CONCERNING HOMOGENEITY IN TOTALLY ORDERED, CONNECTED TOPOLOGICAL SPACE

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Throughout this paper suppose that L denotes a connected, totally ordered topological space in which there is no first or last point, and whose topology is that induced by the order.

A topological space S is said to be homogeneous provided it is true that if $(x, y) \in S \times S$, there is a homeomorphism f from S onto S such that $f(x) = y$. Let H denote the set of all homeomorphisms from L onto L, and let I denote the set of all homeomorphisms which map a closed interval of L onto a closed interval of L. Let $H_0(I_0)$ denote the set of all elements of $H(I)$ which preserve order.

THEOREM 1. *If L is homogeneous, then L satisfies the first axiom of countability.*

Proof. It suffices to show that for some point z of L there exists an increasing sequence $x_1, x_2, \cdots$ and a decreasing sequence $y_1, y_2, \cdots$ such that each of these sequences converges to z. Suppose there is no such point. Let $P_1, P_2, \cdots$ denote an increasing sequence which converges to a point P and $Q_1, Q_2, \cdots$ a decreasing sequence which converges to a point Q. There is an element g in H such that $g(P) = Q$. In view of the preceding supposition, g is order reversing. There is a point R such that $g(R) = R$, and R is the limit of a sequence $R_1, R_2, \cdots$ which is either increasing or decreasing. Suppose the sequence is decreasing. The sequence $g(R_1), g(R_2), \cdots$ is increasing and converges to R. This yields a contradiction. The case where $R_1, R_2, \cdots$ is increasing is similar.

THEOREM 2. *The space L is homogeneous if and only if each pair of closed subintervals of L are topologicaly equivalent.*

Proof. Part 1. Suppose each pair of closed subintervals of L are topologically equivalent and $(x, y) \in L \times L$. There exist elements z and w of L such that $z < x < w$ and $z < y < w$, and an element g of I from $[z, x]$ onto $[z, y]$. If g is order reversing there is an element g' of I_0 from $[z, x]$ onto $[z, y]$ which may be constructed as follows: Let t denote the point of $[z, x]$ such that $g(t) = t$. g' is defined by

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$g'(u) = \begin{cases} u, & z \leqq u \leqq t \\ gg(u), & t < u \leqq x \end{cases}$. In any event, let g' and h' denote elements of I_0 which map $[z, x]$ and $[x, w]$, respectively, onto $[z, y]$ and $[y, w]$, respectively. The function f defined by

$$f(u) = \begin{cases} u\,, & u < z \text{ or } u > w \\ g'(u)\,, & z \leqq u \leqq x \\ h'(u)\,, & x < u \leqq w \end{cases}$$

is an element of H_0 such that $f(x) = y$.

Part 2. Suppose L is homogeneous.

LEMMA 1. *If $(x, y) \in L \times L$, there is an element f of H_0 such that $f(x) = y$. Furthermore, if $f \in I$ there is an element g of I_0 having the same domain and range, respectively, as f.*

Proof. Suppose $g \in H$ and $g(x) = y$, but g is not in H_0. There is a point b such that $b = g(b)$ and an element h of H such that $h(x) = b$. The function $f = gh^{-1}g^{-1}h$ is in H_0 and $f(x) = y$. The proof of the second part of Lemma 1 follows easily from the first part and the proof of Part 1 of Theorem 2.

LEMMA 2. *Suppose $[a, b]$ is a closed interval and f and g are elements of I_0 defined on $[a, b]$ such that $f(a) = g(a)$ $(f(b) = g(b))$, but that $f(x) < g(x)$ for $a < x \leqq b$ $(a \leqq x < b)$. If $f(a) < x_0 < f(b)$ $(g(a) < x_0 < g(b))$ and $x_1, x_2, \cdots$ is a sequence such that $x_n = fg^{-1}(x_{n-1})$ $(x_n = gf^{-1}(x_{n-1}))$ for $n \geqq 1$, then $x_0, x_1, x_2, \cdots$ is a decreasing (increasing) sequence which converges to $f(a)$ $(f(b))$.*

Proof of first part. The inequality $a < g^{-1}(x_0) < f^{-1}(x_0) < b$ implies that $f(a) < x_1 = fg^{-1}(x_0) < x_0 < f(b)$. Suppose it has been established that $f(a) < x_n < x_{n-1} < f(b)$. The preceding implies that $a < g^{-1}(x_n) < f^{-1}(x_n) < b$, which implies that $f(a) < x_{n+1} = fg^{-1}(x_n) < x_n < f(b)$. Therefore, $x_0, x_1, x_2, \cdots$ is a decreasing sequence bounded below by $f(a)$, and thus converges to a point $x \geqq f(a)$. Suppose $x > f(a)$. Since $gf^{-1}(x) > x$, there is a positive integer n such that $gf^{-1}(x) > x_n > x$, which implies that $x > fg^{-1}(x_n) = x_{n+1}$. This yields a contradiction, so $x = f(a)$.

LEMMA 3. *If $c \in L$ there exist an interval $[a, b]$ and elements f and g of I_0 with domain $[a, b]$ such that $f(a) = g(a) = c$ and $f(x) < g(x)$, for $a < x \leqq b$; or if $c \in L$ there exists an interval $[a, b]$ and elements f and g of I_0 with domain $[a, b]$ such that $f(b) = g(b) = c$ and $f(x) < g(x)$, for $a \leqq x < b$.*

Proof. Suppose that for each element (x, y) of $L \times L$ there is a unique element f of H_0 such that $f(x) = y$. Let $u_1, u_2, \cdots$ denote an increasing sequence converging to a point u, and for each n, let f_n denote the element of H_0 such that $f_n(u) = u_n$. If x is an element of L and n a positive integer, then $f_n(x) < f_{n+1}(x) < x$; for if this is not the case, the graph of f_n intersects the graph of f_{n+1}, or the graph of f_{n+1} intersects the graph of the identity homeomorphism, and in either event there is a contradiction to the unique homeomorphism hypothesis. If for some x, the sequence $f_1(x), f_2(x), \cdots$ converges to a point $y < x$, the element g of H_0 such that $g(x) = y$ has the property that its graph either intersects the graph of the identity function or the graph of f_n, for some n. Therefore, for any x in L, the sequence $f_1(x), f_2(x), \cdots$ is increasing and converges to x.

For each positive integer j, let $a_{j1}, a_{j2}, \cdots$ and $b_{j1}, b_{j2}, \cdots$ denote sequences such that (1) $a_{j1} = f_j^{-1}(u)$ and $b_{j1} = f_j(u)$, and (2) $a_{jn} = f_j^{-1}(a_{j,n-1})$ and $b_{jn} = f_j(b_{j,n-1})$, for $n > 1$. Suppose $u < x$ and (r, s) is an open interval containing x. Let n denote an integer such that $r < f_n(x)$ and $x < f_n(s)$. Since $u < x < f_n(s)$, it follows that $a_{n1} = f_n^{-1}(u) < s$. If a_{n1} is not in (r, s), let K denote the set of all a_{nj} such that $a_{nj} < x$ and let $z = \text{l.u.b.}\, K$. If $z \leqq r$, there is an element a_{nj} of K such that $f_n(z) < a_{nj} \leqq z < f_n(x)$, which implies that $z < f_n^{-1}(a_{nj}) = a_{n,j+1} < x$, which is a contradiction. In any event, some a_{nj} is an element of (r, s). The preceding argument clearly indicates that $\sum (a_{ij} + b_{ij})$ is a countable set dense in L, so L is a real line and the unique homeomorphism hypothesis is contradicted.

There exist elements h and k of H_0 and points s and t of L such that $h(s) = k(s)$, but $h(t) < k(t)$. Suppose $s < t$. Let a denote the largest element x of L such that $h(x) = k(x)$ and $x < t$. There is an element p of I_0 with domain $[k(a), k(t)]$ such that $p(k(a)) = c$. The functions $f = p(h)$ and $g = p(k)$ and the interval $[a, t]$ satisfy the first conclusion of the lemma. The case $t < s$ yields the second conclusion.

LEMMA 4. *Suppose $[a, b]$ is a closed interval and c is a point. If $x > c$, there is a point y in (c, x) and an element f of I_0 mapping $[a, b]$ onto $[c, y]$.*

Proof. Let U denote the set of all $x > c$ such that there is a homeomorphism from $[a, b]$ onto $[c, x]$, and let V denote the set of all $x < c$ such that there is a homeomorphism from $[a, b]$ onto $[x, c]$. The sets U and V exist because of the existence of elements h_1 and h_2 of H_0 such that $h_1(a) = c$ and $h_2(b) = c$. Let $u = \text{g.l.b.}\, U$, $v = \text{l.u.b.}\, V$ and suppose that $c < u$.

Case 1. Suppose the first conclusion of Lemma 3 holds There exists a point u_1, an interval $[p, q]$, and elements f and g of I_0 having domain $[p, q]$, and such that (1) $c < u_1 < u$, (2) $f(p) = g(p) = u_1$, and (3) $f(x) < g(x)$, for $p < x \leqq q$. There is a point r such that $p < r < q$, $g(r) < u$, and $g(r) < f(q)$, and an element k of I_0 having domain $[p, q]$ such that (1) $k(r) = u$, and (2) $k(x) \geqq g(x)$ for $x \in [p, q]$. The function h defined on $[p, q]$ by $h(x) = kg^{-1}f(x)$ is an element of I_0 such that (1) $h(q) > u$, (2) $h(p) = k(p)$, and (3) $h(x) < k(x)$, for $p < x \leqq q$. There is a point x_0 such that $u \leqq x_0 < h(q)$ and an element f_0 of I_0 mapping $[a, b]$ onto $[c, x_0]$. Let $x_1, x_2, \cdots$ denote a sequence such that $x_n = hk^{-1}(x_{n-1})$ for $n \geqq 1$, and let $f_1, f_2, \cdots$ denote a sequence of functions defined on $[a, b]$ such that for $n \geqq 1$ (1) $f_n(x) = f_0(x)$, for $a \leqq x \leqq f_0^{-1}(u_1)$, and (2) $f_n(x) = hk^{-1}f_{n-1}(x)$, for $f_0^{-1}(u_1) < x \leqq b$. For each n, f_n is a homeomorphism from $[a, b]$ onto $[c, x_n]$, but, according to Lemma 2, $x_n < u$ for some n. This yields a contradiction, so $u = c$.

Case 2. If the second conclusion of Lemma 3 holds, then it follows, by an argument similar to the one in Case 1, that $v = c$. Let u_1 denote a point between c and u, and g an element of H_0 such that $g(c) = u_1$. There is a point u_2 such that $c < u_2 < u_1$ and an element h of I_0 mapping $[a, b]$ onto $[g^{-1}(u_2), c]$. The function $g(h)$ is an element of I_0 mapping $[a, b]$ onto $[u_2, u_1]$. Let k denote an element of H_0 such that $k(a) = c$. Since $k(b) \geqq u$, there is a point t such that $k(t) = gh(t)$. The function f defined by

$$f(x) = \begin{cases} k(x) \ , & a \leqq x \leqq t \\ gh(x) \ , & t < x \leqq b \end{cases}$$

is an element of I_0 which maps $[a, b]$ onto $[c, u_1]$, so in this case also, the assumption $c < u$ leads to a contradiction.

The proof of the main result now follows easily. Suppose $[a, b]$ and $[c, d]$ are closed intervals and g an element of H_0 such that $g(b) = d$.

Case 1. $g(a) \leqq c$. There is a point e such that $c < e < d$ and an element h of I_0 mapping $[a, b]$ onto $[c, e]$. As in case 2 of Lemma 4, a homeomorphism from $[a, b]$ onto $[c, d]$ may be constructed from g and h.

Case 2. $g(a) > c$. There is a point e such that $a < e < b$ and an element h of I_0 mapping $[c, d]$ onto $[a, e]$. However, h^{-1} is an element of I_0 mapping $[a, e]$ onto $[c, d]$, and a homeomorphism from $[a, b]$ onto $[c, d]$ may be easily constructed from g and h^{-1}.

In order to establish the next theorem it is helpful to use a result of Richard Arens'. A linear homogeneous continuum (LHC) has been defined by G. D. Birkhoff as any set of elements which 1. is simply ordered 2. provides a limit for any monotonely increasing (or decreasing) sequence 3. is isomorphic to every nondegenerate closed subinterval of itself. In [1] Arens shows, among other results, the following (reworded by the author).

THEOREM A. *If I is an LHC and for each positive integer p, I_p denotes I, then the space $I' = I_1 \times I_2 \times \cdots$ with the lexicographic order is also an LHC.*

THEOREM 3. *If L is homogeneous, $[a, b]$ is a closed interval, and for each positive integer p, I_p denotes $[a, b]$, then the space $x = L \times I_1 \times I_2 \times \cdots$ with the topology induced by the lexicographic order is also homogeneous.*

Proof. Let $[u_1, u_2, \cdots; v_1, v_2, \cdots]$ and $[x_1, x_2, \cdots; y_1, y_2, \cdots]$ denote closed subintervals of X. Let u and v denote elements of L such that $u < \min\{u_i, x_i\}$ and $v > \max\{v_i, y_i\}$ for $i = 1, 2, 3, \cdots$, and let g denote an element of I_0 which maps $[u, v]$ onto $[a, b]$. The function F defined by $F(t_0, t_1, t_2, \cdots) = [g(t_0), t_1, t_2, \cdots]$ is an order preserving homeomorphism from $[u, v] \times I_1 \times I_2 \times \cdots$ onto $[a, b] \times I_1 \times I_2 \times \cdots$. Theorem A shows that any two subintervals of the latter are homeomorphic, so it follows that $[x_1, x_2, \cdots; y_1, y_2, \cdots]$ and $[u_1, u_2, \cdots; v_1, v_2, \cdots]$ are homeomorphic. Therefore, by theorem 2, X is homogeneous.

Suppose $L_1, L_2, L_3, \cdots$ denotes a sequence of spaces such that (1) L_1 is the real line, and (2) for each n, L_{n+1} is constructed from L_n by a Theorem 3 type construction. The main theorem of Arens' paper [2] yields the result that if $i \neq j$, then L_i is not homeomorphic to L_j. Is it true that if a homogeneous space L' satisfies the axioms stated on the first page and also has the property that it can be covered by a countable collection of closed intervals, then L' is one of the spaces $L_1, L_2, L_3, \cdots$?

In part 2 of Theorem 2 the construction indicated gives an order preserving homeomorphism from $[a, b]$ onto $[c, d]$. This leads naturally to the following question: If L' satisfies the axioms of L, is homogeneous, and $[a, b]$ is a closed subinterval of L', then is there an order reversing homeomorphism from $[a, b]$ onto $[a, b]$?

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