A NOTE ON UNCOUNTABLY MANY DISKS

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R. H. Bing has shown [2] that E^3 (Euclidean three dimensional space) does not contain uncountably many mutually disjoint wild 2-spheres. J. R. Stallings has given an example [6] to show that E^3 does contain uncountably many mutually disjoint wild disks. It is the goal of this note to show that E^3 does not contain uncountably many mutually disjoint disks each of which fails to lie on a 2-sphere in E^3. (A disk which fails to lie on a 2-sphere is necessarily wild.) For definitions the reader is referred to [1].

THEOREM 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists a disk D of the collection V such that D lies on a 2-sphere in E^3.*

The proof of Theorem 1 follows immediately from the following three lemmas.

LEMMA 1. *If V is an uncountable collection of mutually disjoint disks in E^3 then there exists an uncountable subcollection V^* of V such that if D belongs to V^*, x is an interior point of D, ax is an arc intersecting D only in the point x, and ε is a positive number then there exists an uncountable subcollection V_1 of V^* such that if D_1 is an element of V_1 then* (i) *$D_1 \cap ax \neq \phi$ and* (ii) *there is a homeomorphism of D_1 onto D which moves no point more than ε.*

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3. Let V' denote the subcollection of V defined as follows: D is an element of V' if and only if there exist a point x of Int D, an arc ax intersecting D only in x, and a positive number ε such that there is no uncountable subcollection V_1 of V such that if D_1 belongs to V_1 then (i) $D_1 \cap ax \neq \phi$ and (ii) there is a homeomorphism of D_1 onto D which moves no point more than ε.

It is clear that in order to establish Lemma 1 it is sufficient to show that the collection V' is countable. Suppose that V' is uncountable.

For each element D_α of V' let an arc a_α and a positive number ε_α be chosen such that (i) the common part of D_α and a_α is an end-point of a_α which is on the interior of D_α, and (ii) a_α intersects only a countable number of elements D of V such that there is a homeomorphism of D onto D_α which moves no point by more than ε_α.

Received January 15, 1963. This paper was witten while the author was a post-doctoral fellow of The National Science Foundation.

Let ε be a positive number and V'' be an uncountable subcollection of V' such that if D_α is an element of V'' then $\varepsilon < \varepsilon_\alpha$.

Let E be a disk and v be an arc such that the common part of E and v is an endpoint of v which is on the interior of E. For each element D_α of V'' let h_α be a homeomorphism of $E \cup v$ onto $D_\alpha \cup a_\alpha$. Now $\{h_\alpha; D_\alpha \in V''\}$ with the distance function

$$D(h_\alpha, h_\beta) = \max_{t \in E \cup v} \rho\big(h_\alpha(t), h_\beta(t)\big)$$

is a metric space. In [3] (Theorem 2) Borsuk shows that this metric space is separable. It follows that there exists an element D_{α_0} of V'' such that if δ is a positive number then $\{h_\beta; D(h_\beta, h_\alpha) < \delta\}$ is uncountable. Let h_{α_0} be denoted by h_0, $h_0(E)$ be denoted by D_0, and $h_0(v)$ be denoted by a_0.

Let the endpoints of a_0 be denoted by x and y and assume that the notation is chosen so that $y \in \operatorname{Int} D_0$. Let zyx be an arc such that $a_0 \subset zyx$ and zyx pierces D_0 at y. Let zwx be an arc in $E^3 - D_0$ such that $zwx \cap zyx = \{z, x\}$, and let J denote the simple closed curve $zyx \cup zwx$. Since $J \cup D_0 = \{y\}$ it follows that $Bd\ D_0$ links J.

Now let ε_1 be a positive number such that $2\varepsilon_1$ is less than the minimum of ε, dist $(J, Bd\ D_0)$, and dist (zwx, D_0).

Let H be $\{h_\beta; D(h_\beta, h_0) < \varepsilon_1/2\}$, and let V''' be the set of all elements of V'' such that $D \in V'''$ if and only if there exists an element h of H such that $h(E) = D$. Now if D_1 and D_2 are two elements of V''' then there exists a homeomorphism of D_1 onto D_2 that moves no point more than ε_1.

Suppose that D is an element of V'''. Then since $2\varepsilon_1 <$ dist $(J, Bd\ D_0)$, $Bd\ D_0$ links J, and there is a homeomorphism of D_0 onto D which moves no point more than $\varepsilon_1/2$ it follows that $Bd\ D$ links J, and hence that $J \cap D \neq \phi$. Since $2\varepsilon_1 <$ dist (zwx, D_0), $D \cap zyx \neq \phi$.

Now for each element D_α of V''' let P_α be the greatest point of $D_\alpha \cap zyx$ in the order from z to x on zyx. Now there exists an element D_γ of V''' such that for uncountably many elements D_α of V''', P_α is greater than P_γ. But since $2\varepsilon_1 <$ dist (x, D_0), $2\varepsilon_1 <$ dist $(J, Bd\ D_0)$, and for each element D_α of V''' there is a homeomorphism of $D_0 \cup a_0$ onto $D_\alpha \cup a_\alpha$ which moves no point more than $\varepsilon_1/2$, it follows that a_γ intersects every element D_α of V''' such that P_α is greater than P_γ. This is because a_γ may be completed to a simple closed curve J' which links $Bd\ D_\alpha$ and which intersects D_α only in a_γ. Hence a_γ intersects uncountably many elements of the collection V'''. This is contradictory to the way in which a_γ was chosen and it follows that the collection V' is countable. This establishes Lemma 1.

LEMMA 2. *Suppose that V is an uncountable collection of mutu-*

ally disjoint disks in E^3. *Then there exists a disk* D *of the collection* V *such that* D *is locally tame at each point of* Int D.

Proof. Let V be an uncountable collection of mutually disjoint disks in E^3. Let V^* be an uncountable subcollection of V satisfying the conclusion of Lemma 1. Let D be an element of the collection V^* and let p be an interior point of D. By Theorem 5 of [1] there exists a subdisk D' of D and a 2-sphere S in E^3 such that $p \in \text{Int } D'$ and $D' \subset S$. Without loss of generality it may be assumed that $ap \subset \text{Int } S$ and $pb \subset \text{Ext } S$. Now there exist sequences $D_1D_2\cdots$ and $C_1C_2\cdots$ of disks of the collection V^* such that for each i, (1) $D_i \cap ap \neq \phi$, (2) $C_i \cap pb \neq \phi$, and (3) there exist homeomorphisms f_i and g_i of D_i and C_i, respectively, onto D which move no point more than $1/i$.

Let D'' be a subdisk of D' such that $p \in \text{Int } D''$ and $D'' \subset \text{Int } D'$. Now without loss of generality it may be assumed that each of $f_1^{-1}(D'')$, $f_2^{-1}(D'')\cdots$ lies in Int S and that each of $g_1^{-1}(D'')$, $g_2^{-1}(D'')\cdots$ lies in Ext S. It follows from Theorem 9 of [1] that S is locally tame at p and hence that D is locally tame at p. This establishes Lemma 2.

LEMMA 3. *If* D *is a disk in* E^3 *and* D *is locally tame at each point of* Int D *then* D *lies on a 2-sphere in* E^3.

Proof. Let D be a disk in E^3 which is locally tame at each point of Int D. It follows from [5] that there exists a homeomorphism h of E^3 onto itself such that $h(D)$ is locally polyhedral except on $h(Bd\ D)$. It follows from the proof of Lemma 5.1 of [4] that there exists a 2-sphere S in E^3 such that $h(D) \subset S$. Then $h^{-1}(S)$ is a 2-sphere in E^3 such that $D \subset h^{-1}(S)$. This establishes Lemma 3.

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