SOME REMARKS ON FITTING'S INVARIANTS

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In the paper [2] Fitting introduced a sequence of ideals associated with a finitely generated module M over a commutative ring as follows: if $(E)\ 0 \to K \to F \to M \to 0$ is an exact sequence with F a free module on a basis $e(1), \cdots, e(n)$ and if $k(i) = \Sigma x(ij)e(j)$, i in some index set, generates K then the jth ideal $f(j; M)$ is generated by the $(n-j)x(n-j)$ determinants of the form $(x(uv))$. These ideals are independent of the sequence (E) and have the following properties:

(i) if h is a homomorphism from a ring R to a ring S and if M is a finitely generated R module then $S\cdot h(f(j\colon M))) = f(j; S \otimes_R M)$,

(ii) denoting by $ann(M)$ the annihilator of M we have $f(0\colon M) \leqq ann(M)$ and for sufficiently large m, $[ann(M)]^m \leqq f(0\colon M)$. Note also that $f(j; M) \leqq f(j+1; M)$ and that for j sufficiently large the ideals are all (1). In this paper we wish to make some remarks on the relation between these ideals and the concepts of flat and projective modules.

In the following we shall denote by $F(j; M)$ the R module $R/f(j; M)$ and by $F(M)$ the direct sum of the $F(j; M)$. We remark that the module $F(M)$ is finitely generated and it is free if and only if $F(j; M)$ is free (or zero) for each j. First note that for a free module N we have $F(s; N)$ is free for each s and that for any module (finitely generated) we may write $F(M) = R/f(0; M) \oplus \cdots \oplus R/f(s; M) \oplus \cdots$ where we suppose $f(r; M) \neq (1)$. If $F(j; M)$ is not free for some $j < r$ then $f(r; M) \neq (0)$ and hence $f(r-1\colon F(M)) = f(r; M)$ is neither (0) nor R.

THEOREM 1. *If M is a finitely generated module over a local ring R* (not necessarily noetherian) *then M is free if and only if $F(M)$ if free. If M is free and if I is the maximal ideal of R then*

$$\dim_{R/I}(R/I \otimes_R M) = \text{rank}\,(F(M)) = \text{rank}\,(M)\ .$$

Proof. If M is free then $F(M) = \Sigma_x F(x; M) = \Sigma_{x<n} R$ if M has rank n. Assume $F(M)$ is free and that $0 \to K \to F \to M \to 0$ is exact with F free over R. We may suppose that rank $(F) = \dim_{R/I}(R/I \otimes_R M)$ by the Nakayama lemma. Suppose, therefore, that $K \neq (0)$. Then $F(r-1; M)$, if the rank of F is r, has the form $\Lambda^r F/i(K) \wedge \Lambda^{r-1}F$ where i is the inclusion map of K into F and $\Lambda^r F$ denotes the homogeneous component of degree r in the Grassmann algebra of F.

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We have that if $F(r-1; M)$ is not zero then it is not free. If $F(r-1; M) = (0)$ then $0 = R/I \otimes_R F(r-1; M) = F(r-1; R/I \otimes_R M)$ thus $F(r-1; R/I \otimes_R M) = (0)$ and therefore the dimension of $R/I \otimes_R M$ is less that or equal to $r-1$ which contradicts the choice of F.

REMARK. Villameyor has proved that a finitely generated R module M is flat if and only if M is locally free, i.e. if and only if for each prime I the module $R_I \otimes M$ is free, the tensor product taken over the homomorphism of R into R_I. This result is unpublished. By [1] it suffices to show that a finitely generated flat module over a local ring is free. One checks easily that a cyclic module is flat if and only if for a generator m (fixed) and for a collection a_i, so that $a_i m = 0$ and which span the relations of M, that for each i there are elements $b_j(i)$ of M with $\Sigma_j y_j(i) b(i)$ and $a_i b_j(i) = 0$ for each j. If M is flat then by the Nakayama lemma there is an exact sequence $0 \to K \to F \to M \to 0$ with F free, $IK = K$, I the maximal ideal of R. If F is free on r elements $f(1), \cdots, f(r)$ with images $m(i)$ in M we need only show that the module $0 \neq \Lambda^r M = \Lambda^r F / i(K) \wedge \Lambda^{r-1} F$ is free. Applying the criterion of [1] to a cyclic module it follows that a flat cyclic module is free thus we need only show that $\Lambda^r M$ is flat. A basis for the relations of $\Lambda^r M$ is given by the elements $x(i) f(1) \wedge \cdots \wedge f(r)$ where $\Sigma_i (i) f(i)$ runs over all the relations of M, i.e. over the image of K in F. If M is flat then given a relation $\Sigma x(i) f(i)$ it follows easily from the criterion of flatness in [1] and an easy computation that there are elements $y(ij)$ in R such that $m(i) = \Sigma y(ij) m(j)$ and $\Sigma_i x(i) y(ij) = 0$. In $\Lambda^r M$ set $b^* = m(1) \wedge \cdots \wedge m(r)$ and set $y^* = \det(y(ij))$. Then $y^* b^* = b^*$ and $\Sigma x(i) y(ij) = 0$ implies $x(i) b^* = 0$.

THEOREM 2. *If M is finitely generated then M is flat if and only if $F(M)$ is flat if and only if $F(j; M)$ is flat for each j.*

Proof. If $F(M)$ is flat the module $F(R_I \otimes M)$ is free for each prime I of R and $R_I \otimes M$ is free by the previous theorem which implies that M is flat. Conversely, if M is flat then $R_I \otimes F(M) = F(R_I \otimes M)$ is free which implies $F(M)$ is flat. By the remarks preceding the first theorem $F(M)$ is free if and only if $F(j; M)$ is free for each j which proves the last assertion.

LEMMA 1. *If M is a finitely generated R module then M is projective if and only if it is the covariant extension of a projective module over a noetherian ring.*

Proof. Suppose $0 \to K \to F \to M \to 0$ is exact with F free on a basis $f(k)$, $1 \leqq k \leqq n$ and assume that M is projective. Since K is a direct summand of F it is generated by finitely many elements $k(1), \cdots, k(n)$. Let b denote a homomorphism from M to F such that $ab = \text{Identity}$ and set $k(i) = \Sigma x(ij) f(j)$. Set $b(a(f(i))) = \Sigma_j y(ij) f(j)$ and denote by R^* the subring of R generated by 1 and the elements $x(ij)$ and $y(uv)$. Denote by M^* the module $a(R^* f(1) + \cdots + R^* f(n))$. If we set $F^* = R^* f(1) + \cdots + R^* f(n)$ we have an exact sequence $0 \to K \cap F^* \to F^* \to M^* \to 0$. Since the $y(ij)$ are in R^* the restriction of b to M^* splits this sequence which implies that M^* is projective. If we denote by c the inclusion map of R^* into R we have an exact sequence $0 \to R \otimes_c (K \cap F^*) \to R \otimes_c F^* \to R \otimes_c M^* \to 0$. We may identify $R \otimes_c F^*$ with F by the obvious isomorphism and under this map $R \otimes_c (K \cap F^*)$ maps onto K since $k(i)$ is in $K \cap F^*$ for each i. Therefore, $R \otimes_c M^* = M$.

LEMMA 2. *If $0 \to M' \to M \to M'' \to 0$ is exact with M, M'' finitely generated and M'' flat then $F(M) = F(M'')$ implies $M' = 0$.*

Proof. Suppose I is a maximal prime of R and set $I^* = R_I I$, $k = R_I/I^*$. The sequence $0 \to R_I \otimes M' \to R_I \otimes M \to R_I \otimes M'' \to 0$ is exact. For $N = M', M, M''$ set $R_I \otimes N = N_I$ and note that $F(M_I) = F(M_I'')$. Further M_I'' is free and hence $M_I = M_I'' + M_I'$. We have that $k \otimes M_I$ is a direct sum of M_I''/I^*M_I'' and M_I'/I^*M_I' and $k \otimes F(M_I) = k \otimes F(M_I'')$ implies that $\dim_k k \otimes M_I' = \dim_k (M_I'/I^*M_I')$ thus $M_I'/I^*M_I' = 0$. Since M_I' is a direct summand of a finitely generated module it is finitely generated and thus $M_I' = 0$ whence $M' = 0$.

THEOREM 3. *If M is a finitely generated module then M is projective if and only if $F(M)$ is projective.*

Proof. Suppose $F(M)$ is projective with $0 \to K \to F \to M \to 0$ exact and F free on $f(1), \cdots, f(m)$. Since $F(M)$ is projective so is each $F(j; M)$ and thus we have $R = f(j; M) + A(j)$ as an R module, hence $1 = r(j)b(j) + s(j)a(j)$ where $Rb(j) = f(j; M)$ and $A(j) = R\,a(j) = F(j; M)$. We have there are elements $k(j, w; v)$ in the image of K with v an integer and w a sequence of length j so that if $f(w)$ denotes the multivector $f(w(1)) \wedge \cdots \wedge f(w(j))$, (B) $\Sigma_w k(j, w; 1) \wedge \cdots \wedge k(j, w; n-j) \wedge f(w) = b(j) f(1) \wedge \cdots \wedge f(n)$. Set $k(j, w; v) = 0$ if $b(j)$ is zero, and denote by $K^\#$ the collection of all such k chosen for $0 \leqq j \leqq n$. If $k(1), \cdots, k(n-t)$ are in $K^\#$ and if v is a sequence of length t define $c(k(u); v)$ by $k(u) \wedge f(v) = c(k(u); v) b(t) f(1) \wedge \cdots \wedge f(n)$, $u = (1, \cdots, n-t)$ and set $k(j, w; v) = \Sigma_r x(j, w; vr) f(r)$. Denote by R^* the subring of R generated by 1, $c(k(u), v)$, $x(j, w; vr)$, $b(j)$, $r(i)$, $s(i)$ and $a(i)$ and set $F^* = R^* f(1) + \cdots + R^* f(n)$, $K^* = (K^\#)$ and define M^* by the exact sequence (S) $0 \to K^* \to F^* \to M^* \to 0$. We have $f(j; M^*) \leqq R^* b(j)$ by the definition of K^* and $f(j; M^*) \geqq R^* b(j)$ by (B). Since $1 = r(j)$ $b(j) + s(j)\, a(j)$ and $f(j; M) \cap A(j) = (0)$ we have $f(j; M^*)$ is R^* projective and thus M^* is projective as a flat module over a noetherian ring. The sequence (S) tensored with R considered as an R^* module is exact and identifying $R \otimes F^*$ with F under the map $h(\Sigma r(i) \otimes f(i)) = \Sigma r(i) f(i)$ we have that $h(R \otimes i^*(K^*)) \leqq i(K)$, where i and i^* are the inclusion maps of K and K^* into F and F^* respectively. Therefore, there is an exact sequence $0 \to M'' \to R \otimes M^* \to M \to 0$ with $f(r; R \otimes M^*) = Rf(j; M^*) = f(j; M)$ thus $M'' = (0)$ since M is flat ($F(M)$ is flat) hence M is projective. Conversely, if M is projective it is the covariant extension of a projective module over a noetherian ring, thus so also is $F(M)$ hence $F(M)$ is projective.

COROLLARY 3.1. *Every finitely generated flat module over a ring R is projective if and only if every flat cyclic module is projective.*

LEMMA 3. *For I a prime in a ring R denote by $n(I)$ the collection of all x in R so that $yx = 0$ for some y not in I. If*

$$(0) = Q(1) \cap \cdots Q(t) \cap \cdots \cap Q(s)$$

where $Q(i)$ is primary with radical $p(i)$ and $Q(i) \leqq I$ if and only if $i \leqq t$ then $n(I) = Q(1) \cap \cdots \cap Q(t)$.

LEMMA 4. *If R/a is a flat R module with a an ideal in R then*
(i) *$a = R$ if a contains an element which is not a zero divisor*
(ii) *for any prime $I < R$ if $I \neq R$ and $I \geqq a$ then $n(I) \geqq a$.*
(iii) *if b is an ideal in R and $\theta\colon R \to R/b = R^*$, θ the natural map then the module R^*/a^* is R^* flat with $a^* = \theta(a)$.*
(iv) *for any prime $I \not\geqq a$, $1 = e + n$ where e is in a and n is in $n(I)$*

Proof. We have that $R_I a = (0)$ or (1) for each prime of R. If a contains an x which is not a zero divisor then $R_I a = (1)$ for each I, thus $a = (1)$. For (ii) note that if $I \geqq a$ then $R_I a \neq (1)$ and thus $R_I a = (0)$ or $a \leqq n(I)$. To prove (iii) we need only show that for any maximal ideal $J^* < R^*$ either $R^*_{J^*} a^* = (0)$ or $R^*_{J^*} a^* = (1)$. If $J^* \not\geqq a^*$ then there is an x in a^* with x not in J^*. Thus x is not in $n(J^*)$ hence $R^*_{J^*} a^* = (1)$. If $J^* \geqq a^*$ then $M = \theta^{-1}(J^*)$ is maximal and contains a, thus $n(M) \geqq a$ and hence $n(J^*) \geqq \theta(n(M)) \geqq a^*$, therefore $R^*_{J^*} a^* = (0)$. Turning to (iv) assume $I \not\leq a$ with I a prime. Set $R^* = R/n(I)$, $a^* = \theta(a)$ with θ the natural map from R to R^* and assume $a^* \neq (1)$. Note that $a^* \neq (0)$ since $I \geqq n(I)$. One checks easily that $n(I^*) = (0)$ where $I^* = \theta(I)$. We have, therefore, that $R^*_{I^*} a^* = (1)$ and thus there is an x^* in a^* and a y^* not in I^* with $x^*/y^* = (1)$. Since

$a^* \neq (1)$ we have by (i) that there is an element z^* in R^* such that $0 = z^*x^*, z^* \neq 0$. Since $n(I^*) = (0)$ we have that $z^* = z^* x^*/y^* = 0$ which is a contradiction, thus $a^* = (1)$.

COROLLARY 3.2. *If* $(0) = Q(1) \cap \cdots \cap Q(s)$ *where* $Q(i)$ *is primary with radical* $p(i)$ *then every finitely generated flat module is projective.*

Proof. Since it suffices to prove the assertion for cyclic modules suppose R/a is flat with $p(i) \geqq a$ for $0 \leqq i \leqq t$ (0 if no $p(i)$ contains a). Clearly $n(p(i)) \leqq Q(i)$ and since $n(p(i)) \geqq a$ if $p(i) \geqq a$ it follows that $a \leqq Q(1) \cap \cdots \cap Q(t)$ (if $t = 0$ this intersection is defined to be R). If $p(j) \not\geqq a$ then by the previous Lemma $1 = e(j) + n(j)$ where $e(j)$ is in a and $n(j)$ is in $n(p(j))$. We may set $1 = e + n$ with e in a and n in $Q(t+1) \cap \cdots \cap Q(s)$ by taking the product of the elements $(e(j) + n(j))$ from $t+1$ to s, thus R/a is a direct summand.

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