3\varepsilon^2 < \delta^2/4$ where L is described below,

$$\varepsilon < 1 - \delta/2,\ \ \varepsilon c/1 - \varepsilon < \delta,\ \ \varepsilon\gamma(A) + \varepsilon - \varepsilon^2 < \delta\,.$$

Part 1. Consider x_1 and let $x = x_1$, $r = r_1$, and $P = P(x_1)$. Since $\mu[K^*(x, r)] = 0$ and $\|\tau_i\| \to \mu$, there exists an integer $i_0(r)$ such that for $i \geqq i_0(r)$,

$$\begin{aligned} &M[\tau_i \cap K(x, r)] < c\beta(k)(r/b)^k\,, \\ &M[\tau_i \cap (K(x, r) - S(P, \varepsilon/2\cdot n))] < \varepsilon^2 2^{-k}\beta(k)r^k\,. \end{aligned}$$

For each $i \geqq i_0(r)$, [8; 4.1] implies that

$$\int_{r/b}^{r} M[\partial(\tau_i \cap K(x, s)]ds < Lc\beta(k)(r/b)^k\,,$$

where L is the Lipschitz constant of the function that defines $K(x, s)$. Therefore, by appealing to Fatou's lemma, there exists a number t between r/b and r and a subsequence $\{\tau_{i_j}\}$ (which will still be denoted by $\{\tau_i\}$) such that

$$\mu[K^*(x, t)] = 0$$

and

$$\begin{aligned} M[\partial(\tau_i \cap K(x, t))] &\leqq Lc\beta(k)^1/b - 1(r/b)^{k-1} \\ &\leqq Ld\beta(k)t^{k-1}\,, \quad \text{for all } i\,. \end{aligned}$$

Hence, letting $\sigma_i = \tau_i \cap K(x, t)$ from (1) and (2), we have the following for all elements of a subsequence:

$$
(4)\quad
\begin{aligned}
&M(\sigma_i) \leqq c\beta(k)t^k ,\\
&M(\partial\sigma_i) \leqq Ld\beta(k)t^{k-1} ,\\
&M[\sigma_i \cap (K(x, t) - S(P, \varepsilon/2\cdot t))] \leqq \varepsilon^2 2^{-k}\beta(k)t^k ,\\
&\gamma[K(x, t)] > a\gamma[K(x, r)] ,\\
&W[P \cap K(x, t) - A \cap K(x, t)] < \varepsilon^2/8\cdot\beta(k)t^k ,\\
&\operatorname{spt}\sigma_i \subset \text{closure } K(x, t),\ \operatorname{spt}\partial\sigma_i = K^*(x, t) .
\end{aligned}
$$

Let $U_s = \{x\colon \text{distance}\,(x, P) > s\}$. For each σ_i of the above subsequence, we have from [8; 4.1] that

$$\int_{\varepsilon t/2}^{\varepsilon t} M[\partial(\sigma_i \cap U_s) - (\partial\sigma_i) \cap U_s]ds < \varepsilon^2 2^{-k}\beta(k)t^k ,$$

so that again by appealing to Fatou's lemma, there exists a number s_0 such that $\varepsilon t/2 < s_0 < \varepsilon t$ and a subsequence $\{\sigma_i\}$ such that for all members of this subsequence,

$$(5)\qquad M[\partial(\sigma_i \cap U_{s_0}) - (\partial\sigma_1) \cap U_{s_0}] \leqq \varepsilon 2^{1-k}\beta(k)t^{k-1} .$$

Let $K = K(x, t)$, $U = U_{s_0}$, $N = \text{closure}\,[K \cap (R^n - U)]$ and note that

$$(6)\qquad \operatorname{spt} p_\#[(\partial\sigma_i) \cup N] \subset P \cap K^*$$

where $p\colon R^n \to P$ is the orthogonal projection. If we let
(7) $\theta_i = \partial(\sigma_i \cap N) - (\partial\sigma_i) \cap N = \partial(\sigma_i \cap U) - (\partial\sigma_i) \cap U$ and $\chi_i = p_\#(\sigma_i \cap N)$, then in the notation of [8; 4.6] with $A = \text{closure}\,(P \cap K)$ and $B = P \cap K^*$, we have from (3), (5), and (6),

$$
\begin{aligned}
c_1 M(\partial\chi_i \cap K) = c_1 M[p_\#(\theta_i) \cap K] &\leqq c_1 M[p_\#(\theta_i)]\\
&\leqq c_1 2^{1-k}\varepsilon\beta(k)t^{k-1} \leqq (t/2)^{k-1} .
\end{aligned}
$$

Hence, by [8; 4.6], there exists $\lambda_i \in I_k(A, 2)$ such that

$$
\begin{aligned}
&\operatorname{spt}(\partial\chi_i - \partial\lambda_i) \subset B ,\\
&M(\lambda_i)^{k-1/k} \leqq \varepsilon c_2 4^{k-1} 2^{1-k}\beta(k)t^{k-1} .
\end{aligned}
$$

If we let $\psi_i = \chi_i - \lambda_i$, then we have

$$
(8)\quad
\begin{aligned}
&\operatorname{spt}\psi_i \subset \text{closure } P \cap K ,\\
&\operatorname{spt}\partial\psi_i \subset P \cap K^* ,\\
&M(\psi_i - \chi_i) \leqq \varepsilon^{k/k-1}[c_2 4^{k-1} 2^{1-k}\beta(k)]^{k/k-1} t^k .
\end{aligned}
$$

Hence, we will consider the two following possibilities: $\psi_i = 0$ for all but finitely many i or $\psi_i = P \cap K$ for all i of a subsequence.

Case 1. Suppose $\psi_i = P \cap K$ for some subsequence. Then from (4), (8), and (3),

$$\begin{aligned} M[P \cap K - p_\#(\sigma_i)] &\leqq M[P \cap K - p_\#(\sigma_i \cap N)] + M[p_\#(\sigma_i \cap U)] \\ &\leqq M(\psi_i - \chi_i) + M[p_\#(\sigma_i \cap U)] \\ &\leqq \varepsilon^{k/k-1}[c_2 4^{k-1} 2^{1-k}\beta(k)]^{k/k-1} t^k + \varepsilon^2/2^k \cdot \beta(k) t^k \\ &\leqq \delta^2/8 \cdot \beta(k) t^k . \end{aligned}$$

Therefore, (4) implies $W[p_\#(\sigma_i) - A \cap K] < \delta^2/4 \cdot \beta(k)t^k$.
Also from (4), (3)

$$\begin{aligned} W[p_\#(\sigma_i) - \sigma_i] &\leqq W[p\,(\sigma_i \cap N) - \sigma_i \cap N] + W[p_\#(\sigma_i \cap U) - \sigma_i \cap U] \\ &\leqq \varepsilon t\, N(\sigma_i \cap N) + 2M(\sigma_i \cap U) \\ &\leqq \varepsilon t[c\beta(k)t^k + Ld\beta(k)t^{k-1}] + \varepsilon^2 2^{1-k}\beta(k)t^k \\ &\leqq \delta^2/4 \cdot \beta(k)t^k. \end{aligned}$$

Therefore, from (2), (3), and (4),

$$W[\tau_i \cap K - A \cap K] < \delta^2/2 \cdot \beta(k)t^k \leqq \delta K(x, t)$$

for all members of a suitable subsequence. Now repeat the entire above procedure to the cube $K(x_2, r_2)$, but using the subsequence that was finally obtained at the end of case 1.

Case 2. In the event that $\psi_i = 0$ for all but finitely many i, repeat the entire above procedure to the cube $K(x_2, r_2)$ but using the subsequence that corresponds to $\psi_i = 0$.

Part 2. By repeating the procedure in part 1 m times, we obtain cubes $K(x_1, t_1), \cdots, K(x_j, t_j), K(x_{j+1}, t_{j+1}), \cdots, K(x_m, t_m)$ and a subsequence, $\{\tau_i\}$ such that, for all members of this subsequence,

$$\sum_{l=1}^{j} W[\tau_i \cap K(x_l, t_l) - A \cap K(x_l, t_l)] < \delta^2/2 \cdot \beta(k) \sum_{l=1}^{j} t_l^k \leqq \delta\gamma(A) ,$$

and such that case 2 of step 1 applies to the cubes $K(x_{j+l}, t_{j+l}), \cdots, K(x_m, t_m)$. Now, using the same notation as above except for the addition of superscripts to denote that cube which is under consideration, we have that $\psi_i^q = 0$ for all i and for $q = j+1, j+2, \cdots, m$. This implies that $M[p_\#^q(\sigma_i^q \cap N^q)] < \delta^2/8 \cdot \beta(k)t_q^k$, where $p^q\colon R^n \to P(x_q)$ is the orthogonal projection. Define $A_\delta = \bigcup_{l=1}^{j} K(x_l, t_l)$ and $B_\delta = \bigcap_{l=j+1}^{m} K(x_l, t_l)$ and let $\omega_i^q = h_\#^q[I \times \partial(\sigma_i^q \cap N^q)]$, $\zeta_i^q = h_\#^q[I \times (\sigma_i^q \cap N^q)]$ where $h_\#^q$ is the linear homotopy from the identity to the projection map p^q. If we let

$$\tau_i^\delta = \sum_{q=j+l}^{m} \tau_i \cap [R^n - K(x_q, t_q)] + \sum_{q=j+l}^{m} [\omega_i^q + p_\#^q(\sigma_i^q \cap N^q) + \sigma_i^q \cap U^q] ,$$

then $\partial\tau_i^\delta = 0$ and from (7), (5), (8), (4), (3), and (2),

$$\begin{aligned} M(\tau_i^\delta) &\leqq M(\tau_i) - M(\tau_i \cap B_\delta) + \varepsilon t\Big[\sum_{q=j+1}^{m} 2\varepsilon\beta(k)t_q^{k-1} + Ld\beta(k)t_q^{k-1}\Big] \\ &\quad + \delta^2/4 \sum_{q=j+1}^{m} \beta(k)t_q^k + \varepsilon^2 \sum_{q=j+1}^{m} \beta(k)t_q^k \\ &\leqq M(\tau_i) - M(\tau_i \cap B_\delta) + \delta^2/2 \sum_{q=j+1}^{m} \beta(k)t_q^k \\ &\leqq M(\tau_i) - M(\tau_i \cap B_\delta) + \delta^2/2(1-\varepsilon)\cdot\gamma(\beta_\delta) \\ &\leqq M(\tau_i) - M(\tau_i \cap B_\delta) + \delta\gamma(A) \,. \end{aligned}$$

Since $\|\tau_i\| \to \mu$, there exists a subsequence of the one above such that, for all members of this subsequence,

$$M(\tau_i^\delta) \leqq M(\tau_i) - \mu(B_\delta) + 2\delta\gamma(A) \,.$$

Also,

$$\tau_i - \tau_i^\delta = \sum_{q=j+1}^{m} \partial\zeta_i^q$$

and therefore, from (3),

$$\begin{aligned} W(\tau_i - \tau_i^\delta) &\leqq \sum_{q=j+1}^{m} M(\zeta_i^q) \leqq \varepsilon \sum_{q=j+1}^{m} c\beta(k)t_q^k \\ &\leqq \varepsilon c/1 - \varepsilon\cdot\gamma(B_\delta) < \delta\gamma(A) \,. \end{aligned}$$

Observe, with the help of (4), that

$$\begin{aligned} \gamma(A_\delta \cup B_\delta) &\geqq a\gamma\Big[\bigcup_{i=1}^{m} K(x_i, r_i)\Big] \geqq a[\gamma(A) - \varepsilon] \\ &\geqq (1-\varepsilon)[\gamma(A) - \varepsilon] \geqq \gamma(A) - \delta \,. \end{aligned}$$

Thus, in summary of what has been done so far, we have, for every $\delta > 0$, the existence of sets A_δ and B_δ which are the finite union of disjoint open cubes and the existence of a subsequence $\{\tau_i\}$ and a sequence $\{\tau_i^\delta\}$ such that

$$\begin{aligned} &\gamma[A - (A_\delta \cup B_\delta)] < \delta \,, \\ &W(\tau_i \cap A_\delta - A \cap A_\delta) < \delta\gamma(A) \,, \\ &W(\tau_i - \tau_i^\delta) < \delta\gamma(A) \,, \\ &M(\tau_i^\delta) \leqq M(\tau_i) - \mu(B_\delta) + 2\delta\gamma(A) \qquad \text{for all } i \,. \end{aligned}$$

Now by letting $\delta \to 0$ and by using Cantor's diagonal process, we can infer that $\lim_{\delta\to 0} \sup \mu(B_\delta) = 0$ since $M(\tau_i) \to L(\tau)$. This implies, along with the fact that μ is absolutely continuous with respect to γ, that for every $\delta > 0$, there exists a set A_δ, which is the union of a finite number of disjoint open cubes, and a subsequence $\{\tau_i\}$ such that $\mu(R^n - A_\delta) < \delta$ and $W(\tau_i \cap A_\delta - A \cap A_\delta) < \delta$ for all i.

Part 3. Choose $0 < \delta < 1$ and let $\{\delta_i\}$ be a sequence of real numbers tending to zero with $\sum_{i=1}^{\infty} \delta_i < \delta/3$. Part 2 supplies a set A_{δ_1}. Now repeat the procedures in parts 1 and 2 to the set $A - A_{\delta_1}$ with the restriction that only those cubes that do not intersect the closure of A_{δ_1} should be considered. Since the μ measure of the frontier of A_{δ_1} is zero, those cubes with centers on the frontier of A_{δ_1} need not be considered. Also, the subsequence that is obtained for the set A_{δ_1} is the one that should be used in the procedure for $A - A_{\delta_1}$. Hence, we will obtain a subsequence of the sequence obtained for A_{δ_1} and a set A_{δ_2} such that A_{δ_2} is the finite union of open disjoint cubes with $A_{\delta_2} \subset R^n - A_{\delta_1}$, $\mu[R^n - (A_{\delta_1} \cup A_{\delta_2})] < \delta_2$, and $W(\tau_i \cap A_{\delta_2} - A \cap A_{\delta_2}) < \delta_2$ for all i. Continue this process and let $H_\delta = \bigcup_{i=1}^{\infty} A_{\delta_i}$. Then, $\mu(A - H_\delta) = 0$ and by employing Cantor's diagonal process, we obtain the following: if $S_j = \bigcap_{i=1}^{j} A_{\delta i}$, then there exists an integer j_0 such that for $j \geqq j_0$,

$$\gamma(A \cap H_\delta - A \cap S_j) < \delta/3$$

and

$$\mu(R^n - S_j) = \mu(A \cap H_\delta - A \cap S_j) < \delta/3 .$$

Hence, $M(A \cap H_\delta - A \cap S_{j_0}) < \delta/3$. Since $\|\tau_i\| \to \mu$ and since S_{j_0} is open, there exists one integer $i_0(j_0)$ such that for $i \geqq i_0(j_0)$, $M(\tau_i - \tau_i \cap S_{j_0}) < \delta/3$. Therefore, for $i \geqq i_0(j_0)$,

$$\begin{aligned} W(\tau_i - A \cap H_\delta) &\leqq W(\tau_i - \tau_i \cap S_{j_0}) + W(\tau_i \cap S_{j_0} - A \cap S_{j_0}) \\ &\quad + W(A \cap S_{j_0} - A \cap H_\delta) \\ &\leqq \delta/3 + \sum_{i=1}^{j_0} \delta_i + \delta/3 < \delta . \end{aligned}$$

We now have, for every $\delta > 0$, an open set H_δ and a subsequence $\{\tau_i\}$ such that $\mu(A - H_\delta) = 0$ and $W(\tau_i - A \cap H_\delta) < \delta$ for all i.

Part 4. Choose $\delta > 0$ and again let $\{\delta_1\}$ be a sequence of real numbers tending to zero. After obtaining the set H_{δ_1}, repeat parts 1, 2, and 3 to the set $A \cap H_{\delta_1}$ and to the sequence that was obtained for $A \cap H_{\delta_1}$. Since H_{δ_1} is open, we can require that $H_{\delta_2} \subset H_{\delta_1}$. Continue this process and let $E = \bigcap_{i=1}^{\infty} H_{\delta_1}$ to obtain $\mu(A - E) = 0$. By employing Cantor's diagonal process, we obtain a sequence $\{\tau_i\}$ such that $W(\tau_i - A \cap H_{\delta_j}) < \delta_j$ for large i. Choose j_0 such that $\delta_{j_0} < \delta/2$ and $\gamma(A \cap H_{\delta_{j_0}} - A \cap E) < \delta/2$. Thus

$$\begin{aligned} W(\tau_i - A \cap E) &\leqq W(\tau_i - A \cap H_{\delta_{j_0}}) + W(A \cap H_{\delta_{j_0}} - A \cap E) \\ &\leqq \delta/2 + \gamma(A \cap H_{\delta_{j_0}} - A \cap E) < \delta \quad \text{for large } i , \end{aligned}$$

and therefore the conclusion of the theorem follows.

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