CONJUGATE FUNCTIONS IN ORLICZ SPACES

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1. The purpose of this paper is to prove the following results:

THEOREM 1. *Let*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\tan(1/2)t}\,dt = \lim_{\varepsilon\to+0}\left\{-\frac{1}{\pi}\int_\varepsilon^\pi\right\}.$$

The mapping $f \to \tilde{f}$ is a bounded mapping of an Orlicz space into itself if and only if the space is reflexive.

Beginning with the classical result by M. Riesz for the L_p spaces [6; vol. I, p. 253] several authors have proved this theorem in one direction or the other for various special classes of Orlicz spaces. We mention in particular the papers by J. Lamperti [2] and S. Lozinski [4] and the results given in A. Zygmund's book [6; vol. II, pp. 116–118]. In our proof we use inequalities and techniques due to S. Lozinski [3, 4] to show that boundedness of the mapping implies that the space is reflexive. We use the theorem of Marcinkiewicz on the interpolation of operations [6; vol. II, p. 116] to prove that reflexivity implies the boundedness of $f \to \tilde{f}$. Our results are more general than Lozinski's results since we use the definition of an Orlicz space given by A. C. Zaanen [5] which includes, for example, the space L_1.

Section 2 contains preliminary material about Orlicz spaces. In § 3 we prove that boundedness implies reflexivity and in § 4 we prove the converse.

2. Let $v = \varphi(u)$ be a nondecreasing real valued function defined for $u \geqq 0$. Assume that $\varphi(0) = 0$, that φ is left continuous and that φ does not vanish identically. Let $u = \psi(v)$ be the left continuous inverse of φ. If $\lim_{u\to\infty}\varphi(u) = l$ is finite then $\psi(v) = \infty$ for $v > l$; otherwise $\psi(v)$ is finite for all $v \geqq 0$. The complementary Young's functions Φ and Ψ are defined by

$$\Phi(u) = \int_0^u \varphi(t)dt\,, \qquad \Psi(v) = \int_0^v \psi(s)ds\,.$$

Φ is an absolutely continuous convex function for $0 \leqq u < \infty$ and Ψ is absolutely continuous and convex in the internal where it is finite.

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If $\lim_{u\to\infty}\varphi(u)=\infty$ this internal is $0 \leqq v < \infty$. If $\lim_{u\to\infty}\varphi(u)=l$ is finite we say that Ψ jumps to infinity at $v=l$.

Φ is said to satisfy the Δ_2-condition if there is a constant $k>0$ and a $u_0 \geqq 0$ such that $\Phi(2u) \leqq k\Phi(u)$ for $u \geqq u_0$. This is equivalent to satisfying the inequality $\Phi(lu) \leqq kl\Phi(u)$ for all sufficiently large u, where l is any number greater than one (for a proof and further details see [1; p. 23]).

The Orlicz space $L_\Phi = L_\Phi(0, 2\pi)$ consists, by definition, of all measurable complex functions f defined on the unit circle for which $\|f\|_\Phi = \sup\int_0^{2\pi}|f(t)g(t)|\,dt < \infty$, where the supremum is taken over all functions g with $\int_0^{2\pi}\Psi\,|g(t)|\,dt \leqq 1$. The space L_Ψ is defined by interchanging Φ and Ψ. The Orlicz space $L_{M\Phi}$ is defined to be the set of all measurable complex functions f for which

$$\|f\|_{M\Phi} = \sup\int_0^{2\pi}|f(t)g(t)|\,dt < \infty\ ,$$

where the supremum is taken over all g with $\|g\|_\Psi \leqq 1$. $L_{M\Psi}$ is similarly defined. The spaces L_Φ, L_Ψ, $L_{M\Phi}$ and $L_{M\Psi}$ are all Banach spaces with their respective norms when functions equal almost everywhere are identified. The spaces L_Φ and $L_{M\Phi}$ consist of the same functions and $\|f\|_{M\Phi} \leqq \|f\|_\Phi \leqq 2\|f\|_{M\Phi}$. The same is true replacing Φ by Ψ. The space L_Φ is reflexive with dual space $L_{M\Psi}$ if and only if both Φ and Ψ satisfy the Δ_2-condition.

Two Young's functions Φ_1 and Φ_2 are said to be equivalent ($\Phi_1 \sim \Phi_2$) if and only if there exist positive constants k_1, k_2, and u_0 such that $\Phi_1(k_1u) \leqq \Phi_2(u) \leqq \Phi_1(k_2u)$ for $u \geqq u_0$. It is clear that $\sim$ is an equivalence relation and that the Δ_2-condition is an equivalence class property. If $\Phi_1 \sim \Phi_2$ then L_{Φ_1} and L_{Φ_2} consist of the same functions and the norm $\|\ \|_{\Phi_1}$ and $\|\ \|_{\Phi_2}$ are equivalent. Conversely, if L_{Φ_1} and L_{Φ_2} have the same elements then $\Phi_1 \sim \Phi_2$ [1; p. 112].

3. In this section we will show that if $f \to \tilde{f}$ is bounded then L_Φ is reflexive. Let $S_n(f)$ denote the nth partial sum of the Fourier series of f and write $D_n(t) = \sin[n+(1/2)]t/2\sin(1/2)t$. If $\|\tilde{f}\|_\Phi \leqq C\|f\|_\Phi$ for all $f \in L_\Phi$ then it follows [6; vol. I, p. 266] that $\|S_n(f)\|_\Phi \leqq A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n, where A is a positive constant independent of n and f. Thus, the following result is ostensibly more general than the corresponding part of Theorem 1.

THEOREM 2. *If $\|S_n(f)\|_\Phi \leqq A\|f\|_\Phi$ for all $f \in L_\Phi$ and all n then L_Φ is reflexive.*

The proof of Theorem 2 uses the following two lemmas given by

S. Lozinski in [3]. Lozinski proved these lemmas under more restrictive conditions on φ than we have assumed. Nevertheless, Lozinski's proofs remain valid for the functions as we have defined them.

LEMMA 1. *$(\varphi(u)/250)\log(n/u\varphi(u)) \leqq \| D_n \|_\Phi$ for $u\varphi(u) \geqq 1$.*

LEMMA 2. *If $\| S_n(f) \|_\Phi \leqq A \| f \|_\Phi$ for all $f \in L_\varphi$ and all n then $\| D_n \|_\Phi \leqq 2\pi A(n + \Phi(u))/u$ for $0 < u < \infty$.*

Proof of Theorem 2. Our proof is a variation of the one given by Lozinski in [4]. From Lemmas 1 and 2 we have

$$\varphi(v)\log\frac{n}{v\varphi(v)} \leqq k\frac{n+\Phi(u)}{u} \tag{1}$$

for $v\varphi(v) \geqq 1$ and $0 < u < \infty$. $k = 2\pi A/250$. Our immediate aim is to show that for all sufficiently large $\lambda > 1$

$$\log\left(\frac{\lambda}{2}\right) \leqq 2k\frac{\varphi(v)}{\varphi\left(\dfrac{v}{\lambda}\right)} \tag{2}$$

for $v \geqq v_0$, where v_0 depends upon λ.

For any

$$\lambda > 1,\ \Phi(u) = \int_0^u \varphi(t)dt > \int_{u/\lambda}^u \varphi(t)dt$$

and hence

$$\Phi(u) > \left(u - \frac{u}{\lambda}\right)\varphi\left(\frac{u}{\lambda}\right) = (\lambda - 1)\frac{u}{\lambda}\varphi\left(\frac{u}{\lambda}\right).$$

Thus

$$\log\frac{(\lambda-1)n}{\Phi(v)} < \log\frac{n}{\dfrac{v}{\lambda}\varphi\left(\dfrac{v}{\lambda}\right)}. \tag{3}$$

By combining (3) and (1) we see that

$$\varphi\left(\frac{v}{\lambda}\right)\log\frac{(\lambda-1)n}{\Phi(v)} \leqq k\frac{n+\Phi(v)}{v} \tag{4}$$

whenever $(v/\lambda)\varphi(v/\lambda) \geqq 1$. Let $n = [\Phi(v)] =$ greatest integer in $\Phi(v)$. Then (4) becomes

$$\varphi\left(\frac{v}{\lambda}\right)\log\left\{(\lambda-1)\frac{[\Phi(v)]}{\Phi(v)}\right\} \leqq k\frac{[\Phi(v)]+\Phi(v)}{v} \leqq 2k\frac{\Phi(v)}{v}. \tag{5}$$

For every sufficiently large λ there exist a $v_0 \geqq 0$ such that for $v \geqq v_0$

$$1 < \frac{\lambda}{2} \leqq (\lambda - 1)\frac{[\Phi(v)]}{\Phi(v)} \tag{6}$$

and

$$\frac{v}{\lambda}\varphi\left(\frac{v}{\lambda}\right) \geqq 1 \,. \tag{7}$$

Using (5), (6) and the fact that $\Phi(v) \leqq v\varphi(v)$ we get inequality (2) for $v \geqq v_0$. Since λ can be arbitrarily large (2) implies that $\lim_{u\to\infty}\varphi(u) = \infty$ and hence that Ψ does not jump to infinity. We next show that Ψ satisfies the Δ_2-condition.

Let λ be large but fixed and write $l = (1/2k)\log(\lambda/2)$. Then (2) states that

$$l\varphi\left(\frac{t}{\lambda}\right) \leqq \varphi(t) \tag{8}$$

for $t \geqq v_0$. This implies, on taking inverses, that there is a number s_0 such that for $s \geqq s_0$

$$\psi(s) \leqq \lambda\psi\left(\frac{s}{l}\right) . \tag{9}$$

Thus

$$\int_{s_0}^{v}\psi(s)ds \leqq \lambda\int_{s_0}^{v}\psi\left(\frac{s}{l}\right)ds = \lambda l\int_{s_0/l}^{v/l}\psi(s)ds$$

or

$$\Psi(v) - \Psi(s_0) \leqq \lambda l\left[\Psi\left(\frac{v}{l}\right) - \Psi\left(\frac{s_0}{l}\right)\right] . \tag{10}$$

This shows that for sufficiently large v

$$\Psi(lv) \leqq 2\lambda l\Psi(v) \tag{11}$$

and hence proves that Ψ satisfies the Δ_2-condition.

If $\| S_n(f)\|_\Phi \leqq A\|f\|_\Phi$ for all $f \in L_\Phi$ then it follows that $\| S_n(g)\|_{M\Psi} \leqq A\| g\|_{M\Psi}$ for all $g \in L_{M\Psi}$ or, equivalently, that $\| S_n(g)\|_\Psi \leqq 2A\| g\|_\Psi$ for all $g \in L_\Psi$. Since we have shown that Ψ does not jump to ∞ we can interchange the rôle of Φ and Ψ in the above argument to show that Φ satisfies the Δ_2-condition. This proves that L_Φ is reflexive and completes the proof of Theorem 2.

4. In this section we prove a general result about reflexive Orlicz

spaces which combined with the classical results of M. Riesz [6; vol. I, p. 253 and p. 266] yields the unproved half of Theorem 1 as well as the converse of Theorem 2.

THEOREM 3. *Suppose that T is a bounded linear operator on L_p into L_p for $1 < p < \infty$. Then if L_Φ is reflexive T is defined and bounded on L_Φ into L_Φ.*

Proof. The proof consists of showing that Φ can be replaced by an equivalent function $\Phi_1(\Phi \sim \Phi_1)$ such that Φ_1 satisfies the conditions of the Marcinkiewicz theorem on the interpolation of operations i.e. such that

$$\int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}}\,dt = O\left\{\frac{\Phi_1(u)}{u^\beta}\right\} \tag{12}$$

and

$$\int_1^u \frac{\Phi_1(t)}{t^{\alpha+1}}\,dt = O\left\{\frac{\Phi_1(u)}{u^\alpha}\right\} \tag{13}$$

for $u \to \infty$, where $1 < \alpha < \beta < \infty$.

The assumption that L_Φ is reflexive implies that $\lim_{u\to\infty}\varphi(u) = \infty$ and hence that $\lim_{u\to\infty}\Phi(u)/u = \infty$. By [1; p. 16] Φ is equal for sufficiently large values of u to a function M of the form $M(u) = \int_0^u p(t)\,dt$ where p is a nondecreasing right continuous function with $\lim_{u\to 0} p(u) = 0$ and $\lim_{u\to\infty} p(u) = \infty$. Clearly $\Phi \sim M$.

By [1; p. 46] the function M_1 defined by $M_1(u) = \int_0^u (M(t)/t)\,dt$ is equivalent to M and hence to Φ. The derivative of M_1 is continuous and strictly increasing.

Since L_Φ is reflexive both Φ and Ψ satisfy the Δ_2-condition. Thus both M_1 and its conjugate Young's function N_1 satisfy the Δ_2-condition [1; p. 23]. According to [1; pp. 26–27] this implies the existence of numbers a, b, and $u_0 \geqq 0$ with $1 < a < b < \infty$ such that

$$1 < a < \frac{uM_1'(u)}{M_1(u)} < b$$

for all $u \geqq u_0$. If we define Φ_1 by

$$\Phi_1(u) = \begin{cases} \dfrac{M_1(u_0)}{u_0^a}\,u^a & \text{for } u \leqq u_0 \\ M_1(u) & \text{for } u \geqq u_0 \end{cases}$$

we obtain a function $\Phi_1 \sim \Phi$ such that

$$1 < a \leqq \frac{u\varphi_1(u)}{\Phi_1(u)} \leqq b \tag{14}$$

for all $u \geqq 0$.

We next show that Φ_1 satisfies (12) and (13) for suitably chosen α and β. In particular choose α and β such that $1 < \alpha < a \leqq b < \beta < \infty$. This is clearly possible. In what follows all of the integrals will exist as finite numbers because of (14).

Integration by parts shows that

$$\int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt = \beta \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt - \frac{\Phi_1(u)}{u^\beta} \tag{15}$$

and

$$\int_0^u \frac{\varphi_1(t)}{t^\alpha} dt = \alpha \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt + \frac{\Phi_1(u)}{u^\alpha} . \tag{16}$$

From (14) we obtain

$$\int_u^\infty \frac{\varphi_1(t)}{t^\beta} dt \leqq b \int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \tag{17}$$

and

$$\int_0^u \frac{\varphi_1(t)}{t^\alpha} dt \geqq a \int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt . \tag{18}$$

Combining (15) with (17) and (16) with (18) shows that

$$\int_u^\infty \frac{\Phi_1(t)}{t^{\beta+1}} dt \leqq \frac{1}{\beta - b} \left\{ \frac{\Phi_1(u)}{u^\beta} \right\} \tag{19}$$

and

$$\int_0^u \frac{\Phi_1(t)}{t^{\alpha+1}} dt \leqq \frac{1}{a - \alpha} \left\{ \frac{\Phi_1(u)}{u^\alpha} \right\} . \tag{20}$$

This shows that Φ_1 satisfies (12) and (13). Thus by the Marcinkiewicz theorem and Theorem 10.14 of [6; vol I, p. 174] there exists a constant K_1 such that $\| Tf \|_{\Phi_1} \leqq K_1 \| f \|_{\Phi_1}$ for all $f \in L_{\Phi_1}$. Since $\Phi \sim \Phi_1$ there is a constant K such that $\| Tf \|_\Phi \leqq K \| f \|_\Phi$ for all $f \in L_\Phi$. This completes the proof of Theorem 3.

Statements of the standard corollaries of Theorem 1 can be found in [2].

References

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