AN EXTENSION OF LANDAU'S THEOREM ON TOURNAMENTS

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In an ordinary (round-robin) *tournament* there are n people, $p_1, \cdots, p_n$, each of whom plays one game against each of the other $n-1$ people. No game is permitted to end in a tie, and the *score* of p_i is the total number s_i of games won by p_i. By the *score sequence* of a given tournament is meant the set $S = (s_1, \cdots, s_n)$, where it may be assumed, with no loss of generality, that $s_1 \leqq \cdots \leqq s_n$. Landau [3] has given necessary and sufficient conditions for a set of integers to be the score sequence of some tournament. The object of this note is to show that these conditions are also necessary and sufficient for a set of real numbers to be the score sequence of a *generalized tournament*; a generalized tournament differs from an ordinary tournament in that as a result of the game between p_i and p_j, $i \neq j$, the amounts α_{ij} and $\alpha_{ji} = 1 - \alpha_{ij}$ are credited to p_i and p_j, respectively, subject only to the condition that $0 \leqq \alpha_{ij} \leqq 1$. The score of p_i is given by

$$s_i = \sum_{j=1}^{n}{}' \alpha_{ij} ,$$

where the prime indicates, for each admissible value of i, that the summation does not include $j = i$.

THEOREM. *A set of real numbers* $S = (s_1, \cdots, s_n)$, *where* $s_1 \leqq \cdots \leqq s_n$, *is the score sequence of some generalized tournament if and only if*

$$\sum_{i=1}^{k} s_i \geqq \binom{k}{2} , \tag{1}$$

for $k = 1, \cdots, n$ *with equality holding when* $k = n$.

Proof. The necessity of these conditions is obvious since (1) simply requires that the sum of the scores of any proper subset of the players be at least as large as the number of games played between members of this subset and that the sum of all the scores be equal to the total number of games played.

For terminology and results on flows in networks which will be used in the proof of the sufficiency of the above conditions see Gale [2]. A network N is constructed whose nodes are $x_1, \cdots, x_n$ and

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$y_1, \cdots, y_n$. The node x_i is joined to the node y_j by an arc of capacity $c(x_i, y_j) = 1$, if $i \neq j$. All other ordered pairs of nodes are joined by an arc of capacity zero. A demand d is defined on the nodes of N as follows: $d(x_i) = -s_i$ and $d(y_i) = l_i = (n-1) - s_i$. From (1) it follows that $-s_i \leqq 0 \leqq l_i$ for all i. It is not difficult to show, using the second formulation of the feasibility theorem in [2] (or see Fulkerson [1]), that this demand will be feasible if

$$(2) \qquad \sum_{i=1}^{k} l_i \leqq \sum_{i=1}^{k} \min\{s_i, k-1\} + \sum_{i=k+1}^{n} \min\{s_i, k\},$$

for $k = 1, \cdots, n$. We shall show that (1) implies (2) and that from the feasibility of the demand d we can ultimately infer the existence of a generalized tournament having S as its score sequence.

For each of the above values of k let $t = t(k)$ be the largest integer less than or equal to k such that $s_t \leqq k-1$; let $t(k) = 0$ if $s_1 > k-1$. It follows from (1) that

$$(3) \qquad \sum_{i=1}^{k} \min\{s_i, k-1\} = \sum_{i=1}^{t} s_i + (k-1)(k-t)$$
$$\geqq \binom{t}{2} + (k-1)(k-t) = \binom{k}{2} + \binom{k-t}{2} \geqq \binom{k}{2}.$$

Also, for each such value of k, let $h = h(k)$ be the largest integer less than or equal to $n-k$ such that $s_{k+h} \leqq k$; let $h(k) = 0$ if $s_{k+1} > k$. Then

$$(4) \qquad \sum_{i=k+1}^{n} \min\{s_i, k\} = \sum_{i=1}^{k+h} s_i - \sum_{i=1}^{k} s_i + k(n-k-h)$$
$$\geqq \binom{k+h}{2} + k(n-k-h) - \sum_{i=1}^{k} s_i$$
$$= \binom{h}{2} - \binom{k}{2} + k(n-1) - \sum_{i=1}^{k} s_i,$$

using (1) again and rearranging slightly. Combining (3) and (4) and using the definition of l_i we see that (2) will hold if

$$k(n-1) = \sum_{i=1}^{k} l_i + \sum_{i=1}^{k} s_i \leqq \binom{h}{2} + k(n-1),$$

which certainly holds for all k.

By definition the feasibility of the demand d means that there exists a flow f on the network such that

$$(5) \qquad \sum_{i=1}^{n} f(x_i, y_j) \geqq l_j, \qquad j = 1, \cdots, n,$$

and

$$\sum_{j=1}^{n} f(y_j, x_i) \geqq -s_i, \qquad i = 1, \cdots, n, \tag{6}$$

where $f(u, v)$ denotes the flow along the arc from u to v and such that $f(u, v) \leqq c(u, v)$ and $f(u, v) + f(v, u) = 0$ for all ordered pairs of nodes. These constraints and the fact that $\sum_{j=1}^{n} l_j = \sum_{i=1}^{n} s_i$ imply that equality holds throughout in (5) and (6), that $0 \leqq f(x_i, y_j) \leqq 1$ for all $i \neq j$, and that $f(x_i, y_i) = 0$ for all i.

Let

$$\alpha_{ij} = \tfrac{1}{2}(f(x_i, y_j) - f(x_j, y_i) + 1),$$

for all $i \neq j$. From the above properties of f it may easily be verified that

$$0 \leqq \alpha_{ij} \leqq 1,$$
$$\alpha_{ij} + \alpha_{ji} = 1,$$

and that

$$\sum_{j=1}^{n}{}' \alpha_{ij} = \tfrac{1}{2}[s_i - l_i + (n-1)] = s_i, \qquad \text{for } i = 1, \cdots, n.$$

These three properties of the α_{ij}'s are precisely those which are used to define a generalized tournament whose score sequence is S. Hence the existence of the flow f implies the existence of a generalized tournament having S as its score sequence, which suffices to complete the proof of the theorem.

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References

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3. H. G. Landau, *On dominance relations and the structure of animal societies. III. The condition for score structure*, Bull. Math. Biophys. **15** (1953), 143–148.

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