ORTHOGONAL DEVELOPMENTS OF FUNCTIONALS AND RELATED THEOREMS IN THE WIENER SPACE OF FUNCTIONS OF TWO VARIABLES[1]

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1. Introduction. Let C_w be the Wiener space of functions of two variables, i.e. the collection of real valued continuous functions $f(x, y)$ defined on $Q: 0 \leqq x, y \leqq 1$ and satisfying $f(0, y) = f(x, 0) = 0$. Let $F[f]$ be a complex valued functional defined almost everywhere on C_w and having Wiener measurable[1] real and imaginary parts, and let $L_2(C_w)$ be the Hilbert space of functionals $F[f]$ satisfying

$$\int_{C_w} |F[f]|^2 d_w f < \infty$$

with the inner product

$$(F_1, F_2) = \int_{C_w} F_1[f]\overline{F_2[f]} d_w f \, .$$

The contents of this paper are:

1. An extension of the Cameron-Martin translation theorem, Theorem III, [11].
2. An extension of the Paley-Wiener theorem to C_w. Our proof is different from that of Paley and Wiener given for the Wiener space of functions of one variable and is based on the extended Cameron-Martin translation theorem, and
3. Construction of complete orthonormal systems in $L_2(C_w)$.

2. THEOREM I[2]. *Let $p(x, y)$ be of bounded variation[3] on Q, $p(0, y)$, $p(1, y)$, $p(x, 0)$, $p(x, 1)$ be of bounded variation on the respective unit interval. Let $p(x, y)$ be continuous a.e. and bounded on Q. Let*

$$q(x, y) = \int_{Q_{xy}} p(s, t) ds\, dt \text{ where } Q_{xy} = [0, x] \times [0, y] \, , \quad 0 \leqq x, y \leqq 1 \, . \tag{2.1}$$

Let $\Gamma \subset C_w$ be Wiener measurable and a translation T be defined by

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[1] For the definition of the Wiener measure and Wiener measurable functionals, see [10] or [11].

[2] Theorem 1 can also be derived from Theorem 3, [9].

[3] For functions of bounded variation in n variables and Riemann-Stieltjes integrals with respect to them, see [11].

$$TΓ = \{f \in C_w; f = g - q, g \in Γ\} . \tag{2.2}$$

Then

$$m(Γ) = \exp\left\{-\int_Q p^2 dxdy\right\}\int_{TΓ} \exp\left\{-2\int_Q pd^2 f\right\} d_w f \tag{2.3}$$

and if $F[g]$ *is a real valued measurable functional defined on* $Γ$,

$$\int_Γ F[g]d_w g = \exp\left\{-\int_Q p^2 dxdy\right\}\int_{TΓ} F[f+q] \exp\left\{-2\int_Q pd^2 f\right\} d_w f . \tag{2.4}$$

The proof of this theorem for the most part consists in justification for passing to the limit under the various integral signs involved and is lengthy and we shall only give an outline in the following leaving the details to the reader.

The linearization L_p of a function $p(x, y)$ defined on Q corresponding to a partition $\mathfrak{P}$ of Q: $0 = x_0 < x_1 < \cdots < x_m = 1$, $0 = y_0 < y_1 < \cdots < y_n = 1$ is the continuous function defined on Q which agrees with $p(x, y)$ at (x_i, y_j), $i = 0, 1, 2, \cdots, m$; $j = 0, 1, 2, \cdots, n$ and is linear on each of the $2mn$ closed rectangular triangles having either (x_{i-1}, y_{j-1}), (x_i, y_{j-1}), (x_i, y_j) or (x_{i-1}, y_{j-1}), (x_{i-1}, y_j), (x_i, y_j) as corners.

The following theorem can be proved more or less in the same way Helly's 2nd theorem is proved.

THEOREM. *Let* $p(x, y) \in B.V.(Q)$ *and let* $\{\mathfrak{P}_k\}$ *be a sequence of partitions of* Q *such that*

1° $\lim_{k\to\infty} \sum_i \sum_j | p(x_i^{(k)}, y_j^{(k)}) - p(x_i^{(k)}, y_{j-1}^{(k)}) - p(x_{i-1}^{(k)}, y_j^{(k)}) + p(x_{i-1}^{(k)}, y_{j-1}^{(k)}) | = 0$

2° *each* $\mathfrak{P}_k$ *is a refinement of its predecessor and* $\lim_{k\to\infty} |\mathfrak{P}_k| = 0$

where $|\mathfrak{P}_k| = \max_{i\,j}\{|x_i - x_{i-1}|, |y_j - y_{j-1}|\}$, *and let* $\{L_k p\}$ *be the sequence of linearizations of* p *corresponding to* $\{\mathfrak{P}_k\}$. *Let* $\{f_k(x, y)\}$ *be a sequence of continuous functions which converges uniformly to* $f(x, y)$ *on* Q. *Then*

$$\lim_{k\to\infty} \int_Q f_k(x, y) d^2 L^k p(x, y) = \int_Q f(x, y) d^2 p(x, y) .$$

Now let

$$q_k(x, y) = \int_{Q_{xy}} L_k p(s, t) dsdt .$$

Theorem II, [11] holds with q_k replacing f_0 since q_k satisfies the conditions on f_0. Letting $k \to \infty$ and justifying passing to the limit under the integral signs by bounded convergence and utilizing our theorem above, we derive Theorem I the way Theorem III, [11] was derived from Theorem II, [11].

COROLLARY. *If p satisfies the conditions of Theorem I, then for every complex number λ*

$$\int_{C_w} \exp\left\{\lambda\int_Q pd^2f\right\}d_wf = \exp\left\{\frac{\lambda^2}{4}\int_Q p^2dxdy\right\}. \tag{2.5}$$

Proof. Let a translation T be defined by (2.2). Then $TC_w = C_w$. Since $m(C_w) = 1$, (2.3) reduces to

$$1 = \exp\left\{-\int_Q p^2dxdy\right\}\int_{C_w} \exp\left\{-2\int_Q pd^2f\right\}d_wf. \tag{2.6}$$

Let λ be a real number. If we replace p by $-(\lambda/2)p$ which satisfies the conditions on p, (2.6) becomes (2.5), and (2.5) holds for real λ.

Now let $\lambda \in C$, the complex plane. The right hand side (2.5) is a holomorphic function on C. According to the identity theorem of holomorphic functions, to prove (2.5) we only have to show that the left hand is also holomorphic on C. This is done by means of Morera's theorem.

Let Γ be a smooth contour in C and parametrize it by its arc length s, $0 \leqq s \leqq 1$ so that $|\lambda'(s)| = 1$. Consider

$$\int_\Gamma\left[\int_{C_w}\exp\left\{\lambda\int_Q pd^2f\right\}d_wf\right]d\lambda = \int_0^l\left[\int_{C_w}\exp\left\{\lambda(s)\int_Q pd^2f\right\}\lambda'(s)d_wf\right]ds.$$

To apply Fubini's theorem, we show that the iterated integrals of the absolute values of the real and imaginary parts of the integrand are finite. Let λ_0 be a real number such that $|\lambda| \leqq \lambda_0$ for $\lambda \in \Gamma$. For any real u, $|e^{\lambda u}| < e^{\lambda_0 u} + e^{-\lambda_0 u}$ when $\lambda \in \Gamma$ and hence

$$\begin{aligned}\int_0^l\left[\int_{C_w}\left|Re\exp\left\{\lambda(s)\int_Q pd^2f\right\}\lambda'(s)\right|d_wf\right]ds &\leqq \int_0^l\left[\int_{C_w}\left\{\exp\left\{\lambda_0\int_Q pd^2f\right\} + \exp\left\{-\lambda_0\int_Q pd^2f\right\}\right\}d_wf\right]ds \\ &= 2l\exp\left\{\frac{\lambda_0^2}{4}\int_Q p^2dxdy\right\}\end{aligned}$$

which is finite. The same argument goes for the imaginary part. Thus by Fubini's theorem

$$\int_0^l \left[\int_{C_w} \exp\left\{\lambda(s)\int_Q p d^2 f\right\}\lambda'(s) d_w f\right] ds$$
$$= \int_{C_w}\left[\int_\Gamma \exp\left\{\lambda \int_Q p d^2 f\right\} d\lambda\right] d_w f = \int 0 d_w f = 0$$

because $\exp\left\{\lambda\int_Q pd^2f\right\}$ is holomorphic on C. Thus by Morera's theorem, the left hand side (2.5) is holomorphic on C and (2.5) holds.

3. THEOREM II. *Let $\{p_j(x, y)\}$, $j = 1, 2, \cdots, n$ be an orthonormal set of functions on Q with each p_j satisfying the condition on p in Theorem 1. If $\Phi(u_1, \cdots, u_n)$ is a complex valued Lebesgue measurable function on R_n, the functional*

$$F[f] = \Phi\left[\int_Q p_1 d^2 f, \cdots, \int_Q p_n d^2 f\right] \tag{3.1}$$

is Wiener measurable on C_w and

$$\int_{C_w} F[f] d_w f = \pi^{-n/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\Phi(u_1, \cdots, u_n)\exp\left\{-\sum_{j=1}^n u_j^2\right\} du_1 \cdots du_n \tag{3.2}$$

in the sense that the existence of one side implies that of the other together with the equality.

REMARK. Let $-\infty < \alpha < \beta < \infty$ and let k be so large that $\alpha + (1/k) < \beta$. Let $\Phi_{\alpha\beta k}(u)$ be the trapezoidal function defined by

$$\Phi_{\alpha\beta k}(u) = \begin{cases} 0 & u \leqq \alpha \\ k(u-\alpha) & \alpha \leqq u \leqq \alpha + \dfrac{1}{k} \\ 1 & \alpha + \dfrac{1}{k} \leqq u \leqq \beta \\ -k(u-\beta)+1 & \beta \leqq u \leqq \beta + \dfrac{1}{k} \\ 0 & \beta + \dfrac{1}{k} \leqq u \,. \end{cases} \tag{3.3}$$

Then

$$\Phi_{\alpha\beta k}(u) = \frac{k}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iuv}}{v^2}\{-e^{-i\alpha v} + e^{-i[\alpha+(1/k)]v} + e^{-i\beta v} - e^{-i[\beta+(1/k)]v}\} dv \,. \tag{3.4}$$

This follows immediately from the following result in the calculus of residues:

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha z}}{z^2} dz = \begin{cases} -2\pi\alpha & \alpha \geqq 0 \\ 0 & \alpha \leqq 0 \end{cases}$$

where the left hand side stands for the limit as $R \uparrow \infty$, $\delta \downarrow 0$ of the integral of the same integrand along $\Gamma_{R\,\delta}$ on the z-plane defined by

$$\Gamma_{R\,\delta} = \{z \mid \delta \leqq |x| \leqq R, y = 0\} \cup \{z \mid |z| = \delta, y \leqq 0\} .$$

Proof of Theorem II. (1) Consider the Lebesgue measurable function defined on R_n.

$$\Phi(u_1, \cdots, u_n) = \exp\left\{i \sum_{j=1}^{n} \lambda_j u_j\right\}, \qquad \lambda_j, \qquad \text{real} \tag{3.5}$$

and the corresponding functional defined by (3.1)

$$F[f] = \exp\left\{i \sum_{j=1}^{n} \lambda_j \int_Q p_j d^2 f\right\} .$$

To show that $F[f]$ is Wiener measurable on C_w it suffices to show that $\int_Q p_j d^2 f$ is for each j. Now since $\int_Q p_j d^2 f$ exists for all $f \in C_w$, let us choose independently of f a sequence of partitions $\{\mathfrak{P}_k\}$ of Q with $\lim_{k\to\infty} |\mathfrak{P}_k| = 0$. For definiteness we may also agree to choose $(\xi_i^{(k)}, \eta_j^{(k)})$, $i = 1, 2, \cdots, m(k)$, $j = 1, 2, \cdots, n(k)$ in the Riemann-Stieltjes sum to be always $(x_i^{(k)}, y_j^{(k)})$.

Since $\sum_{z=1}^{n} \lambda_j p_j$ satisfies the condition on p of the Corollary of Theorem I, § 2, we have, by the orthonormality of $\{p_j\}$

$$\begin{aligned} \int_{C_w} F[f] d_w f &= \int_{C_w} \exp\left\{i \sum_{j=1}^{n} \lambda_j \int_Q p_j d^2 f\right\} \\ &= \exp\left\{-\frac{1}{4} \int_Q \left\{\sum_{j=1}^{n} \lambda_j p_j\right\}^2 dx dy\right\} = \exp\left\{-\frac{1}{4} \sum_{j=1}^{n} \lambda_j^2\right\} . \end{aligned}$$

On the other hand

$$\begin{aligned} &\pi^{-n/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \Phi(u_1, \cdots, v_n) \exp\left\{-\sum_{j=1}^{n} u_j^2\right\} du_1 \cdots du_n \\ &= \pi^{-n/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \exp\left\{i \sum_{j=1}^{i} \lambda_j u_j\right\} \exp\left\{-\sum_{j=1}^{n} u_j^2\right\} du_1 \cdots du_n \\ &= \pi^{-n/2} \prod_{j=1}^{n} \int_{-\infty}^{\infty} \exp\{i\lambda_j u_j - u_j^2\} du_j \\ &= \pi^{-n/2} \prod_{j=1}^{n} \int_{-\infty}^{\infty} \exp\left\{-\left(u_j - \frac{i\lambda_j}{2}\right)^2\right\} d\left(u_j - \frac{i\lambda_j}{2}\right) \exp\left\{-\frac{\lambda_j^2}{4}\right\} \\ &= \exp\left\{-\frac{1}{4} \sum_{j=1}^{n} \lambda_j^2\right\} . \end{aligned}$$

Thus both sides of (3.2) exist and (3.2) holds.

(2) Let $\Phi(u_1, \cdots, u_n)$ be the characteristic function $\chi_I(u_1, \cdots, u_n)$ of an interval I in R_n, i.e. let

$$I = I_1 \times \cdots \times I_n \text{ with } I_j = \{u_j \in R_1 ; -\infty \leqq \alpha_j < u_j \leqq \beta_j \leqq \infty\} , \quad j = 1, 2, \cdots, n .$$

Consider the case where I is bounded, i.e. $-\infty \leqq \alpha_j, \beta_j \leqq \infty$, $j = 1, 2, \cdots, n$. Now

$$\Phi(u_1, \cdots, u_n) = \chi_I(u_1, \cdots, u_n) = \prod_{j=1}^{n} \Phi_j(u_j) \quad \text{with } \Phi(u_j) = \chi_{I_j}(u_j),\ j = 1, 2, \cdots, n .$$

Let $\Phi_{j,k}(u_j) = \Phi_{\alpha_j \beta_j, k}(u_j)$ as defined in Remark and let

$$\begin{aligned} \Phi_k(u_1, \cdots, u_n) &= \prod_{j=1}^{n} \Phi_{j,k}(u_j) , \\ F_k[f] &= \Phi_k\left[\int_Q p_1 d^2 f, \cdots, \int_Q p_n d^2 f\right] , \\ F[f] &= \Phi\left[\int_Q p_1 d^2 f, \cdots, \int_Q p_n d^2 f\right] . \end{aligned}$$

The functionals $F_k[f]$, $F[f]$ are Wiener measurable. Now by Remark

$$\begin{aligned} \int_{C_w} F_k[f] d_w f &= \int_{C_w} \prod_{j=1}^{n} \Phi_{jk}\left[\int_Q p_j d^2 f\right] d_w f \\ &= \int_{C_w} \frac{k^n}{(2\pi)^n} \left[\int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \exp\left\{i \sum_{j=1}^{n} v_j \int_Q p_j d^2 f\right\} \prod_{j=1}^{n} \Psi_{j,k}(v_j) dv_1 \cdots dv_n\right] \end{aligned}$$

where

$$\begin{aligned} \Psi_{j,k}(v_j) = \frac{1}{v_j^2}\Big[&-\exp\{-i\alpha_j v_j\} + \exp\left\{-i\left(\alpha_j + \frac{1}{k}\right)v_j\right\} \\ &+ \exp\{-i\beta_j v_j\} - \exp\left\{-i\left(\beta_j + \frac{1}{k}\right)v_j\right\}\Big] . \end{aligned}$$

Now since

$$\left|\exp\left\{i \sum_{j=1}^{n} v_j \int_Q p_j d^2 f\right\} \prod_{j=1}^{n} \Psi_{j,k}(u_j)\right| \leqq \prod_{j=1}^{n} |\Psi_{j,k}(u_j)| \leqq 4^n \prod_{j=1}^{n} \frac{1}{v_j^2}$$

so that the repeated integral of the absolute value of the integrand is finite and Fubini's theorem is applicable. Thus

$$\int_{C_w} F_k[f]d_w f$$
$$= \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\frac{k^n}{(2\pi)^n}\left[\int_{C_w}\exp\left\{i\sum_{j=1}^{n}v_j\int_Q p_j d^2 f\right\}d_w f\right]\prod_{j=1}^{n}\Psi_{j\ k}(v_j)dv_1\cdots dv_n$$
$$= \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\frac{k^n}{(2\pi)^n}\left[\pi^{-n/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\exp\left\{i\sum_{j=1}^{n}u_jv_j\right\}\exp\left\{-\sum_{j=1}^{n}u_j^2\right\}\right.$$
$$\left.\cdot\prod_{j=1}^{n}\Psi_{j,k}(v_j)du_1\cdots du_n\right]$$

where the second equality is from (1). Applying Fubini's theorem again

$$\int_{C_w} F_k[f]d_w f = \int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\pi^{-n/2}\prod_{j=1}^{n}$$
$$\cdot\left[\frac{k}{2\pi}\int_{-\infty}^{\infty}\exp\{iu_jv_j\}\Psi_{j,k}(v_j)dv_j\right]\exp\left\{-\sum_{j=1}^{n}u_j^2\right\}du_1\cdots du_n$$
$$= \pi^{-n/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\prod_{j=1}^{n}\Phi_{j\ k}(u_j)\exp\left\{-\sum_{j=1}^{n}u_j^2\right\}du_1\cdots du_n$$
$$= \pi^{-n/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\Phi_k(u_1,\cdots,u_n)\exp\left\{-\sum_{j=1}^{n}u_j^2\right\}du_1\cdots du_n\ .$$

Since $F_k[f]$ and $\Phi_k(u_1, \cdots, u_n)$ converge nondecreasingly to $F[f]$ and $\Phi(u_1, \cdots, u_n)$ respectively, both sides of (3.2) exist and (3.2) holds if we let $k \to \infty$ in the last equality according to Levi's monotone convergence theorem.

When I is unbounded, we take an increasing sequence of bounded intervals $\{I_r\}$ which converges to I. Then $\chi_{I_r}(u_1, \cdots, u_n) \uparrow \chi_I(u_1, \cdots, u_n)$ as $r \to \infty$ on R_n and hence $\chi_{I_r}[f] \uparrow \chi_I[f]$ as $r \to \infty$ on C_w. Thus $\chi_I[f]$ is Wiener measurable. For each $\chi_{I_r}(u_1, \cdots, u_n)$, both sides of (3.2) exist and (3.2) holds. By Levi's monotone convergence theorem, the same is true of $\chi_I(u_1, \cdots, u_n)$.

(3) To complete the proof of the theorem we show that both sides of (3.2) exist and are equal when Φ is the characteristic function of an 0-set, an 0_δ-set (i.e. the intersection of countably many 0-sets), a null set and finally a measurable set in R_n. We then show that the same is true when Φ is an integrable simple function and when it is an extended positive valued measurable function defined on R_n. We conclude by showing that when Φ is an extended real valued or a complex valued measurable function on R_n the existence of one side of (3.2) implies that of the other and the equality of the two.

COROLLARY. *Let p satisfy the conditions of Theorem I. Then*

$$\int_{C_w}\left[\int_Q p d^2 f\right]^2 d_w f = \frac{1}{2}\int_Q p^2 dxdy .$$

Proof. In case $\int_Q p^2 dxdy = 0$, $p(x, y) = 0$ a.e. on Q, $\int_Q p d^2 f = 0$ for all $f \in C_w$, and hence $\int_{C_w}\left[\int_Q p d^2 f\right]^2 d_w f = 0$ and the corollary holds trivially.

Suppose $\int_Q p^2 dxdy \neq 0$. Let $\varphi(x, y) = (1/\lambda)p(x, y)$ where $\lambda = \left\{\int_Q p^2 dxdy\right\}^{1/2}$. By Theorem II

$$\int_{C_w}\left[\int_Q p d^2 f\right] d_w f = \lambda^2 \int_{C_w}\left[\int_Q \varphi d^2 f\right]^2 d_w f$$
$$= \lambda^2 \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} u^2 e^{-u^2} du = \frac{\lambda^2}{2} = \frac{1}{2}\int_Q p^2 dxdy .$$

4. DEFINITION 1. Let $\{e_m(u)\}$, $m = 0, 1, 2, \cdots$, be a C.O.N. system in the separable Hilbert space $L_2(-\infty, \infty)$ with

$$e_0(u) = \pi^{-1/4} \exp\left\{-\frac{u^2}{2}\right\} \tag{4.1}$$

and let

$$G_m(u) = \pi^{1/4} \exp\left\{\frac{u^2}{2}\right\} e_m(u) , \qquad m = 0, 1, 2, \cdots . \tag{4.2}$$

As immediate consequences of the definition, we have

$$G_0(u) \equiv 1 \tag{4.3}$$

$$\pi^{-1/2}\int_{-\infty}^{\infty} G_m(u) G_n(u) \exp\{-u^2\} du = \delta_{mn} \tag{4.4}$$

$$\pi^{-1/2}\int_{-\infty}^{\infty} G_m(u) \exp\{-u^2\} du = 0 , \qquad m = 1, 2, \cdots . \tag{4.5}$$

As an example of $G_m(u)$, let us name the partially normalized Hermite polynomials

$$G_m(u) = (-1)^m 2^{-m/2} (m!)^{-1/2} \exp\{u^2\} \frac{d^m}{du^m} \exp\{-u^2\} , \qquad m = 0, 1, 2, \cdots .$$

DEFINITION 2. Let $\{p_k(x, y)\}$ be a C.O.N. system of real valued functions in $L_2(Q)$ with each p_k satisfying the conditions on p in Theorem I. Let

$$\Phi_{m\,k}[f] = G_m\left[\int_Q p_k d^2 f\right], \qquad \text{for } f \in C_w \tag{4.6}$$

(4.7) $$\Psi_{m_1, \dots, m_r}[f] = \Phi_{m_1,1}[f] \cdots \Phi_{m_r,r}[f] .$$

Then by (4.3)

(4.8) $$\Phi_{0\,k}[f] = 1$$

(4.9) $$\Psi_{m_1,\dots,m_r,0,\dots,0}[f] = \Psi_{m_1,\dots,m_r}[f] .$$

DEFINITION 3. Let $\{\Psi_\alpha\}$ be the collection of the functionals of the form (4.7), i.e. $\alpha \in A$, which is the collection of sequences of natural numbers with finitely many nonzero entries.

THEOREM III. *The series expansion of any $F[f] \in L_2(C_w)$ in $\{\Psi_\alpha\}$ converges to $F[f]$ in the $L_2(C_w)$ sense, i.e.*

(4.10) $$\lim_{N\to\infty} \int_{C_w} \left| F[f] - \sum_{m_1 \cdots, m_N=0}^{N} A_{m_1,\dots,m_N} \Psi_{m_1,\dots,m_N}[f] \right|^2 d_w f = 0$$

where $A_{m_1,\dots,m_N}$ are the Fourier coefficients

(4.11) $$A_{m_1\cdots m_N} = \int_{C_w} F[f]\Psi_{m_1,\dots,m_N}[f] d_w f .$$

The proof is based on the following lemmas.

LEMMA 1. *$\{\Psi_\alpha\}$, is a O.N. system over C_w.*

COROLLARY. *The Bessel inequality and the best approximation theorem hold with $\{\Psi_\alpha\}$.*

DEFINITION 5. We say that $F[f] \in S_n\{p_k\}$ if $F[f]$ is defined on C_w and

(4.12) $$F[f] = \Phi\left[\int_Q p_1 d^2 f, \cdots, \int_Q p_n d^2 f\right]$$

where

(4.13) $$\Phi(u_1, \cdots, u_n) \exp\left\{-\frac{1}{2}(u_1^2 + \cdots + u_n^2)\right\} \in L_2(R_n) .$$

LEMMA 2. *Let $F[f] \in S_n\{p_k\}$, then with $m_r \neq 0$,*

(4.14) $$\int_{C_w} F[f]\Psi_{m_1,\dots,m_r}[f] d_w f = \begin{cases} 0 & \text{if } r \neq n \\ \varphi_{m_1\cdots m_n} & \text{if } r = n \end{cases}$$

where

$$(4.15) \qquad \varphi_{m_1,\cdots,m_n} = \pi^{-n/2}\int_{-\infty}^{\infty}(n)\int_{-\infty}^{\infty}\Phi(u_1, \cdots, u_n) \cdot \exp\left\{-\sum_{k=1}^{n} u_k^2\right\}\left\{\sum_{k=1}^{n} G_{m_k}(u_k)\right\}du_1 \cdots du_n .$$

LEMMA 3. *If* $F[f] \in S_n\{p_k\}$ *for some* n, *the orthogonal development of* $F[f]$ *in* $\{\Psi_\alpha\}$ *converges in the* $L_2(C_w)$ *sense to* $F[f]$.

LEMMA 4. *Functionals of the type given in Definition* 5 *are dense in* $L_2(C_w)$ *in the sense of Hilbert metric, i.e. if* $F[f] \in L_2(C_w)$, *for every* $\varepsilon > 0$ *there exists a positive integer* l *and a functional* $F^*[f] \in S_l\{p_\lambda\}$ *such that*

$$(4.16) \qquad \left\{\int_{C_w} | F[f] - F^*[f] |^2 d_w f\right\}^{1/2} < \varepsilon .$$

These lemmas are proved by means of Theorem II in the same way the corresponding lemmas for the Wiener space of functions of one variable are proved in [4] with relevant modifications made for the two variable case. We wish to point out that in the proofs in [4] only those properties of partially normalized Hermite polynomials that are our (4.2), (4.3), (4.4), (4.5) are utilized.

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