EXTREMAL ELEMENTS OF THE CONVEX CONE OF SEMI-NORMS

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1. Introduction. Let L be a real linear space and let p be a real function on L such that (1) $p(\lambda x) = |\lambda| p(x)$ for all x in L and all real λ, and $p(x_1 + x_2) \leqq p(x_1) + p(x_2)$ for all x_1 and x_2 in L, i.e. is a *semi-norm* on L. Since the sum of two semi-norms, $p_1 + p_2$ and the positive scalar multiplication of a semi-norm, λp, $\lambda > 0$ are semi-norms, the set of semi-norms on L, C form a convex cone. Those $p \in C$ such that if $p = p_1 + p_2$ where p_1 and $p_2 \in C$ we have p_1 and p_2 proportional to p are *extremal element* of C, [1]. In this paper it is shown that $p = |f|$, where f is a real linear functional of L is an extremal element of C. For L, the plane it is shown that these are the only extremal elements of C. Since norms are semi-norms, C includes this interesting class of functionals.

2. The main results. The convex cone C and the convex cone $-C$, the negatives of the elements of C have only the zero semi-norm in common since semi-norms are nonnegative. The zero semi-norm is an extremal element if one wishes to allow in the definition the vertex of a convex cone to be an extremal element. Below only the nonzero elements are considered.

The following lemma which characterizes the nature of certain semi-norms will be used in obtaining the two main theorems.

LEMMA 1. *If p is a semi-norm on L such that the co-dimension of $N(p) = 1$, then p is of the form $p = |f|$ where f is a linear functional on L.*

Proof. Let $b \in L \backslash N(p)$, where $N(p)$ is the *null space* of p. Then any element $x \in L$ can be written $x = z + \lambda b$ where $z \in N(p)$ and λ is real. Let $f(x) = \lambda p(b)$. Then clearly f is a linear functional on L. It shall now be shown that $|f(x)| = p(x)$ for all $x \in L$. Notice that

$$|f(x)| = |f(z + \lambda b)| = |\lambda p(b)| = |\lambda| p(b).$$

Thus

$$|f(x)| = p(\lambda b) = p(z) + p(\lambda b) \geqq p(z + \lambda b) = p(x).$$

The proof will be complete if it can be shown that the inequality

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cannot be a strict inequality for $\lambda \neq 0$.

Consider the case of the strict inequality occurring at $z' + \lambda_0 b$ where $\lambda_0 > 0$ and $z' \in N(p)$. The set $U = \{x : p(x) \leqq \lambda_0 p(b)\}$ is a convex circled set containing $N(p)$ and $\lambda_0 b$. It follows that there exists $\gamma \geqq 1$ such that

$$p(\gamma(z' + \lambda_0 b)) = \gamma p(z' + \lambda_0 b) = \lambda_0 p(b)$$

and hence $\gamma(z' + \lambda_0 b) \in U$. Take $\beta = (\gamma(1 - \alpha))/\alpha$ where $\alpha = (\gamma - 1)/(2\gamma)$. Then $0 < \alpha < 1$ and

$$\alpha[\beta(-z')] + (1 - \alpha)[\gamma(z' + \lambda_0 b)] = (1 - \alpha)\gamma\lambda_0 b$$

belongs to U since $-z'$ and $\gamma(z' + \lambda_0 b) \in U$ and U is convex. Now

$$p((1 - \alpha)\gamma\lambda_0 b) = (1 - \alpha)\gamma p(\lambda_0 b) > \lambda_0 p(b)$$

since $(1 - \alpha)\gamma = (1/2)(1 + \gamma) > 1$, a contradiction since $(1 - \alpha)\gamma\lambda_0 b \in U$. Thus $|f(x)| = p(x)$ for $\lambda_0 > 0$. Now for the case $\lambda_0 < 0$ it follows from the above

$$|f(x)| = |f(z + \lambda_0 b)| = |-f(-z - \lambda_0 b)| = |f(-z - \lambda_0 b)|$$

and

$$|f(-z - \lambda_0 b)| = p(-z - \lambda_0 b) = p(z + \lambda_0 b).$$

Thus $p(x) = |f(x)|$ for all $x \in L$.

It is now possible to prove the following theorem which shows that the absolute value of a real linear functional is an extremal element of C.

THEOREM 1. *If f is a real linear functional on L, then $|f|$ is an extremal element of C.*

Proof. It is easy to check that $|f|$ is subadditive and absolutely homogeneous and hence $|f| \in C$.

Suppose $|f| = p_1 + p_2$ where p_1 and $p_2 \in C$. Since p_1 and p_2 are nonnegative $0 \leqq p_i \leqq |f|$, $i = 1, 2$. Thus when $f(x) = 0$, $p_i(x) = 0$, $i = 1, 2$ and $N(f) \subset N(p_i)$, $i = 1, 2$. Hence the co-dimension of p_1 and p_2 is less than or equal to one. If the co-dimension of $N(p_1)$ is zero, then clearly p_1 and p_2 are proportional to $|f|$. If the co-dimension of $N(p_1)$ is one then by Lemma 1, there exists a real linear functional f_1 such that $p_1 = |f_1|$. Since $N(f_1) = N(p_1) \supset N(f)$ it follows that $\lambda_1 f = f_1$ for some real $\lambda_1 \neq 0$. Hence $|\lambda_1||f| = p_1$. Thus p_1 (and consequently p_2) is proportioned to $|f|$, and hence $|f|$ is an extremal element of C.

The following theorem shows that for the case $L = E^2$, the Euclidean plane, the only extremal elements for C are the semi-norms given in Theorem 1.

THEOREM 2. *Let $L = E^2$, then if p is an extremal element of C, there exists a linear functional f on L such that $p = |f|$.*

In order to prove this theorem it will be necessary to show that for p a semi-norm on L and p not of the form $p = |f|$ then there exists semi-norms p_1 and p_2 on L such that $p = p_1 + p_2$ and p_1 (and consequently p_2) is not proportional to p.

It follows from Lemma 1 that for a semi-norm p on L to not be of the form $|f|$, where f is a linear functional on L that the co-dimension of $N(p)$ must be greater than one. Hence for arbitrary L and p an extremal element of C other than those of Theorem 1, then p must have the co-dimension of $N(p) > 1$. For $L = E^2$ and $p \in C$ such that the co-dimension of $N(p) > 1$, then p is a norm. Thus for the proof of Theorem 2 a non-proportional decomposition must be provided for all norms on E^2.

For p a norm on $E^2 = \{(x_1, x_2)\}$, the unit ball $U(p) = \{x : p(x) \leqq 1\}$ is a convex circled set containing the origin as a core point. There is no loss in generality in assuming that the segment $(-1, 0)$, $(1, 0)$ is a diameter of $U(p)$. This will mean that $U(p)$ is contained in the closed unit disk with center at the origin. Let $b_p(x_1) = \sup \{x_2 : (x_1, x_2) \in U(p)\}$, the function giving the upper boundary of $U(p)$. Then b_p is a concave function on $[-1, 1]$ and $b_p(+1) = 0$. The lower boundary is given by $b'_p(x_1) = -b_p(-x_1)$ since $p(-x) = p(x)$. The next lemma gives a non-proportional decomposition of norms p such that the set $U(p)$ is a parallelogram.

LEMMA 2. *Let p be a norm on E^2 such that $b_p(a_1) = b_1 > 0$ for some a_1, $-1 \leqq a_1 \leqq 1$ and $b(x_1)$ is linear on $[-1, a_1]$ and on $[a_1, 1]$, then p is not an extremal element of C.*

Proof. Let $p_1((x_1, x_2)) = (1/b_1)\,|\,b_1x_1 - a_1x_1|$ and let $p_2((x_1, x_2)) = (1/b_1)\,|\,x_2|$. Then p_1 and $p_2 \in C$ since they are positive multiples of the absolute values of linear functionals. In order to see $f = p_1 + p_2$ it is sufficient to show that $p_1((x_1, b_p(x_1))) + p_2((x_1, b_p(x_1))) = 1$ for all $x_1 \in [-1, 1]$. This can be easily checked directly by substituting in the equations of the appropriate straight lines for b_p. Clearly p_1 and p_2 are not proportional to p.

The next lemma will give a non-proportional decomposition of a norm p such that the set $U(p)$ is a six-sided polygon.

LEMMA 3. *Let p be a norm on E^2 such that $b_p(a_i) = b_i > 0$,*

$i = 1, 2$, where $-1 < a_1 < a_2 < 1$ and b_p is linear on $[-1, a_1]$, $[a_1, a_2]$ and on $[a_2, 1]$, then p is not an extremal element of C.

Proof. Let $p_i((x_1, x_2)) = \alpha_i \mid a_i x_2 - b_i x_1 \mid$, $i = 1, 2$ and let $p_3((x_1, x_2)) = \alpha_3 \mid x_2 \mid$ where

$$\alpha_1 = (b_2/\Delta)(b_1 - b_2 + \mid b_2 a_1 - a_2 b_1 \mid),$$
$$\alpha_2 = (b_1/\Delta)(b_2 - b_1 + \mid b_2 a_1 - a_2 b_1 \mid),$$
$$\alpha_3 = ((\mid b_2 a_1 - a_2 b_1 \mid)/\Delta)(b_1 + b_2 - \mid b_2 a_1 - a_2 b_1 \mid),$$

and

$$\Delta = 2 b_1 b_2 \mid b_2 a_1 - a_2 b_1 \mid.$$

Then $p = p_1 + p_2 + p_3$ gives a non-proportional decomposition of p.

Although an extension of this method will not be used in the proof of Theorem 2 it is worth noting at this point that this method of decomposing p can be used on any norm p such that $U(p)$ is a polygon. For a polygon with $2n + 2$ sides then $b_p(x)$ is a concave polygonal function having vertices at $\{(a_i, b_i)\}$, $i = 1, 2, \cdots, n$ where $b_i > 0$ and $-1 < a_1 < a_2 < \cdots < a_n < 1$. In this case set.

$$p(x) = \sum_{i=1}^{n} \alpha_i \mid a_i x_2 - b_i x_1 \mid + \alpha_{n+1} \mid x_2 \mid.$$

By substituting each of the points (a_i, b_i), $i = 1, 2, \cdots, n$ and $(1, 0)$ in this equation we have $n + 1$ linear equations in $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ since $p((a_i, b_i)) = p((1, 0)) = 1$ for all i. By solving for the α_i and nothing that they are nonnegative we get the required decomposition of p. Notice that p is a finite sum of extremal elements of C.

For any norm p on E^2 such that $U(p)$ is not a polygon of less than six sides, that is p is a norm different from those considered in Lemmas 2 and 3, then there exist points of E^2, $x^{(1)} = (a_1, b_p(a_1))$, $x^{(2)} = (a_2, b_p(a_2))$, $-1 \leqq a_1 < a_2 \leqq 1$, $a_2 - a_1 < 2$ such that b_p is not piecewise linear on $[a_1, a_2]$ on three or fewer non-overlapping segments whose union is $[a_1, a_2]$. This means that p restricted to the line segment $[x^{(1)}, x^{(2)}]$ is a strictly positive convex function that is not piecewise linear on three or fewer non-overlapping segments whose union is $[x^{(1)}, x^{(2)}]$.

Let C_{12} be the convex cone in E^2 with vertex at the origin that is generated by $[x^{(1)}, x^{(2)}]$ and let $-C_{12}$ be the negatives of the vectors in C_{12}. Let $U(p')$ be the closed convex hull of $U(p) \backslash (C_{12} \cup (-C_{12}))$. Let t_1 and t_2 be the tangent half-lines to $U(p)$ at $x^{(1)}$ and $x^{(2)}$ respectively. These tangent half-lines are to be taken from the interior of C_{12}. Their intersection $x^{(3)}$ will be a point in C_{12}. Let $U(p'')$ be the closed convex circled set whose boundary $U(p) \backslash (C_{12} \cup (-C_{12}))$ is the same as $U(p)$ and whose boundary in C_{12} is $[x^{(1)}, x^{(3)}] \cup [x^{(3)}, x^{(2)}]$.

Let p' and p'' be the semi-norms whose unit ball is $U(p')$ and $U(p'')$ respectively. Since $U(p') \subset U(p) \subset U(p'')$ we have $p'(x) \leqq p(x) \leqq p''(x)$ for all $x \in E^2$. Then if there exist semi-norms q_1 and q_2 on E^2 such that $p'(x) \leqq q_i(x) \leqq p''(x)$, $i = 1, 2$ for all $x \in E^2$ and such that on $C_{12} \cup (-C_{12})$,

$$\alpha q_1(x) + (1 - \alpha) q_2(x) = p(x),$$

$0 < \alpha < 1$, q_1 (and hence q_2) is not equal to p on $C_{12} \cup (-C_{12})$, then $p_1 = \alpha q_1$ and $p_2 = (1 - \alpha) q_2$ will be semi-norms on E^2 such that $p_1 + p_2 = p$ and p_i, $i = 1, 2$ is not proportional to p. Thus the problem reduces to showing the existence of these semi-norms q_1 and q_2.

Notice that it must be that $q_1(x) = q_2(x) = p(x)$ on $E^2 \backslash (C_{12} \cup (-C_{12}))$ and hence it remains to show that the definition of q_1 and q_2 can be satisfactorily extended as required above to all of E^2. If q_i, $i = 1, 2$, restricted to the closed line segment $[x^{(1)}, x^{(2)}]$ is defined to be a convex function such that $q_i \neq p$ restricted to this same segment but agreeing with p at $x^{(1)}$ and $x^{(2)}$ and $q_i \geqq p'$ restricted to this same segment then q_i can be extended to a semi-norm on E^2. Consider the following: For $x \in C_{12}$, $x \neq 0$, there is a $\lambda > 0$ such that λx belongs to $[x^{(1)}, x^{(2)}]$. Then take $q_i(x) = (1/\lambda) q_i(\lambda x)$. For $x \in (-C_{12})$ take $q_i(x) = q_i(-x)$ and take $q_i(0) = 0$. Now $U(q_i)$ is a closed convex circled set since the central projection of a convex curve is convex. Hence q_i is a semi-norm. Notice $U(p') \subset U(q_i) \subset U(p'')$ and thus $p'(x) \leqq q_i(x) \leqq p''(x)$, $i = 1, 2$ and $x \in E^2$. Notice also that the slopes of the half-tangents to q_i, $i = 1, 2$ restricted to $[x^{(1)}, x^{(2)}]$ are finite even at the end-points. The possibility of defining q_i, $i = 1, 2$ on $[x^{(1)}, x^{(2)}]$ as required above is assured by the following lemma.

LEMMA 4. *Let f be a real convex function on $[a, b]$ such that the right-hand derivative at a, $f'_+(a)$ and the left-hand derivative at b, $f'_-(b)$ are finite. Suppose further that f is not piecewise linear on three or fewer non-overlapping segments whose union is $[a, b]$. Then there exist real convex functions f_1 and f_2 on $[a, b]$ that differ from f on $[a, b]$, but have the same values and derivatives as f at the end-points and for some α, $0 < \alpha < 1$, $\alpha f_1(x) = (1 - \alpha) f_2(x) + f(x)$ for all $x \in [a, b]$*

Proof. Let $h(x) = f'_+(a)(x - a) + f(a)$. Then $F = (1/m)(f - h)$, where m is the left-hand derivative of $f - h$ at b, is a nonnegative convex function on $[a, b]$ such that $F(a) = 0$, $F'_+(a) = 0$, and $F'_-(b) = 1$. The right-hand derivative of F, F'_+ is a nondecreasing right continuous function on $[a, b]$. Let F'_+ be defined at b by $F'_+(b) = F'_-(b)$. Since f is not piecewise linear on three or fewer non-overlapping segments whose union is $[a, b]$ then the range of F'_+ has at least four values, that is two besides 0 and 1. If there exist two non-decreasing right continuous functions F_i, $i = 1, 2$ on $[a, b]$ such that $F_i(a) = 0$, $F_i(b) = 1$, $F_i \neq F'_+$ on some subinterval of $[a, b]$,

$$\alpha F_1(x) + (1 - \alpha)F_2(x) = F'_+(x),$$

$0 < \alpha < 1$ on $[a, b]$, and

$$\int_a^b F_i(x)dx = \int_a^b F'_+(x)dx$$

then the required functions f_i are given by

$$f_i(x) = h(x) + m\int_a^x F_i(t)dt,$$

$i = 1, 2$.

Consider first the case of F'_+ having at least three discontinuities. Let F'_+ have positive jump discontinuities of θ_i at c_i, $i = 1, 2, 3$ where $a < c_1 < c_2 < c_3 < b$. Take $\theta = (1/2) \min(\theta_1, \theta_2, \theta_3)$. Let

$$F_1(x) = F'_+(x) - \sigma_1,$$

when $c_1 \leqq x < c_2$,

$$F_1(x) = F'_+(x) + \sigma_2,$$

when $c_2 \leqq x < c_3$, and $F_1(x) = F'_+(x)$ elsewhere; and let

$$F_2(x) = F'_+(x) + \sigma_1,$$

when $c_1 \leqq x < c_2$,

$$F_2(x) = F'_+(x) - \sigma_2,$$

when $c_2 \leqq x < c_3$, and $F_2(x) = F'_+(x)$ elsewhere. Take σ_i, $i = 1, 2$ such that $0 < \sigma_i < \theta$, $\sigma_1(c_2 - c_1) = \sigma_2(c_3 - c_2)$. It follows that F_1 and F_2 satisfy the above requirement for $\alpha = (1/2)$.

Now for the case where F'_+ has less than three points of discontinuity it follows from the condition that F'_+ has at least four range values that there exists a subinterval of $[a, b]$ on which F'_+ is continuous and non-constant. If now F_1 and F_2 can be defined on $[a_1, b_1]$ as it was required that they be on $[a, b]$ then F_1 and F_2 can be extended to $[a, b]$ by taking $F_1(x) = F_2(x) = F'_+(x)$ for $x \in [a, b] \backslash [a_1, b_1]$. It will follow that F_1 and F_2 obtained in this manner satisfy the above requirements. Thus it is sufficient to show the existence of F_1 and F_2 where F'_+ is continuous on $[a, b]$.

Let us perform one further simplification. Let $\bar{a}=\sup\{x : F'_+(x)=0\}$ and let $\bar{b} = \inf\{x : F'_+(x) = 1\}$. Then $a \leqq \bar{a} < \bar{b} \leqq b$. Since F_1 and F_2 are non-decreasing, $F_i(a)=0$, and $F_i(b)=1$, and since $\alpha F_1+(1-\alpha)F_2= F'_+$ it follows that $F_i(x) = 0$ on $[a, \bar{a}]$ and $F_i(x) = 1$ on $[\bar{b}, b]$, i $= 1, 2$. Thus we may assume that $0 < F'_+(x) < 1$ on the interior of the interval of definition. Take the interval $[\bar{a}, \bar{b}]$ to be $[0, 1]$ since there is no loss in generality in doing so.

The problem is now reduced to the following: Given F (instead of F'_+ for simplicity) a continuous non-decreasing function on $[0, 1]$ such that $F(0) = 0$, $F(1) = 1$ and $0 < F(x) < 1$ for $0 < x < 1$. Show that there exist two functions F_1 and F_2 that have the same properties as F but are not F (that is, they differ from F at one point) and such that for some α, $0 < \alpha < 1$, $\alpha F_1 + (1 - \alpha)F_2 = F$ and such that

$$\int_0^1 F_i\,dx = \int_0^1 F\,dx$$

$i = 1, 2$. Take η_1, η_2, η_3 such that $0 < \eta_1 < \eta_2 < \eta_3 < 1$ and let ξ_i, $i = 1, 2, 3$ be such that $F(\xi_i) = \eta_i$. Then let

$$F_1(x) = (\eta_2/\eta_1)\min(F(x), \eta_1),$$

when $0 \leqq x \leqq \xi_2$ and

$$F_1(x) = ((1 - \eta_2)/(1 - \eta_3))(\max(F(x),\eta_3) - \eta_3) + \eta_2,$$

when $\xi_2 < x \leqq 1$. Let

$$F_2(x) = (\eta_2/(\eta_2 - \eta_1))(\max(F(x), \eta_1) - \eta_1),$$

when $0 \leqq x \leqq \xi_2$ and

$$F_2(x) = ((1 - \eta_2)/(\eta_3 - \eta_2))(\min(F(x), \eta_3) - \eta_2) + \eta_2,$$

when $\xi_2 < x \leqq 1$. Now F_1 and F_2 are continuous non-decreasing on $[0, 1]$ such that $F_i(0) = 0$, $F_i(1) = 1$, $i = 1, 2$ and $F_i \neq F$. Then

$$(\eta_1/\eta_2)F_1 + ((\eta_2 - \eta_1)/\eta_2)F_2 = F$$

on $[0, \xi_2]$ and

$$((1 - \eta_3)/(1 - \eta_2))F_1 + ((\eta_3 - \eta_2)/(1 - \eta_2))F_2 = F$$

on $(\xi_2, 1)$. Take $\eta_1 = (1/2)\eta_2$ and $\eta_3 = (1/2)(1 + \eta_2)$. Then it follows that $f = (1/2)F_1 + (1/2)F_2$ on $[0, 1]$, with η_2 arbitrary. It remains only to be shown that η_2 can be chosen such that

$$\int_0^1 F_i\,dx = \int_0^1 F\,dx,$$

$i = 1, 2$ but this is assured if there exists a ξ_2, $0 < \xi_2 < 1$ such that

$$G(\xi_2) = \int_0^{\xi_2} (F_1 - F)\, dx = \int_{\xi_2}^1 (F - F_1)\, dx = H(\xi_2).$$

It can easily be checked that $G(0) = H(1) = 0$, G is a not identically zero non-decreasing continuous function on $[0, 1)$ and H is a not identically zero non-increasing continuous function on $(0, 1]$. Hence there exists ξ_2, $0 < \xi_2 < 1$ such that $G(\xi_2) = H(\xi_2)$.

3. **Remarks.** The argument in E^2 that shows that the norms in E^2 are not extremal elements of C shows also that for L general and $p \in C$ such that the co-dimension of $N(p) = 2$, then p is not an extremal element of C. Thus for L general any extremal element of C other than those mentioned in Theorem 1 must be such that the co-dimension of its null space is greater than or equal to two.

References

1. N. Bourbaki, *Espaces vectoriels topologiques.* Act. Sci. Ind. no. 1189, Paris, 1953.
2. G. Choquet, *Theory of capacities.* Ann. Inst. Fourier, Grenoble (1953–1954), pp. 131–295, 1955.

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